ESSAYS IN CORPORATE FINANCE AND INVESTMENT

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Abstract

This thesis consists of two essays that examine several problems in corporate finance and mechanism design. The central theme is endogenous agency conflicts and their impact on dynamic investment decisions. The first essay features auctions of assets and projects with embedded real options, and subsequent exercises of these investment options. The essay shows timing and security choice of auctions endogenously misalign incentives among agents and derives the optimal auction design and exercise strategy. The second essay studies implications of endogenous learning on irreversible investment decisions, in particular, how learning gives rise to asymmetric information between managers and shareholders in decentralized firms. Depending on the quality of the project, the optimal contract between principal and agent distorts investments in ways that has not been examined in the literature.

Specifically, in Chapter 1 of the dissertation, I study how governments and corporations auction real investment options using both cash and contingent bids. Examples include sales of natural resource leases, real estate, patents and licenses, and start-up firms with growth options. I incorporate both endogenous auction initiation and post-auction option exercise into the traditional auctions framework, and show that common security bids create moral hazard because the winning bidder’s real option differs from the seller’s. Consequently, investment could be either accelerated or delayed depending on the security design. Strategic auction timing affects auction initiation, security ranking, equilibrium bidding, and investment; it should be considered jointly with security design and the seller’s commitment level. Optimal auction design aligns investment incentives using a combination of down payment and royalty payment, but inefficiently delays sale and investment. I also characterize informal negotiations as timing and signaling games in which bidders can initiate an auction and determine the forms of bids. I show that post-auction investments are efficient and bidding equilibria are equivalent to those of cash auctions. However, in
this setting, bidders always initiate the informal auctions inefficiently early. In addition, I provide suggestive evidence for model predictions using data from the leasing and exploration of oil and gas tracts, which leads to several ongoing empirical studies. Altogether, these results reconcile theory with several empirical puzzles and imply novel predictions with policy relevance.

In Chapter 2, I examine learning as an important source of managerial flexibility and how it naturally induces information asymmetry in decentralized firms. Timing of learning is crucial for investment decisions, and optimal strategies involve sequential thresholds for learning and investing. Incentive contracts are needed for learning and truthful reporting. The inherent agency conflicts alter investment behavior significantly, and are costly to investors and welfare. But contracting on learning restores efficiency with low future uncertainty or sufficient liquidity. Unlike prior studies, the moral hazard of learning accelerates good projects and delays bad projects. Even the best type’s investment is distorted, and only when learning is contractible can adverse selection dominate learning.
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Chapter 1

Auctions of Real Options

1.1 Introduction

On September 8, 2003, Regeneron, a New York based biotech company, granted the pharmaceutical giant Aventis a license to develop and commercialize Regeneron’s Vascular Endothelial Growth Factor (VEGF) Trap, a potential cancer therapeutic. In return, Aventis would pay an $80 million upfront fee—the highest to date for a product in phase I clinical trials—and $430 million in stock purchase and milestone awards, and would cover all development costs and split future product revenue evenly with Regeneron. Ten years later in an oil lease auction that netted the U.S. Government $1.2 billion, Exxon-Mobil emerged as the highest bidder and was entitled, but not obliged, to explore and drill on seven of the 320 auctioned tracts in the Central Gulf of Mexico for 5–10 years, and was to pay 18.75% royalty of future revenues from oil production to the U.S. Department of Interior.

In these two deals, both the license and the lease constitute classic examples of a broader class of projects and assets with embedded real options whose sale and exercise underlie some of the most crucial decisions for entrepreneurs, firm executives, and government

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1 In fact, Aventis acquired a portfolio of options: the timing option to observe the progress of Genentech’s drug Avastin before developing VEGF, the exit option if VEGF fails clinical tests, and the operations option to use research knowledge of VEGF to anchor into the anti-angiogenesis therapeutic market.


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officials. Moreover, these transactions routinely involve competing bids in combinations of cash and payment contingent on the asset’s future cash flows, and can be effectively viewed through the lens of security-bid auctions. Why did Regeneron negotiate for both cash and royalty payments? Why are nearly 72% of oil and gas tracts offshore and 56% of those on federal lands neither producing nor under active exploration despite the imbalance in supply and demand? More fundamentally, how should a seller trade off rent extraction using security bids with incentive provision for option exercise? How to jointly decide security choice and auction timing? What is the role of the seller’s commitment and who initiates the auction in equilibrium?

To simultaneously address these important questions, this paper builds a model of auctions of real options, endogenizing auction initiation and tying together option exercises with selling mechanisms. I derive the following main results under the unifying intuition that economic agents enjoy different optionalities in the sale and operation of an asset: First, common security bids cause inefficient and often suboptimal investments. Second, strategic auction timing affects auction initiation, security choice, bidding equilibrium, and post-auction investment, and should be considered jointly with security design and option exercise. Third, optimal auction design entails delayed auction initiation and investment, but aligns investment incentives using cash and royalties, as is prevalent in real-life business practice. Fourth, when a seller lacks commitment to auction design, bidders always initiate the auction and bidding equilibria are equivalent to those in cash auctions. These findings imply that many conclusions from traditional auction and real options models need to be modified in dynamic settings with learning and strategic interactions. I argue that the model sheds light on several puzzling empirical observations, and further show that oil exploration

4 Oil leases have been auctioned using cash contracts, bonus-bid contracts, royalty contracts and pure profit share contracts. In transfer and licensing of technologies, as in the example of pharmaceutical industry, rival bidders offer various contingent contracts (Vishwasrao (2007) and Bessy and Brousseau (1998)). In sales of large assets such as the wireless spectrum auction for FCC bandwidth, aggressive bidders can declare bankruptcy and the bids are essentially debts (Board (2007a) and Zheng (2001)). Equities, preferred convertibles, call options, and debts are frequently used in M&A and venture capital financing (Martin (1996), Kaplan and Stromberg (2003), and Hellmann (2006)). Other examples include advance and royalty payments in publishing contracts (Dessauer (1981) and Caves (2003)), motion picture deals (Chisholm (1997)), business licenses such as electronic gambling machines with pre-specified profit tax, and military procurement contracts (McAfee and McMillan (1987b)).

5 Oil and Gas Lease Utilization - Onshore and Offshore. Report to the President” by U.S. Department of the Interior dated May 2012. This revelation has triggered a huge public outcry and heated debate in congress on the reason for the purported sluggish development of natural resources, and has policy implications in the backdrop of Obama’s proposal to increase onshore royalty by 50%.
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data and business anecdotes corroborate model implications.

Auctions of real options are prevalent in licensing and patent acquisitions, leasing of natural resources, real estate development, M&A deals, venture capital and private equity markets, and privatization of large national enterprises. They entail tremendous financial resources and mainly come in two categories in practice. While in formal settings such as oil lease auctions or wireless spectrum auctions, the seller specifies explicitly and commits to the time and rules of the auction and an ordered set of security bids, many other cases can be thought of as informal auctions where the seller lacks such commitment. Prominent examples of informal auctions include corporate takeovers and project finance, where bidders decide what to offer and often can initiate the contact or negotiation. Still others such as licensing agreements and contracts in the entertainment industry appear in both categories. This paper addresses both formal and informal auctions.

While prior studies offer insights into auctions of real options, existing applications typically take auction initiation as exogenous and do not consider post-auction dynamics, leaving out the evolving market environment and the dependence of an asset’s payoff on post-auction actions. They also analyze the sales and exercises of real options in isolation, and exogenously specify the agency conflicts in the latter. This paper differs by taking into consideration that auction initiation affects the security choice, and the security design shapes the investment decision.

This study therefore attempts to bridge the gap between auction theory and corporate finance, and adds to the emerging literature both on agency conflicts in real options and on auction initiation and security bids. The first key contribution is recognizing the optionality and dynamic nature of auctions. This paper shows equilibrium behaviors can differ significantly from static settings and underscores that auction timing, security design, and real option exercise are interdependent and should not be studied in isolation in this setting.

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6 Bolton, Roland, Vickers, and Burda (1992) describe the privatization policies in Central and Eastern Europe. Pakes (1986) and Schwartz (2004) discuss patents as real options. One prominent M&A case involves Microsoft’s $8.5 billion cash acquisition of the voice-over-IP service Skype—its largest acquisition to date—for the portfolio of options such as Windows phone integration, and technology merger with outlook which it exercised in 2013. Google’s largest acquisition with $12.5 billion for Motorola in 2012 also gave it the option to develop the portfolio of patents Motorola held, and other options such as the one to sell the set-top box business which it did in 2013 to Arris Group. Both deals are against the backdrop of potential rival bids and can be effectively viewed as auctions.

7 In the Gulf of Mexico alone, the oil and gas leases auctioned by the U.S. federal government in 1954-2007 have exceeded 300 billion, and annual licensing deals by pharmaceutical giants exceed $20 billion even in the aftermath of the financial crisis. M&A volume world-wide is also in the trillions of dollars annually.

8 DeMarzo, Kremer, and Skrzypacz (2005) first introduce the concept.
second key contribution is in linking agency issues in real option exercises to selling mechanisms. Auction design and competitive bidding endogenously induce agency conflicts. In addition, this paper casts auction initiation as an optimal stopping problem to capture a new dimension of strategic interaction amongst sellers and bidders, and empirically examines its implications on corporate investment.

Specifically, this paper models the sale and exercise of a typical investment option with endogenous participation in both formal and informal settings. The model involves a seller and multiple potential bidders who are risk-neutral and maximize their expected payoffs. They interact in continuous time in three sequential stages. In the first stage, the seller (or potentially a bidder in informal auctions) strategically initiates the auction. In the second stage, participating bidders bid cash and contingent securities and the seller allocates the asset. In the final stage, the winning bidder rationally times the exercise of the investment option and delivers the contingent payment to the seller.

There are two key frictions in the model. The first is the non-contractibility of the bidders’ private information. Contingent payment does not account for the bidders’ private costs and thus misaligns investment incentives in the third stage. This leads to a tradeoff for the seller between the post-auction moral hazard in investments and the benefits of contingent bids such as enhanced rent extraction. As contingent bids become increasingly prevalent, this tradeoff could have a first-order impact on projects with high option values, such as the development of real estate and natural resources, as well as the transfer and licensing of technologies. Moreover, there is no “one-size-fits-all” in security ranking, as any comparison has to be made in conjunction with considerations of auction timing and the market environment.

The second friction is the cost associated with the ownership transfer, such as legal fees for underwriting contingent securities, initial opportunity cost to the winning bidder, the seller’s discontinued benefit from the asset’s alternative use, or irreversible loss of the option for more efficient allocation of the asset in future. Delaying the auction saves the

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9 Although the paper focuses on the post-auction action of investment timing, which is also fitting for oil lease auctions and licensing agreements, the key intuitions certainly manifest themselves in many other types of post-auction actions. While moral hazard associated with capital allocation has been well-studied (see Stein (2003)), that associated with investment timing is equally important and deserves attention.

10 Prima facie, the type of bids should not matter as there is always a cash equivalent. One advantage to contingent bids is that they enhance the seller’s revenue by effectively linking payoff to a variable affiliated with bidders’ private information—the “linkage” principle in Milgrom (1985). Contingent bids also mitigate liquidity or legal constraints and reduce valuations gaps amongst various parties.
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time value of money on these costs and encourages greater participation, but risks missing
the opportune exercise of the investment option. These tradeoffs endogenize auction tim-
ing in the first stage. The seller times the auction to maximize the option value less the
information rent, and a bidder times the auction to maximize the information rent, neither
of which maximizes social welfare. Thus the seller inefficiently delays the auction and
the bidders prefers earlier initiation than the seller. I find that strategic auction timing is a
salient feature in real-life business practice and is integral to auction design.

By combining cash and royalty payments, optimal security design eliminates post-
auction moral hazard, but inefficiently delays auction and investment. The intuition is that
the seller faces a real option with an added exercise cost that is the information rent; she thus
needs a contingent payment to pass this cost to the winning bidder, who then invests at the
optimal investment threshold which is higher than the efficient one. Cash payments com-
plements the design by ensuring individual rationality and incentive-compatibility. This
result is consistent with the popular use of negotiated royalty payment and down payment
in sales of marketing rights, licensing agreements, publishing and movie contracts, and
many other franchise business practices.

It further turns out that the seller’s commitment level to auction timing and security de-
sign significantly impacts the bidding and investment outcomes. Absent such commitment,
bidding equilibria are equivalent to those in cash auctions. The intuition is that cash-like
bids allow a bidder to generate the maximum social surplus, and at the same time outbid
competitors in the cheapest way. For example, a bidder with higher valuation finds it easier
to outbid others using cash than equity shares because the same shares cost him more than
they cost someone with a lower valuation. The auction timing game is also complicated by
Bayesian updates of beliefs on the types present absent initiation. In equilibrium, bidders
always initiate and invest efficiently.

These results imply the following: first, post-auction investments can be both ineffi-
ciently delayed or accelerated depending on the security design. In particular, the security
design in oil and gas lease auctions causes the winning bidder to delay exploration beyond
efficient rational waiting due to optionality. A greater royalty rate in highly uncertain envi-
ronment exacerbates the delay, which suggests that the large number of idle tracts reported
can be the consequence of inefficient security design. Anecdotal evidence and empirical
studies corroborate these predictions.\footnote{See Humphries (2009), Opaluch, Grigalunas, Anderson, Trandafir, and Jin (2010), and Cong (2013).}
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Second, auction timing is an integral part of auction design and depends crucially on the seller’s commitment. While many studies have focused on security design or security ranking, their conclusions are sensitive to auction timing, whose impact could be larger. Moreover, when the seller cannot commit to auction timing, bidders always initiate when they expect to exercise the investment option. This is consistent with Fidrmuc, Roosenboom, Paap, and Teunissen (2012) which finds that strategic acquisitions are more often bidder-initiated. I also use the data on leasing and exploration of oil and gas tracts in the Gulf of Mexico to show that when bidders could initiate lease auctions before the implementation of Area-Wide Leasing in May 1983, they did so, and carried out first exploratory drills at least 10% faster.

Finally, and perhaps most surprisingly, my findings stand in contrast to conventional predictions derived from several classic studies. For example, it is well-established that having more bidders is beneficial to a seller (Bulow and Klemperer (1996)), which makes it all the more puzzling why the sellers in some corporate auctions restrict the number of bidders (Hansen (2001) and French and McCormick (1984)). I show that depending on the security design, more bidders could decrease revenue and social welfare due to aggressive bidding and increased moral hazard. Another widely held belief is that security bids generate higher revenue than cash, but this paper argues cash dominates common securities as the bidders’ market becomes very competitive. In addition, most auctions traditionally deemed efficient (including cash auctions) are actually not once we consider endogenous initiation.

This paper builds on the literature on security-bid auctions and their applications in corporate finance. DeMarzo, Kremer, and Skrzypacz (2005) give an extensive exposition of security-bid auctions, showing that “steeper” securities lead to higher expected value to the seller. Samuelson (1987) suggests that adverse selection and moral hazard complicate the effect. Che and Kim (2010) and Rhodes-Kropf and Viswanathan (2000) demonstrate, respectively, that adverse selection could reverse the ranking of securities and lead to inefficiencies in bankruptcy reorganizations and privatizations. This paper examines
post-auction moral hazard—the second issue Samuelson (1987) emphasized. Kogan and Morgan (2010) compare equity and debt auctions under moral hazard in an experimental study. McAfee and McMillan (1987a) is another related study which derives optimal linear incentive contract under competition, information asymmetry, and moral hazard. This paper is unique in that it considers post-auction moral hazard in a dynamic setting and jointly with endogenous auction timing.

This paper also complements the emerging literature on agency issues and auction initiation in a real options framework. Maeland (2002), Grenadier and Wang (2005), and Cong (2012) study distortion of investment incentives due to adverse selection and moral hazard. Board (2007b) derives optimal selling mechanisms of options. This paper differs primarily in considering auction timing and linking the agency conflicts to a broader class security design. Gorbenko and Malenko (2013b) examine bidder-initiated takeover attempts in cash and stocks with heterogeneous cash constraints. Gorbenko and Malenko (2013a) is another detailed study on auction initiation, focusing on time-varying types and signaling in cash auctions. This paper complements by examining initiations driven by aggregate market conditions and Bayesian learning. In addition, I highlight the role of seller’s commitment, and link auction initiation to post-auction investment.

The remainder of the paper proceeds as follows. Section 1.2 introduces the model and illustrates its main features. Section 1.3 analyzes optimal investment strategies. Section 1.4 derives bidding equilibria and optimal design of formal auctions. Section 1.5 characterizes informal auctions as signaling and timing games. Section 1.6 discusses implications and provides empirical evidence. Section 1.7 considers extensions and limitations. Section 1.8 concludes. Appendix A contains all the proofs. Appendix B lists technical conditions for “well-behaved”. Appendix C describes institutional details, data construction, and empirical tests based on sales and explorations of oil and gas tracts in the Gulf of Mexico.

1.2 A Model of Auctions of Real Options

This section describes the economic environment and sets up the model, before illustrating the main features of the model using examples of cash and bonus-bid auctions.

13 Though studies such as Hackbarth and Morellec (2008) discuss timing of mergers, they typically ignore the competitive effect.


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1.2.1 Setup

A risk-neutral seller with discount rate \( r > 0 \) owns a project with an embedded option. Once developed, the project generates a verifiable lump sum cash flow whose value \( P_t \) is publicly observed and evolves stochastically according to a geometric Brownian motion (GBM)

\[
dP_t = \mu P_t dt + \sigma P_t dB_t,
\]

where \( B_t \) is a standard Brownian Motion under the equivalent martingale measure, \( \mu < r \) is the instantaneous conditional expected percentage change per unit time in \( P_t \), and \( \sigma \) is the instantaneous conditional standard deviation per unit time.\(^{14}\)

The seller does not have the expertise to exploit the option but can sell the project to \( N \) risk-neutral potential bidders with the same discount rate \( r \) who have the expertise to exploit the option.\(^{15}\) Bidder \( i \) can develop the project by paying a private investment cost \( \theta_i \).\(^{16}\) The distribution of the \( \theta_i \)'s are i.i.d. with positive support \([\theta, \bar{\theta}]\), and follows either Uniform Distribution or Generalized Pareto Distribution.\(^{17}\) Denote the cumulative distribution and density function by \( F_\theta(\theta) \) and \( f_\theta(\theta) \) respectively. Similar to DeMarzo, Kremer, and Skrzypacz (2005), the winning bidder has to pay an up-front cost \( X \geq 0 \), which we can interpret as the initial resources required by the project.\(^{18}\) The project is worthless to him if it is never developed. The seller loses a reservation value \( Y \) when the asset is sold.\(^{19}\)

---

\(^{14}\) \( r > \mu \) ensures finite value of the option. See McDonald and Siegel (1986) or Dixit and Pindyck (1994).

\(^{15}\) For clarity, I refer to the seller as female and the bidders as male.

\(^{16}\) The analysis also applies when bidders differ in other quantities, such as production capacities.

\(^{17}\) This specification includes common distributions used in the literature, with both bounded and unbounded supports. It renders the auction design problem “regular”, standard in auction literature, and rules out unrealistic mathematical constructions in discussing auction timing. Results in the paper hold more generally for “well-behaved” distributions. Appendix B details the technical requirements.

\(^{18}\) Examples include illiquid human capital, the cost of underwriting in the case of security issuance, or simply an opportunity cost for the winning bidder.

\(^{19}\) For example, the seller may derive a continuous stream of cash flow \( rY \geq 0 \) through alternative uses of the asset till it is sold. In pharmaceutical licensing, the alternative use could be sales without marketing. In leasing natural resources, it could be environmental benefits or income from utilizing the land as a national park. Though not modeled in this paper, the value from alternative use can also depend on \( P \), for example, \( Y(P) = y_0 + y_1P \) with \( y_0 \geq 0 \) and \( y_1 \in [0, 1] \). The intuition for auction timing carries through as long as \( Y(P) \) and \( P \) evolve differently. \( Y \) can also represent the option value of more efficient allocation and use of the asset when technology improves. \( X \) and \( Y \) represent in reduced-form auction frictions that are commonplace. Startups, project finance, etc, all have high reservation values and the legal fees are high. Potential bidders may incur cost of due diligence, which can be modeled as an entry cost. Holding the auction may incur heavy costs such as that of revealing proprietary information to rivals (Hansen (2001)). The losers in an M&A could have diminished value (Gorbenko and Malenko (2013)). Other pre-contract
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When the auction is held at time $t_a$, bidders compete by offering security bids that are combinations of contingent payments from the cash flow of the project and non-contingent payments which, for simplicity, can be viewed as upfront cash at the time of the auction. Unless stated otherwise, the discussion focuses on standard security bids as defined next.

**DEFINITION.** A *standard security bid* is an upfront cash payment $C \in \mathbb{R}$ and a contingent payment $S(P_\tau) \in \mathbb{R}$ paid at the time of investment $\tau$, where $S(P)$ is continuous in $P$.\(^{20}\)

Standard security bids are simple and intuitive, and as discussed in section 1.4.2, can implement the optimal auction design even in the augmented universe of security bids. They admit most securities and contracts used in practice. For example, with equity bids, the seller receives a fraction $\alpha$ of the payoff: $S(P) = \alpha P$; with call option bids, the seller can pay a strike price $k$ for the project cash flow: $S(P) = [P - k]^+$; with bonus bids on fixed royalty rate $\phi$, the seller receives bonus $C$ and royalty payment $S(P) = \phi P$.

The agents interact in continuous time as shown in Figure 1.1. To analyze the dynamics, I work backward to first solve for the optimal investment strategy for the winning bidder, then derive the bidding equilibrium given the bidders’ valuations based on their investment strategies, and then study the impact of strategically timing the auction.

$t_0 = 0$: Interaction starts.

$\leftarrow t_a$: Auction held at $P_a$, project allocated, cash part $C$ of bid paid, auction frictions $X$ & $Y$ incurred.

$\leftarrow \tau$: Project invested at $P_\tau$, $\theta$ incurred, contingent part $S(P_\tau)$ of bid paid.

Figure 1.1: Timeline

The seller’s commitment to auction design plays a fundamental role in equilibrium. In formal auctions, the seller decides the timing of the auction and pre-specifies a set of permissible bids ordered by an index (the bid with the highest index wins). In other settings, the seller cannot commit to pre-specified bids or auction timing, thus considers all types of costs are common too (French and McCormick (1984)).

\(^{20}\)This definition rules out directly contracting on private cost $\theta$, which is familiar in real-life practices. This is frequently due to important practical problems in validating profits reported. Consequently, payments are usually contingent on top-line revenue in the development of natural resources, contracts on marketing and licensing rights, as well as franchise chain operations.
offers and potentially allows the bidders to initiate the auction. I follow DeMarzo, Kremer, and Skrzypacz (2005)'s convention to call such cases informal auctions and discuss them in Section 1.5. Throughout the paper, I focus on First-Price Auctions (FPAs) and Second-Price Auctions (SPAs) where the bidder with the highest bid wins and pays the highest bid or the second-highest bid respectively.\(^{21}\)

### 1.2.2 Two Simple Examples

#### Cash Auctions

The bidding strategies and post-auction investments in cash auctions serve as a useful benchmark for later sections. Upon winning, a bidder of type \(\theta\) owns the project entirely, and optimally develops the project at time \(t \geq t_a\) to maximize \(E[r t P_a^\theta q_s]\). This is a standard problem in the real options literature.\(^{22}\) The optimal strategy involves immediate investment upon reaching an upper threshold \(P^*(\theta)\). Let \(P_a\) denote the cash flow level when the auction is held. The value of the investment option \(W\) and \(P^*(\theta)\) are independent of \(X\) and \(t\), and are given by

\[
P^*(\theta) = \max \left\{ P_a, \frac{\beta}{\beta - 1} \theta \right\}, \quad \text{where} \quad \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \quad \text{and}
\]

\[
W(P_a; \theta) = D(P_a; P^*(\theta))(P^*(\theta) - \theta), \quad \text{where} \quad D(P; P') = \left(\frac{P}{P'}\right)^\beta \quad \text{for} \quad P \leq P'.
\]

Appendix A.1 shows that \(D(P_t; P')\) corresponds to the time-\(t\) price of an Arrow-Debreu security that pays one dollar the first moment threshold \(P' \geq P_t\) is reached. The option value of the project is simply the total value of Arrow-Debreu securities that replicate the payoff of the investment option at exercise. It can also be viewed as the value of the exercise payoff \(P^*(\theta) - \theta\) discounted by the “expected discount factor” \(D(P_a; P^*(\theta))\).

Bidder \(i\)'s private valuation in cash auctions is then \(W(P_a; \theta_i) - X\), which is decreasing.

---

\(^{21}\)Following Skrzypacz (2013) and Ding and Wolfstetter (2011), I assume the seller commits to no renegotiation, and to no contracting or resale to losing or non-participating bidders. Section 6 discusses relaxations of these assumptions.

\(^{22}\)For example, in McDonald and Siegel (1986) and Dixit and Pindyck (1994).
There exists a cutoff type for participation \( \theta_c = \min\{\bar{\theta}, \theta_{BE}\} \), where the break-even type \( \theta_{BE} \) solves \( W(P_a; \theta) - X = 0 \) and is given explicitly as

\[
\theta_{BE} = (\beta - 1)(P_a^\beta \beta^{-\beta} X^{-1}) \frac{1}{\beta - 1} \mathbb{I}(P_a \geq \beta X) + (P_a - X) \mathbb{I}(P_a \leq \beta X)
\]  

(1.2.4)

Types with costs higher than \( \theta_c \) do not participate. Since there is no misalignment of incentives in post-auction investments, FPAs and SPAs generate equivalent revenues to the seller, and efficiently allocate the project to type \( \theta(1) \) if \( \theta(1) \leq \theta_c \), where \( \theta(j) \) is the \( j \)th lowest realized \( \theta \). The cases with reserve price or entry fee are similar.

**Bonus-bid Auctions**

In many countries, the predominant design for leasing natural resources involves fixing a royalty rate \( \phi \) and having the contractors bid up-front payment \( C \)—the so-called “bonus”\(^{23}\).

The winning bidder owns a fraction \( 1 - \phi \) of the project and has a real option value \( L(\theta) = \max_{\tau} \mathbb{E}[e^{-\tau r}((1-\phi)P_{\tau} - \theta)] - X \). Scaling the cash flow in Eq. (1.2.2) gives the optimal investment threshold \( P^{\text{bonus}}(\theta) = \max\{P_a, \frac{\theta}{\beta - 1 - \phi}\} \). In SPAs, every participant bids up to the expected value \( L(\theta) \), which is decreasing in \( \phi \). \( L(\theta_{BE}) = 0 \) gives the breakeven type. The seller has a real option value \( \max_{\tau} \mathbb{E}[e^{-\tau r} \phi P_{\tau}] \) and prefers immediate investment. But the winning bidder only invests when \( P^{\text{bonus}} \) is first hit. The expected total revenue is simply \( R^{\text{bonus}} = \mathbb{E}\left[ L(\theta(2)) 1_{\{\theta(2) < \theta_c\}} + [D(P_a; P^{\text{bonus}}) \phi P^{\text{bonus}}(\theta(1)) - Y] 1_{\{\theta(1) < \theta_c\}} \right] \).

The cut-off type, revenue and social welfare in FPA bonus-bid auctions are the same, because this is essentially a cash auction for \( 1 - \phi \) fraction of the project and revenue equivalence applies.

**1.2.3 Post-auction Investment and Pre-auction Timing**

There are a few observations from these two examples. When using contingent payment in bonus-bid auctions, the winning bidder’s real option differs from the seller’s, leading to post-auction misalignment of incentives. This distortion in investment timing is costly to

\(^{23}\)In the US, the Minerals Lands Leasing Act prescribes the base share of royalty rate at 1/8 the value of production for onshore leases, and the Outer Continental Shelf Lands Act used 1/6 for offshore leases. The offshore rate for leasing beginning in 2008 is set at 18.75%. This form of payments are also common for technology license and marketing rights. See [Rothkopf and Engelbrecht-Wiggans (1992)](#), [Reece (1979)](#), [Hendricks, Porter, and Tan (1993)](#), and [Haile, Hendricks, and Porter (2010)](#) for more details.
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social welfare and the seller’s revenue. Moreover, since the real option value, break-even type, and cash bids all depend on $P_a$, the timing of the auction clearly matters.

Cost of Inefficient Investment and Post-auction Moral Hazard

As $P^{bonus}(\theta) \geq P^a(\theta)$, investment is inefficiently delayed in bonus-bid auctions, which could potentially explain the higher idle rate of oil tracts observed—a topic I revisit in Section 5 on model implications. This delay is costly to the seller ex post the auction. Whether it is costly ex ante depends on the tradeoff between the additional rent extracted with contingent payments and the cost of post-auction moral hazard. For example, bonus-bid auctions yield the seller higher revenues than cash auctions only when $X$ is relatively small and either $\beta$ or $\phi$ is also small.

Let us examine a situation absent moral hazard to understand how contingent securities enhance the seller’s revenue. Suppose $\theta$ were contractible, the seller can use a profit share SPA to completely align investment incentives, i.e., $S(\alpha, P) = \alpha(P - \theta)$. The winning bidder simply faces a scaled optimization problem, and because the individual rationality constraints for participation is the same as in a cash-bid auction, the cut-off type is identical. If $\theta(2) < \theta_c$, she bids up to its valuation, i.e., $(1 - \alpha(\theta(2)))W(P_a; \theta(2)) = X$. The seller’s expected revenue $E[(\alpha(\theta(2))W(P_a; \theta(1)) - Y) \mathbb{1}_{\{\theta(2) < \theta_c\}}]$ can be expressed as

$$E\left[(W(P_a; \theta(2)) - X - Y) \mathbb{1}_{\{\theta(2) < \theta_c\}}\right] + E\left[\alpha(\theta(2))(W(P_a; \theta(1)) - W(P_a; \theta(2))) \mathbb{1}_{\{\theta(2) < \theta_c\}}\right].$$

The first term represents the seller’s expected revenue in cash auctions and the second term is the linkage benefit: for every realization that $\theta(2)$ participates, the seller recovers a portion $\alpha(\theta(2))$ of the winning bidder’s information rent. This leads to higher revenues than in cash auctions.

In reality, $\theta$ is not observable or difficult to contract upon. Either revenue extraction or efficiency loss can dominate. Table 1.1 gives an illustration. The profit share auction yields higher revenue and the same welfare compared to the cash auction, as anticipated. A bonus-bid auction with $\phi = 1/8$ also results in higher revenue, but welfare is reduced due to inefficient investment. But both equity-bid auction and bonus-bid auction with $\phi = 1/4$.

---

24Profit share auctions are rare due to high monitoring costs, limited comparability, and landowner’s risk aversion (Robinson (1984)). Past experience in oil lease auctions has shown considerable difficulties in reaching agreement on the proper profits (Opaluch, Grigalunas, Anderson, Trandafir, and Jin (2010)).
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yield significantly lower revenues and welfares compared to the cash auction. Depending on the exact real option involved, one could lose more than 90% of revenue and welfare using a security-bid auction compared to a cash auction.

\[ P_a = 15, \ N = 18, \ \beta = 1.8, \ \theta \sim Unif[10, 50], \ X = 3, \ Y = 0 \]

<table>
<thead>
<tr>
<th>Security Design</th>
<th>Cash-bid Revenue</th>
<th>Profit Share 1.8384</th>
<th>Bonus-bid ( \phi = 1/8 ) 1.7403</th>
<th>Equity-bid 1.6416</th>
<th>Bonus-bid ( \phi = 1/4 ) 1.4429</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>1.6578</td>
<td>1.8384</td>
<td>1.7403</td>
<td>1.6416</td>
<td>1.4429</td>
</tr>
<tr>
<td>Welfare</td>
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<td>2.2565</td>
<td>2.1862</td>
<td>2.0597</td>
<td>1.6027</td>
</tr>
</tbody>
</table>

Table 1.1: Expected Welfare and Seller’s Revenue.

Impacts of Auction Timing

Auction timing affects the post-auction investments, bidding behaviors, and agents’ revenues. First, auctions held at a higher \( P_a \) results in less delays in post-auction investment. In particular, when \( P_a \) is higher than the investment thresholds in both cash and bonus-bid auctions, investment follows the auction immediately. Second, conditional on the security design, the bidders’ valuations are higher for a higher \( P_a \); thus bidders participate more and bid more cash or bonus. Third, while the seller receives more at an auction held at a higher \( P_a \), she has to discount the payoff, since she needs to wait to hold the auction.

Auction timing has both direct effects independent of security design, and interaction effects with security design. Figure 1.2 plots time zero present values of the expected revenues from cash and bonus-bid auctions held when \( P_t \) first reaches \( P_a \). Surprisingly, auction timing affects seller’s revenue even in cash auctions. \( P_a \) determines which security design dominates and changing it leads to substantial variations in the seller’s revenue. For example, Cash dominates only for \( P_a \leq 44 \), and optimal timing of bonus-bid auctions can increase revenue by more than 60%.

1.3 Post-auction Investments under Uncertainty

This section takes the standard security bid in equilibrium as given to formally derive the optimal investment strategy, and shows how it is shaped by both auction timing and security design.
Suppose the winning bidder of type \( \theta \) pays cash \( C \) at the time of auction \( t_a \) and pays contingent security \( S(P_t) \) when project is invested at time \( t \geq t_a \), his private valuation at \( t_a \) is
\[
\tilde{V}(C, S(\cdot), \theta) = \max_{\tau \geq t_a} E_P[e^{-r(\tau-t_a)}(P_\tau - S(P_\tau - \theta)] - X - C,
\]
(1.3.1)
where \( \tau \) is any stopping time. The key innovation from traditional real options models is the term \( S(P) \) which is of general form. It is not clear if \( V(\theta) \) is well-defined a priori. The following proposition shows that an optimal investment strategy exists and characterizes it.

**Proposition 1.1.** For any standard security bid, there exists a threshold investment strategy that is optimal among all stopping times. Moreover, the valuation \( \tilde{V}(C, S(\cdot), \theta) \) is continuously decreasing in \( \theta \).

As detailed in Appendix A.2, the optimal strategy is independent of \( X \) and \( C \), which are sunk costs when deciding on the investment. The strategy generally involves both upper and lower thresholds that are dependent on \( P_a \)—auction timing clearly matters. The remainder of this section turns to the following special case.

**Condition (Conv):** The function \( P^{-\beta}[P - S(P) - \theta] \) of \( P \) is quasi-concave with a maximum achieved at \( \bar{P}(\theta) \). For \( P \geq \bar{P}(\theta) \), \( S(P) \) is piecewise twice-differentiable with only positive
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jumps in \( S'(P) \) and \( S''(P) \) (when exists) satisfying \( PS''(P) \geq (1-\beta)[1 - S'(P)] \).

Condition (Conv) requires that the security is not too “concave” at large \( P \), and is non-restrictive as it holds for most common securities used in real life and in equilibria in this paper, such as equities or call options. The following Lemma simplifies discussions in later sections:

**Lemma 1.1.** With (Conv), the optimal investment follows an upper threshold strategy with threshold \( \tilde{P}(\theta) \). If in addition \( P - S(P) \) is non-decreasing in \( P \), \( \tilde{V}(C, S(\cdot), \theta) \) is non-decreasing in \( P \).

The tradeoff is similar to that in traditional real options models. At \( P_t \), the gain of waiting to reach \( P_t + dP \) is the increase in payoff \( d[P_t - S(P_t)] = [1 - S'(P_t)]dP + o(dP) \), but the payoff is also discounted by \( D(P_t; P_t + dP) \), resulting in a fractional loss \( 1 - D(P_t; P_t + dP) = \beta P_t^{-1} + o(dP) \) of the original payoff. The net benefit decreases in \( P_t \) and becomes negative beyond \( \tilde{P}(\theta) \). Thus, waiting is beneficial if and only if \( P_t < \tilde{P}(\theta) \).

Security bidding generally causes inefficient investments and the distortion is closely tied to the shape of the security. This is best seen with \( S(P) \) differentiable at \( \tilde{P} \), in which case

\[
\tilde{P}(\theta) = \frac{\beta}{\beta - 1 + S''(\tilde{P})}[\theta + S(\tilde{P})] \quad (1.3.2)
\]

Compared to the traditional real options investment threshold \( \frac{\beta}{\beta-1}\theta \), the security payment has two effects. Intuitively, the bidder faces an additional cost \( S(P) \), and thus requires a higher threshold. But at the same time, the sensitivity of the security payment to cash flow implies a smaller option premium and prompts an earlier investment. Depending which effect dominates, the threshold could be either higher or lower than the efficient threshold. The following proposition describes the direction of distortion:

**Proposition 1.2.** With condition (Conv), relative to what is socially efficient,

(a) A project invested at \( P \) is weakly delayed if \( \beta S(P) - PS'(P) > 0 \), and weakly accelerated if \( \beta S(P) - PS'(P) < 0 \), regardless of the winning bidder’s type;

(b) A winning bidder of type \( \theta \) will invest weakly late if \( (\beta - 1)S \left( \frac{\beta}{\beta-1}\theta \right) > \theta S' \left( \frac{\beta}{\beta-1}\theta \right) \),

and weakly early if \( (\beta - 1)S \left( \frac{\beta}{\beta-1}\theta \right) < \theta S' \left( \frac{\beta}{\beta-1}\theta \right) \).

25As in [Grenadier (2002)] and [Grenadier and Malenko (2011)], option premium is the NPV of investment at the moment of exercise divided by the total cost: \( OP(\theta) = \frac{P(\theta) - S(\tilde{P}(\theta)) - \theta}{S(\tilde{P}(\theta)) + \theta} \).
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These results follow directly from Lemma 1.1 and (Conv). While prior literature mostly points to investment delays due to agency conflicts, the timing distortion really depends on the security design. The cash flow elasticity of security (CES) $E_S = \frac{P^{S(P)}S}{S(P)}$ is thus informative:

**Corollary 1.1.** Investment is inefficiently delayed (accelerated) if $E_S < (>) \beta$.

For example, $E_S = 1$ in bonus-bid auctions, resulting in inefficient delays in investments.

1.4 Formal Auctions

The knowledge of optimal investment strategies allows bidders to value the real option. This section continues to analyze bidding equilibria in formal auctions. There are three key findings. First, bidding and investment equilibria exist under mild regularity conditions. Second, bidding outcome, investment, and security ranking are all sensitive to auction timing and the market condition. Third, optimal auction designs entail combinations of cash and royalty payments, and lead to inefficiently delayed investments auction timing.

In formal auctions, the seller commits to a pre-specified auction timing and a well-ordered set of allowed bids. Thus the bidders compete by offering allowed security bids, which in real life are ranked by simple, easily implementable rules. A variant of the definition in DeMarzo, Kremer, and Skrzypacz (2005) formalizes this notion of well-orderedness:

**DEFINITION** An ordered set of securities ranked by index $s$ is defined by a left-continuous map $\Pi(s) = \{C(s), S(s, \cdot)\}$ from $[s_L, s_H] \subset \mathbb{R}$ to the set of standard security bids such that for each voluntary participant of type $\theta$, $V(s, \theta) \equiv \tilde{V}(C(s), S(s, \cdot), \theta)$ is non-negative and non-increasing in $s$ on $[s_L, \tilde{s}]$ and negative on $(\tilde{s}, s_H]$ for some $\tilde{s} \in [s_L, s_H]$.

In addition to being standard, an ordered set of securities admits ranking with index $s$ for any payoff from the project and permissible bids cover a range wide enough such that each participant earns non-negative profit by bidding low enough but earns no profit bidding too high. The seller allocates the project to the bidder with the highest index. The winning bidder pays a security using the highest-bid index in FPAs or the next-highest-bid index in SPAs. This notion of formal auctions is consistent with real-life practice: $s$ could be the
fraction of shares $\alpha$ in a pure equity auction $\{C(\alpha) = 0, S(\alpha, P) = \alpha P\}$, the (negative) strike price $k$ in a call option auction $\{C(-k) = 0, S(-k, P) = \max\{P - k, 0\}\}$, or the bonus $b$ in a bonus-bid auction with royalty rate $\phi$ fixed $\{C(b) = b, S(b, P) = \phi P\}$. Such securities are routinely used in M&As, VC contracts, and lease auctions where the winning bidder is indeed the one offering the highest $s$.

1.4.1 Equilibrium Bidding Strategies

Using the fact that $V(s, \theta)$ is well-defined (Proposition 1.1), I characterize equilibrium bidding strategies, assuming any indifference in bidding is resolved by bidding a higher index.

**Proposition 1.3.** In first-price auctions, when $\ln V(s, \theta)$ is absolutely continuous in $s$ with the derivative (when exists) decreasing in $\theta$, there exists a unique symmetric Bayesian Nash equilibrium that is decreasing, differentiable, and is characterized by:

$$s'(\theta) = \frac{(N - 1)f(\theta)V(s(\theta), \theta)}{1 - F(\theta)V_1(s(\theta), \theta)} \quad (1.4.1)$$

for $\theta \leq \hat{\theta}$ with the boundary condition $s(\hat{\theta}) = \sup\{s \in [s_L, s_H] \mid V(s, \hat{\theta}) = 0\}$. The cut-off type for participation is $\hat{\theta} = \sup\{\theta \leq \bar{\theta} \mid \max_s V(s, \theta) \geq 0\}$.

Notice I have assumed “Single-Crossing” here, which is standard in the literature.

Next for SPAs, the equilibrium bidding strategy is characterized by

**Proposition 1.4.** In second-price auctions, the unique Bayesian Nash equilibrium in weakly undominated strategies is for type $\theta$ to bid $s(\theta) = \sup\{s \in [s_L, s_H] \mid V(s, \theta) \geq 0\}$, which is decreasing in $\theta$. The cut-off type for participation is $\hat{\theta} = \sup\{\theta \leq \bar{\theta} \mid \max_s V(s, \theta) \geq 0\}$.

The next corollary follows directly from the fact that the bidding strategies are monotone.

**Corollary 1.2.** In security-bid FPAs and SPAs as described above, the investment option is allocated, if at all, to a bidder with the least investment cost. Moreover, the level of

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26In M&As with acquirer’s stocks as bids, $C$ simply corresponds to the value of acquirer’s cash flows that are independent of the acquisition, $X$ corresponds to the opportunity cost of incorporating the target firm, and $P$ is the payoff from the acquired assets and projects, and the synergy created.
participation is the same for FPAs and SPAs, and is weakly smaller than that in cash auctions.

In addition, the amount of competition as indicated by $N$, the initial commitment cost $X$, and the timing of the auction $P_a$ all have fundamental impacts on bidding behavior.

**Proposition 1.5.** Bidders bid more aggressively (weakly greater $s$ for all types, and strictly greater $s$ for a positive measure of types) in FPAs with security bids as $N$ increases or $X$ decreases, or if $V$ and $V/V_1$ are increasing in $P_a$, as $P_a$ increases. They bid more aggressively in SPAs with security bids as $X$ decreases or if $V$ is increasing in $P_a$, as $P_a$ increases.

Consequently, the winner bids a greater index with more competition, smaller initial cost, or higher threshold for auction. Intuitively, a smaller $X$ or higher $P_a$ correspond to higher valuations of the project by the bidders, which allows them to promise more to the seller to increase their chances of winning. When $N$ is bigger in FPAs, one has to increase the bid to outbid more competitors. However, this does not apply in SPAs because one bidder’s bidding strategy is independent of others’ bids.

These results allow the characterization of many common auctions with standard securities in addition to cash and bonus-bid auctions. Below are two examples.

**Equity Auctions and Investment Delays**

Suppose $\alpha$ is the fraction of shares the winning bidder has to pay and $Y$ is the reserve price, the $S(\alpha, P) = \alpha P$ and $C(\alpha) = Y$. By Lemma 1.1 a participant’s present value conditional on winning and exercising at $P_\tau$ is $\mathbb{E}_P[D(P_a; P_\tau)[(1 - \alpha)P_\tau - \theta] - X - Y]$. First and second order conditions give the following corollary,

**Corollary 1.3.** In auctions with equity bids, the winning bidder invests when cash flow first reaches $P^{\text{equity}}(\theta) = \max\left\{P_a, \frac{\beta \theta}{(\beta - 1)(1 - \alpha)}\right\}$.

Comparing the threshold to $P^*(\theta)$, the investment is inefficiently delayed—undesirable to the seller because her revenue $D(P_a; P)S(\alpha, P)$ is decreasing in $P$. Ex post the auction, investing some time earlier could improve both seller’s revenue and welfare.

Bidding equilibria exist for both FPAs and SPAs, and the cut-off type $\hat{\theta}$ is identical to that in a cash auction with the same reserve price. In SPAs, the bidder $\theta$ increases $\alpha$ until
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\[ V(\alpha, \theta) = 0. \] In FPAs, since \( \frac{\partial^2 \ln V(\alpha, \theta)}{\partial \alpha \partial \theta} < 0 \) is well-defined except on the boundary \( P_a = \frac{\beta \theta}{(\beta - 1)(1 - \alpha)} \), Proposition 1.3 applies and \( \alpha(\theta) \) is continuous and decreasing. For example, when \( X = 0 \),

\[ \alpha(\theta) = \int_{\theta}^{\hat{\theta}} \left( \frac{N - 1}{\beta(1 - F(\theta'))} \right) \exp \left[ \int_{\theta'}^{\theta} \frac{(N - 1) f(\theta')}{\beta(1 - F(\theta'))} d\theta' \right] d\theta', \quad \text{for} \quad \theta \leq \hat{\theta}. \quad (1.4.2) \]

With uniform distribution, this translates to \( \alpha(\theta) = 1 - \left( \frac{N - \theta}{1 - \theta} \right)^{\frac{N - 1}{\beta}} \). Clearly bidders bid more shares when \( N \) or \( P_a \) increases or \( X \) decreases (cutoff \( \hat{\theta} \) is increasing in \( P_a \) and decreasing in \( X \)). Note inefficient delays of investments also follow directly from Corollary 1.1 since \( E_S = 1 < \beta \).

Call Option Auctions and Investment Accelerations

Consider call option bids with no reserve price. Let \( k \) be the strike price the winning bidder of type \( \theta \) contracts, then \( S(-k, P) = \max\{ P - k, 0 \} \) and \( C(-k) = 0 \). Bidder \( \theta ' \)'s present value conditional on winning and exercising at \( \tau \) is \( \mathbb{E}_P[D(P_a; P_{\tau})(P_{\tau} - \max\{ P_{\tau} - k, 0 \} - \theta)] - X \). If a bidder of type \( \theta \) bids a strike less than \( X + \theta \), with non-trivial probability he wins with a required strike \( k < X + \theta \) in both FPA and SPA, and fails to break even. So he is better off bidding \( k \geq X + \theta \). If he bids a strike greater than \( P^*(\theta) \), the required strike conditional on winning in either FPA or SPA satisfies \( k > P^*(\theta) \). He always invests with the threshold \( P^*(\theta) \) and the call is never exercised. But he could bid lower \( k \) to increase the chance of winning. Hence \( k \in [X + \theta, P^*(\theta)] \). The investment threshold maximizes the winning bidder’s value,

**Corollary 1.4.** In auctions with call option bids, a bidder of type \( \theta \) always bids \( k \in [X + \theta, P^*(\theta)] \), and upon winning, invests when the cash flow first reaches \( P^{\text{call}}(\theta) = \max\{ P_0, k \} \).

Notice \( P^{\text{call}}(\theta) \leq P^*(\theta) \) and the equality holds when \( P_a > \frac{\beta}{\beta - 1} \theta \). Inefficiency therefore lies in the potential acceleration of investments.\(^{27}\) Basically, if the call option is going to be exercised, there is no incentive for the bidder to keep timing the market because that delays his payment \( k \). When \( P_a \leq k \), \( V = \frac{P_a^\beta (k - \theta)}{k^\beta} - X \), otherwise \( V = k - \theta - X \).

\(^{27}\)Existing dynamic agency models of investment often predict decreased or delayed investments (e.g., Grenadier and Wang (2005) and DeMarzo and Fishman (2007)). It really depends on the security design.
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Proposition 1.3 applies for the bidding equilibrium. Next for SPAs, for those who participate, they bid up to their valuations, in other words, \( k = \theta + X \) if \( \theta < P_a - X \) or \( k \) solves \( \frac{k^3(k-\theta)}{\beta} = X \) otherwise. In either case, \( k < \frac{\beta}{\beta-1} \theta \) and the investment is strictly accelerated. In fact, the seller makes a profit only when \( \theta < P_a - X \), otherwise the strike price is simply the value of the project, netting her no profit. The cutoff type is the same as that in FPAs.

The numerical illustrations to follow sometimes include another common form of security: a fixed promise of payment \( B \) from the project’s payoff—essentially debts without interests, \( S(B, P) = \min(P, B) \), also known as friendly debt, or in Islamic finance, Qard/Qardul hassan. Since \( E_S < \beta \) for friendly debt, investments are delayed.

### 1.4.2 Optimal Auction Design

DeMarzo, Kremer, and Skrzypacz (2005) show that “steeper” securities yield higher revenues for the seller. This ranking breaks down due to post-auction moral hazard: a “steeper” security extracts more from the winning bidder’s information rent, but it also reduces his incentive investment efficiently post-auction. As seen in the examples of cash and bonus-bid auctions, a “steeper” security (bonus with royalty) does not always dominate. This subsection approaches security ranking from a mechanism design perspective and allow general structures of security payments.

By direct revelation principle, it suffices to examine a truth-telling mechanism corresponding to the auction. Let \( Q(\tilde{\theta}_i, \theta_{-i}) \) be the probability of allocating the project to bidder \( i \), where \( \tilde{\theta}_i \) is the reported type by \( i \) and \( \theta_{-i} \) are other participants’ reported types. A general security payment then has the form \( S(\tilde{\theta}_i, \theta_{-i}, I_t) \) at time \( t \), where \( I_t \) is the set of contractible information up to time \( t \). The expected utility at time zero to type \( \theta_i \) upon

\[
\text{Note } \gamma \ln V_{\beta - k} \gamma 0. \text{ For } k \leq P_a, \quad \gamma^2 \ln V_{\beta - k} \gamma 0 = -\frac{(\beta - 1)}{\beta - 1} \theta - X \leq \left( \frac{P_a - \theta}{\beta - 1} \right) \left( \frac{P_a - \theta - X}{\beta - 1} \right) = -\frac{(\beta - 1)}{\beta - 1} \theta < 0 \text{ using the fact } k \leq \frac{\beta}{\beta - 1} \theta.
\]

Interestless debts are used frequently in contractual agreements in Islamic banking and microfinance, and are equivalent to granting the winning bidder instead of the seller call options - the exact opposite situation to that for call option bids.

\[29\] Though return in the form of a flow payment, \( S \) could be a lump-sum payment when it is a Delta function. For standard security bids, \( I_t \) contains cash flow from the project \( P_\tau \) when invested at \( \tau \), but in general \( I_t \) could include the history of \( P \) up to \( t \), and \( t \) itself if they are contractible. For example, contracting on investment timing (when feasible) at some \( \tilde{P} \) is included by setting a lump-sum payment at option exercise \( S(P) = P I_{P \leq \max(P, \tilde{P})} + K I_{P > \max(P, \tilde{P})} \) where \( K < P \), in which case the bidder only gets paid
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participating and optimally investing is

\[ U(\theta, \tilde{\theta}) = \mathbb{E}_{\theta_{-i}} \left[ Q(\tilde{\theta}, \theta_{-i}) \max_{\tau \geq t_a} \mathbb{E}_P \left[ e^{-rt} (P_\tau - \theta) - \int_{t_a}^{\infty} e^{-rt} S(\tilde{\theta}, \theta_{-i}, \mathcal{T}_t) \, dt - e^{-rt_a} X \right] \right]. \]

As \( S(\tilde{\theta}, \theta_{-i}, \mathcal{T}_t) \) could be artificially constructed that an optimal stopping time for exercising the real option may not exist, it is reasonable to focus attention on the set of \( S(\tilde{\theta}, \theta_{-i}, \mathcal{T}_t) \) such that an optimal stopping time exists for all types under a direct mechanism. With this restriction, let \( \tau^* (\theta, \tilde{\theta}, \theta_{-i}) \) denote the optimal stopping time that is almost surely bigger than \( t_a \), and \( \tau^*_i = \tau^* (\theta, \tilde{\theta}, \theta_{-i}) \). Incentive compatibility requires \( U(\theta, \tilde{\theta}) = U(\theta, \tilde{\theta}) \geq U(\theta, \tilde{\theta}) \) and the individual rationality requires \( U(\theta, \tilde{\theta}) \geq 0 \). Equivalently,

**Lemma 1.2.** Any incentive compatible and individually rational mechanism satisfies

\[ U(\theta, \tilde{\theta}) = \mathbb{E}_{\theta_{-i}} \left[ \int_{\tilde{\theta}}^{\tau^*_i} Q(\theta, \theta_{-i}) \mathbb{E}_P [e^{-rt^*_j}] \, d\theta_j \right] + U(\tilde{\theta}) \]  

where \( U(\tilde{\theta}) \geq 0 \). Moreover \( \tau_i \geq t_a \forall i \) for time consistency.

The design of the auction naturally decomposes into two parts: auction timing and security design. I now examine the ranking of security design given \( t_a \), and derive the optimal security. Auction timing in the optimal auction design is the subject of next subsection. For notational convenience, let \( z(\theta) = \theta + F(\theta) / f(\theta) \) following a threshold trigger \( \tilde{P} \).

### Ranking Security Design

**Proposition 1.6.** The seller’s revenue in FPAs and SPAs with standard security bids is given by

\[ \mathbb{E} \left[ \mathbb{1}_{\theta_{(1)} \leq \theta} \left[ e^{-r(\tau^*_1 - t_a)} (P_{\tau^*_1} - z(\theta_{(1)})) - X - Y \right] \right] \]  

where \( \theta_{(1)} \) is the smallest realized cost, and \( \tau^*_1 \) is the bidder’s corresponding optimal stopping time for investment. Cutoff type \( \hat{\theta} \) is as given in Propositions 3 and 4.

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31 An extension with positive reserve utility or entry cost/fee is straightforward.
32 \( z(\theta) \) is increasing - a standard assumption in the auctions literature, for example, see [Krishna (2009)](https://example.com). One sufficient condition is the “inverse hazard function” \( F(\theta)/f(\theta) \)’s being non-decreasing.
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The seller’s payoff thus depends on the “virtual valuation” of the best type rather than the actual valuation. The seller essentially owns the best type’s real option with an additional cost, which is stochastic. In general, the winning bidder’s optimal investment timing differs from the seller’s. This proposition, together with the bidding equilibria for formal auctions derived earlier, allow the ranking of various security designs.

Security ranking depends on parameters such as $N$ and $\sigma$. In particular, different auction timings lead to different security ranking. This phenomenon is best seen in Figure 1.3(a): Among several pure contingent securities, equity gives the highest expected revenue and call option the lowest at $P_a = 280$, whereas call option is the highest and debt is the lowest at $P_a = 360$.

Optimal Security Design

Despite the complexity in security ranking, one can find the optimal security design. To maximize revenue, the seller wants $\tau_i^*$ to be the first-hitting time of threshold $P_i(z(\theta_i))$, in which case she allocates the project to the bidder with the lowest $\theta$.

Proposition 1.7. An optimal security design exists and is implemented by FPAs using well-ordered securities indexed by $s$: Denote $\hat{\theta}$ the solution to

$$P_a^\beta (\beta - 1)^{\beta - 1} = (X + \ldots

33This payoff is equivalent to the expected marginal revenue (MR), see Bulow and Roberts (1989).
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\[ Y) \beta^\beta z(\theta)^{\beta^{-1}}, \]

\[ C(s) = \begin{cases} \frac{P_0^\beta}{\beta P^\beta(z(s))^{\beta^{-1}}} - X - \int_{s_L}^s \left[ \frac{1-F(-s')}{1-F(-s)} \right]^{N-1} \left[ \frac{1}{P^\beta(z(s))} \right] \beta ds', & \text{if } s \in [s_L, -\theta] \\ C(-\theta) + \theta + s, & \text{if } s > -\theta \end{cases} \]

\[ S(s, P) = \phi(s) P, \quad \text{where} \quad \phi(s) = \frac{(\beta - 1) F(-s) \mathbb{1}_{s \in [s_L, -\theta)}}{1 - \beta f(-s)}, \]

and \[ s_H = \infty, \quad s_L = \max\{-\theta, -\hat{\theta}\}. \]

The optimal security can be interpreted as a cash down payment plus a royalty payment, which is frequently used in the sales of licensing or marketing rights and contracts in publishing or movie production. In equilibrium, type \( \theta \) bids \( s = -\theta \). Recall \( P^{\text{bonus}} = \frac{\beta}{\beta - 1} \frac{\theta}{1 - \theta} \), which implies \( S(s, P^{\text{bonus}}) = \frac{F(\theta)}{f(\theta)} \), i.e., the information rent in Equation (1.4.4). This equates the bidder’s marginal benefit of waiting to his marginal cost of waiting. The winner and the seller then face the same optimization problem for investment. The contingent payment thus aligns the winner’s post-auction incentives with the seller’s, and the cash payment ensures incentive compatibility.\(^{34}\) The contingent security also satisfies limited liability \( S(\cdot, P) \in [0, P] \) and double monotonicity (\( S(P) \) and \( P - S(P) \) being non-decreasing), as typically required in the security design literature.\(^{35}\)

The project is allocated, if at all, to the best type. However, the investment threshold for type \( \theta \) satisfies \( P^* (z(\theta)) \geq P^* (\theta) \), and it can be verified that some projects of positive social value are not allocated and invested. Thus despite the elimination of moral hazard, investments are weakly delayed or missed entirely relative to the socially efficient outcome.

Moreover, to ensure incentive compatibility, the royalty payment is increasing in \( \theta \) while the cash payment is decreasing in \( \theta \). A high-cost type has less incentive to mimic a low-cost type because he has to pay more cash, and has a contingent residual that is more sensitive to his investment timing which is more distorted in equilibrium. This extends McAfee and McMillan (1987a)’s work on optimal linear incentive contracts in the following ways: First, my setup include realistic cases where the contractible output are generated post-auction. Second, the result shows linear incentive contracts are rather robust to time discounting, especially that the discounted project payoff is actually decreasing.

\(^{34}\) Another interpretation of the optimal security is a cash payment to acquire the real option, plus a strike payment \( \frac{P^*(z(s))}{f(s)} \) to the seller for exercising the option. This corresponds to the insight in Board (2007b) to use a revenue-independent contingent payment to align incentives.

in contractible output $P$. Moreover, linear incentive contracts are also robust when we include all dynamic payments based on available contractible information. Finally, optimal security involves negatively correlated cash down payment and contingent royalty payment, a novel and testable prediction that is of interests for empirical studies.

**Optimal Auction Timing**

In addition to security design, the seller has the liberty to decide when to hold the auction. Figure 1.3(a) illustrates how auction timing affects the seller’s revenue for several common securities. This subsection derives the auction timing in an optimal auction.

The seller’s present expected utility holding the auction with optimal security design at time $t_a$ is $E[e^{-rt_a}[W(P_a; z(\theta(1))) - X - Y]^+]$, where $P_a$ is the cash flow level at $t_a$. The seller essentially owns a timing option with irreversible cost $X + Y$.

**Proposition 1.8.** There is a unique threshold strategy for timing the auction that is optimal among all stopping times. The threshold is the largest root $P_a$ to

$$
\int_{\hat{\theta}_z}^{\theta} f(\theta) [1 - F(\theta)]^{N-1} \left[ \beta (X + Y + z(\theta)) - (\beta - 1) P^* (z(\theta)) \right] d\theta = 0, \quad (1.4.5)
$$

where $P^*(\theta) = \max\{P_a, \frac{\beta P_a}{\beta - 1}\}$, and cutoff type $\hat{\theta}_z$ solves $W(P_a; z(\hat{\theta}_z)) = X + Y$. In particular, the auctioneer never sells the project when she expects no chance of immediate investment.

Intuitively, option values erode as $P_a$ increases, thus it would not be optimal to postpone the auction indefinitely. If the seller sells at a level where she expects no bidder to invest right away, she can profitably deviate by delaying the incidence of cost $X + Y$ and waiting for greater participation. The bigger $X + Y$ is, the more the seller endogenously delay the auction. When $X + Y \to \infty$, the project is never sold.$^{36}$

Since the seller owns an investment option with a greater exercise cost $z(\theta(1)) \geq \theta(1)$ due to information rent, the threshold to incur $X + Y$ to sell the option is higher:

**Proposition 1.9.** Optimal formal auctions happen weakly later than efficient formal auctions.

$^{36}$Though not considered in the current setup, $X + Y < 0$ simply implies that the seller holds the auction as soon as it is feasible to avoid missing the investment threshold.
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I show this in Appendix A.11 by first proving that both optimal and efficient security designs have unique threshold for holding the auction, and then arguing that at the efficient threshold, it is better to wait further using the optimal security.

As in Myerson (1981)’s analysis on optimal auction design in a static setting, there is a wedge between the seller’s revenue and welfare, but with post-auction actions and auction timing, the optimal design here has several distinct features. Although the seller still excludes bidders, the auction is held under better market condition (higher $P_a$), which encourages participation and mitigates the exclusion. This implies that in real life one may not see sellers excluding bidders as much using entry fee or reserve price, because she has the alternative tool of choosing a more propitious time to hold the auction. Moreover, the auction and investment are inefficiently delayed - inefficiencies in dynamic settings are multi-dimensional.

While in formal auctions the seller commits to the auction timing and security design, in reality bidders often play a more active role as discussed next.

1.5 Informal Auctions

Many economic interactions such as corporate takeovers, competition for supply contracts, and talent recruitment can have characteristics of auctions. The discussion in formal auctions restricts bids to a pre-specified ordered set. Yet the seller often cannot ignore offers from outside the set, and thus considers all bids and chooses the most desirable one ex post, especially in informal auctions. This essentially leaves the security design of the auction to the bidders as they can bid any contingent payment. Moreover, the seller may have to consider offers before she puts the asset up for sale, in which case both the auctioneer and the bidders can strategically initiate the auction.\footnote{When an asset of a Delaware corporation is for sale after being approached by a buyer, the Revlon rule imposes upon directors a duty to solicit bids and conduct an auction.}

It turns out investments are always efficient when the seller cannot commit to auction timing and pre-specified security design. As such, the conclusion in DeMarzo, Kremer, and Skrzypacz (2005) still applies: cash is the cheapest way for a better type to separate from worse types and, in equilibrium, every bid is equivalent to cash. This section further shows that when only the seller can initiate, the auction is inefficiently delayed (ex ante).
When the bidders can initiate the auction as well, the auctioneer always waits for an offer in equilibrium, and the auction is inefficiently accelerated (ex post).

1.5.1 The Signaling and Timing Game

If the seller commits to neither a pre-specified timing of the auction nor a bidding and allocation rule, she chooses the bid that gives her the highest expected payoff based on her beliefs regarding the type of each bidder at the time the auction is held. The auction therefore exhibits features of a signaling and timing game of the following form:

1. Either the seller or a bidder initiates the auction at some time $t_a \geq 0$.

2. Participating bidders submit informal bids simultaneously. An informal bid $\Pi^i$ by bidder $i$ is a cash payment $C^i$ and a security payment $S^i(\cdot)$ contingent on the project cash flow subject to limited liability $S^i(P) \in [0, P]$.

3. The seller chooses the winning bidder rationally according to the valuation function $R(\Pi^i) = C^i + \mathbb{E}[R_\theta(S^i)|\Theta(\Pi^i)]$ provided she values the bid more than the reservation value $Y$. $\Theta(\Pi^i)$ is her belief of bidder $i$’s type upon seeing the bid, and $R_\theta(S^i) = \mathbb{E}[e^{-\tau^i_\theta}S^i(P_{\tau^i_\theta})]$ where $\tau^i_\theta$ is the optimal stopping rule for type $\theta$ when bidding $\Pi^i$, i.e. $\tau^i_\theta = \arg\max_{\tau \geq t_a} \mathbb{E}[e^{-\tau^i_\theta}(P_{\tau} - S^i(P_{\tau}) - \theta)] - X - C^i$.

4. The winning bidder $i$ pays the upfront cash $C^i$ and the initial cost $X$ at $t_a$, then invests rationally at $\tau^i_\theta$ and makes the contingent payment.

Note that the seller’s valuation $R(\Pi^i)$ is not necessarily the same as the value of the security to bidder $i$, $C^i + R_\theta(S^i)$. One may question if the setup of the game misses out any informal offers, such as contracting on the timing of investment when feasible. The results are robust to additional side contracts because one can enlarge the security space to $\int_{t_a}^{\infty} S(\mathcal{I}_t) dt$ where $\mathcal{I}_t$ is the entire contractible information set, as long as limited liabilities hold. The proofs apply with minor changes in notations.

For informal auctions, FPAs simply mean allocating the project to the bidder with the highest bid according to the valuation function, and SPAs are understood as a variant of English auctions where the seller announces the valuation of bids as bidders continuously adjust their offers until all have stopped.

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38 Any indifference between bidding or not when knowing that a better type has initiated can be resolved by having a liquidity shock arriving at Poisson rate $\lambda \to 0$ that precipitates the auction.
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1.5.2 Bidding Equilibrium in Informal Auctions

Taking the cash flow level at $t_a$ as given, there is an essentially unique bidding equilibrium. Moreover, equilibrium investments are efficient. The proofs for the following results are robust to the belief on the distribution of types, thus are independent of how learning takes place in the timing game. This allows a nice separation of auction timing and signaling by security bids, rendering the problem tractable.

**Lemma 1.3.** In a bidding equilibrium, a participating bidder $i$ has $\tau_{\theta_i}^i = \tau_i^*$ where $\tau_i^*$ is the stopping time corresponding to the threshold strategy with investment trigger $P^*(\theta_i)$.

The intuition is that if a bidder does not invest efficiently upon winning, he can always deviate to a bid that results in efficient investment, and offer more cash to the seller to increase his marginal probability of winning without reducing the payoff upon winning.

**Lemma 1.4.** Informal auctions only admit fully-separating equilibria.

As every bidder upon winning invests efficiently, a better type has greater valuation than worse types and can separate. This also implies that no two bidders place the same bid.

**Proposition 1.10.** There is an essentially unique bidding equilibrium for the informal auction, which is equivalent, in terms of allocation outcome and expected payoffs, to a first-price cash auction with reserve price $Y$. In particular, post-auction investment is efficient.

Basically, the bids are all cash-like in equilibrium. A better type finds it cheaper to use a security that is less sensitive to the true type and creates more social surplus. Take equity bids for example. Not only do they inefficiently delays investment, but a better type finds them costly to use because his $\alpha$ shares are worth more than a worse type’s. Cash-like securities that ensure efficient investment and cheap separation are the most attractive. Because a better type is indifferent from mimicking a marginally worse type in equilibrium, all bidders must be using cash-like securities.

1.5.3 Endogenous Timing of Informal Auctions

I assume that when forming initiation and bidding strategies, any indifference in timing is resolved by initiating later, and in that participation, by participating.

39This makes the equilibrium more robust and can be formally justified by assuming a small probability of costly initiation failure and a small Poisson arrival rate of exogenous auction initiation, then taking their limits to zero.
the simpler game where only the seller can initiate the auction. Since in equilibrium, the payoffs are equivalent to a first-price cash auction and FPAs and SPAs with cash generate the same revenue, the seller’s timing problem is equivalent to the strategic timing of a second-price cash auction. Her expected utility for holding the auction at time \( t_a \) with cash flow \( P_a \) is 
\[
E[e^{-r t_a}[W(P_a; \theta(2)) - X - Y]^+] \]
Similar to timing an formal auction design,

**Proposition 1.11.** When only the seller can initiate an informal auction, there is an optimal timing with auction threshold \( P_a \) inefficiently high.

Cash auctions and informal auctions timed by the seller are thus inefficiently delayed. The intuition is that a welfare maximizer initiates only when the auction payoff exceeds the cost of ownership transfer by a certain threshold. But the seller faces the additional cost in the form of information rent paid to the winning bidder, thus times her option value starts to erode only with higher \( P_a \), commanding a higher option premium for holding the auction.

In real life, especially in M&As and patent sales, we often see bidders initiating the auction. To fully endogenize the auction timing in an informal auction, one has to consider the Bayesian equilibrium in the optimal stopping game where both seller and bidders can initiate the auction. This timing game is complicated by two facts: first, all parties dynamically update their beliefs about the distribution of types; second, this learning process potentially renders FPAs different from SPAs because the bidding strategy in FPAs is dependent on the belief of other bidders’ types whereas in SPAs the undominated strategy is to bid one’s own value.

**Proposition 1.12.** The Bayesian auction game with informal bids admits an auction timing equilibrium where bidders always initiate with threshold \( P_1(\theta) \) increasing in \( \theta \) that uniquely solves

\[
\int_{\theta}^\theta \frac{d}{dP} \frac{W(P; \theta) - W(P; \theta')}{P^\beta} \bigg|_{P = P_1} f(\theta') [1 - F(\theta')]^{N-2} d\theta' = 0. \tag{1.5.1}
\]

This is an essentially unique equilibrium in SPAs, and the unique monotone equilibrium in FPAs.

The intuition is simpler in SPAs. If by cash flow level \( P \) the auction has not been initiated, everyone updates their beliefs about types that are present. The seller times the auction to

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40See Fidrmuc, Roosenboom, Paap, and Teunissen (2012), and Gorbenko and Malenko (2013b).
maximize the second-highest valuation, whereas type $P_2^{-1}(P)$ times the auction to maximize the present value of informational rent (difference between the his valuation and the second highest valuation). The latter starts to erode earlier than the former does as the initiation threshold $P_s$ increases. Therefore the seller always waits in such an equilibrium. In FPAs this may not hold because bidding strategies depend on the dynamically updated beliefs about the types present. Fortunately, in a monotone timing equilibrium, the bidders truncate the support of $\theta$ from lower values corresponding to better types, but form bidding strategies using beliefs on types that are worse. This ensures that the bidding strategies are not affected by dynamic learning.

The prediction that bidders initiate is broadly consistent with empirical evidence. For example, patent holders rarely organize an auction and instead are often approached by acquirers. Also, acquisitions by strategic bidders in informal negotiations are primarily bidder-initiated.\footnote{For example, Fidrmuc, Roosenboom, Paap, and Teunissen (2012) document almost 80% are bidder-initiated. Note the current model is more applicable to strategic acquisitions where bidders are more likely to have private information regarding valuation than in financial acquisitions.}

**Corollary 1.5.** *Bidder-initiated informal auctions are inefficiently accelerated ex post.*

Conditional on knowing the least-cost type, the auction should be held later to maximize social welfare. To see this, a bidder gets the difference between his valuation and the second highest valuation, and does not bear the cost of ownership transfer $X + Y$ unless he is the only participant, because this cost impacts the highest valuation and the second highest valuation in the same way. It in turn means that when $X + Y$ increases, both the seller and the bidders would prefer a higher threshold for initiation, but the seller is differentially affected more. Numerical simulations show that in general the bidder-initiated informal auctions does better than the seller-initiated informal auctions. The intuition is that when bidders initiate, there is dynamic learning and their private information is utilized for the timing decision, which improves welfare, and often the revenue to the seller. When the seller initiates, she only uses the prior belief on the distribution of types, which differs from the realized types.

Finally, since a bidder would not initiate until the investment trigger for his real option is reached, the following corollary ensues.

**Corollary 1.6.** *The real option is exercised more quickly in informal auctions that are bidder-initiated than in those that are not bidder-initiated.*
In practice, the seller’s level of commitment may lie in a continuous spectrum. For examples, the seller may commit to the security design but not the auction timing. Though not explicitly discussed here, the tradeoffs regarding auction timing are similar and many results generalize. In particular, Corollary 1.6 holds for bonus-bid auctions.

1.6 Model Implications and Empirical Evidence

The model has three main implications. First, security bids cause misaligned incentives in post-auction investment; the distortion depends on auction timing and security design. Second, strategic timing of auctions affects auction initiation, security choice, investment and equilibrium payoffs, and is fundamental to auction design. Third, in settings rich in dynamics and information asymmetry, conventional wisdom should be applied with caution.

1.6.1 Inefficiencies in Investment Timing

Earlier sections have demonstrated that investments are delayed in equity auctions, bonus auctions, and friendly debt auctions, and are accelerated in call option auctions. Even the revenue-maximizing design delays investment. Figure 1.4 gives an illustration of investments at very different times under various security designs. Using the properties of the Wald distribution, the time delay $t_D$ when the investment threshold is increased from $P_1$ to $P_2$ has PDF

$$f_{GBM}(t_D; P_1, P_2) = \frac{\ln P_2}{\sqrt{2\pi\sigma^2 t_D}} \exp \left[ -\frac{\left( \ln \frac{P_2}{P_1} - \left( \frac{\mu - \frac{\sigma^2}{2}}{2\sigma^2 t_D} \right) t_D \right)^2}{2\sigma^2 t_D} \right] \quad (1.6.1)$$

with the mean and shape parameters $m = \ln \left( \frac{P_2}{P_1} \right) / [\mu - \frac{\sigma^2}{2}]$ and $y = \left( \ln \left( \frac{P_2}{P_1} \right) / \sigma \right)^2$. In the case of royalty auctions or bonus-bid auctions, the expected investment lag is $\Gamma = -\ln(1 - \phi) / [\mu - \frac{\sigma^2}{2}]$, where we have assumed $\mu - \frac{\sigma^2}{2} > 0$ for the expectation to exist.

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42. The setup in Gorbenko and Malenko (2013b) resembles such a case.
43. See Karlin and Taylor (1975) pg.363, and Chhikara and Folks (1989).
44. If $\mu < \sigma^2/2$, the median lag $M$ can be considered instead. It satisfies $\Phi \left[ \ln \left( 1 - \phi \right) + \left( \mu - \frac{\sigma^2}{2} \right) \frac{M}{\sigma \sqrt{M}} \right] + \left( 1 - \phi \right)^{\frac{1}{2}} \phi \left[ \ln \left( 1 - \phi \right) - \left( \mu - \frac{\sigma^2}{2} \right) \frac{M}{\sigma \sqrt{M}} \right] = \frac{1}{2}$, and numerical simulations yield the same qualitative results under a wide range of parameters.
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Figure 1.4: Investment thresholds under various security designs. Simulated with $\mu = 0.06$, $\sigma = 0.2$, $r = 0.16$, $\theta \sim Unif[1.5, 5]$, $X = 0.4$, $Y = 0$, $P_a = 3$. Horizontal lines indicate investment thresholds, vertical lines indicate the calendar time of investment.

Next, I examine the effects of changing the underlying parameters on the expected investment lags in bonus-bid auctions. The expected development delays are independent of $N$, $X$, $P_a$, and $r$, but $\frac{\partial \Gamma}{\partial \phi} > 0$, $\frac{\partial \Gamma}{\partial \mu} < 0$ and $\frac{\partial \Gamma}{\partial \sigma} > 0$. In addition, $\frac{\partial^2 \Gamma}{\partial \phi^2} > 0$, $\frac{\partial^2 \Gamma}{\partial \mu^2} > 0$, $\frac{\partial^2 \Gamma}{\partial \sigma^2} > 0$. Not only do more volatile market or high royalty rates result in longer delays, but they are mutually reinforcing, with increasing marginal effects. What is the social cost of the investment lag? It can be shown that the option value is a fraction $(1 - \phi + \phi \beta) (1 - \phi)^\beta$ of the socially efficient value, and the fractional loss $L$ satisfies $\frac{\partial L}{\partial \phi} > 0$, $\frac{\partial^2 L}{\partial \phi^2} > 0$. Again, royalty rate has a compounding effect on social cost.

These predictions are consistent with available empirical evidence. The US Department of the Interior experimented with royalty auctions in 1978–1983, where the government fixed a small up-front “bonus” payment and allowed the bidders to compete on royalty rates. Many bidders bid extremely high royalty rates and the tracts were never drilled. Oil price and volatility were indeed extremely high during that period. Moreover, Humphries (2009) reports that the royalty relief programs in the 1990s significantly increased interest

45 See Dougherty and Lohrenz (1980) and Binmore and Klemperer (2002).
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in deepwater leases, and oil production increased sharply. Opaluch, Grigalunas, Anderson, Trandafir, and Jin (2010) also conclude that increased royalty rates would have a net negative effect on the social value of offshore development.

By highlighting the destructive powers of moral hazard, this paper suggests a potential explanation for why large tracts of land remain idle.46 Without prescribing detailed policy changes, this paper suggests that there is always the trade-off between social efficiency and revenue extraction. Any useful policy recommendations should first focus on reducing the post-auction moral hazard that is inimical to both the revenue and social welfare. Moreover, instead of uniformly raising the royalty rate, allowing bidders to self select into differential rates as described in Proposition 1.7 could be a more effective way in increasing revenue to the government.

1.6.2 Strategic Timing of Auctions

As seen in propositions 1.5, 1.8-1.9, 1.11, and 1.12, auction timing affects bidding strategy, auction outcome, and investment decision. More importantly, this shows many aspects of auction design should be analyzed in conjunction with endogenous auction timing. In Figure 1.3(a), the worst security design at $P_a = 300$ outperforms the best security design at $P_a = 220$ by at least 1.5. Similar phenomena are observed in FPAs (Figure 1.2) and for welfare (Figure 1.3(b)) too. In this regard, strategic timing is equally important as security design.

There could be factors exogenous to the model that affect the strategic timing of auctions. For one, delaying the auction risks creating market uncertainty and delaying the introduction of new technology or development, and potentially losing the “first-mover” advantage.47 Another motivation for timing the auction is the coordination of market players.48 This paper shows that strategic timing due to cost of auction is an important complement to other factors in explaining the timing of auctions observed in real life.

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46 In reality, many other strategic interactions among the bidders complicate the issue. For example, Beshears (2011) shows alliances in oil and gas drilling perform better than solo bidders; Hendricks and Porter (1996) attributes the delays in exploratory drilling to free-rider problem and war of attrition. The above analysis complements these studies.

47 The sale of the British 3G telecom licences is an illustrating case (Klemperer (2002), Binmore and Klemperer (2002)).

48 Auctions of futures contracts on electricity provision is a good example where an early auction allows the winning bidder ample time to construct facilities for generating and delivering electricity.
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Empirical Test of Auction Initiation and Investment

In addition to auction timing, who initiates an auction also matters. When the seller lacks commitment to auction timing, the bidders always initiate and the real option is exercised more quickly (Proposition 1.12 and Corollary 1.6). Since bidders time the auction to maximize information rent, they only initiate when their option value starts to erode, implying that the investment option is, on average, exercised faster when bidders initiate. I test this using data on leasing and exploration of oil and gas tracts in the Gulf of Mexico. Prior to the introduction of Area Wide Leasing (AWL) in May 1983, energy firms could nominate oil and gas tracts to be auctioned. AWL eliminated the nomination process and made most of the offshore lands available in every sale. The auction environment thus underwent an important break from one where bidders can initiate to one where leases are always on sale in a region decided by the seller. The model predicts that, bidders explore and drill faster when they initiate sales than when seller initiates. I use the Cox proportional hazard model to study the time to first exploratory drill for over 20,000 leases sold in the Gulf of Mexico. The Cox model does not impose restrictions on the baseline hazard rate and allows time-varying covariates and censoring of observations, and has become standard for duration analysis.

Table 1.2 reports the estimations of the model \( \kappa(t) = \psi(t) \exp(\vec{X}(t)^T \vec{\gamma}) \), where \( \kappa(t) \) is the hazard rate of exploratory drill at time \( t \) conditional on lack of drill until time \( t \), and \( \psi(t) \) is the baseline hazard rate that is unrestricted. \( \vec{X}(t) \) is a vector of independent variables, including the variable of interest AWL, which is 0 before May 1983 and 1 after, and other controls for firm, lease, and market effects (Appendix C.2 details their constructions). \( \vec{\gamma} \) are their coefficients to be estimated. The hazard ratio associated with AWL is consistently negative and significant, indicating that ceteris paribus, the rate to explore decreased from that in the era with bidder initiations. I consider the years 1978-1989, though the results are robust to the size of the window. AWL consistently reduces the likelihood of exploratory drilling by at least 10\%. I also estimate the model with year dummies instead of AWL, and Figure 1.5 clearly shows the break the estimated coefficients, which corresponds to a reduction of 40\% in the hazard ratio to explore and drill for leases auctioned after the implementation of AWL.
This table presents estimates from a Cox regression with time-varying covariates. The dependent variable is time-to-exploratory-drill, which measures the number of days from the lease auction to the first exploration. The independent variables are the variable of interest \( AWL \) indicating the absence of bidder initiation, oil and gas price measure \( P(t) \), price volatility \( VOL(t) \), drilling cost \( COST(t) \), royalty rate \( RTY \), water depth \( DEPTH \), lease length \( DUR \), tract size \( SIZE \), market demand \( MKT \), firm fixed effects \( FIRM \) f.e., and control for information externality \( INFO \). If the dependent variable is observed without any realization, it is treated as a censored event. Model \( \chi^2 \) reports the joint significance of the estimates. Hazard ratio indicates the impact of \( AWL \) on likelihood to drill.

<table>
<thead>
<tr>
<th>Model</th>
<th>AWL</th>
<th>MKT</th>
<th>RTY</th>
<th>DUR</th>
<th>DEPTH</th>
<th>SIZE</th>
<th>P(t)</th>
<th>VOL(t)</th>
<th>COST(t)</th>
<th>Info.Ext.</th>
<th>Firm f.e.</th>
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<td>0.0118***</td>
<td>-0.0200</td>
<td>-0.0073*</td>
<td>-0.0005†</td>
<td>-0.0003</td>
<td>0.0214†</td>
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<td>No</td>
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<td>(0.0698)</td>
<td>(0.0054)</td>
<td>(0.0227)</td>
<td>(0.0043)</td>
<td>(0.0001)</td>
<td>(0.0002)</td>
<td>(0.0058)</td>
<td>(0.0159)</td>
<td>(0.0004)</td>
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<td>(Full)</td>
</tr>
<tr>
<td>Model2</td>
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<td>-0.0591*</td>
<td>-0.0106**</td>
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<td>-0.0022†</td>
<td>0.0212†</td>
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<td>(0.0005)</td>
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<td>-0.2728</td>
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<td>-0.0022†</td>
<td>-0.0495</td>
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<td>-0.0112</td>
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<td>(0.0005)</td>
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<td>0.9153†</td>
<td>-0.0611*</td>
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<td>-0.0003</td>
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<td>(0.0073)</td>
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<td>-0.0022†</td>
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<td>-0.3014*</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>(0.0889)</td>
<td>(0.0061)</td>
<td>(0.0004)</td>
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</tbody>
</table>

Standard errors in parentheses. * \( p < 0.10 \), ** \( p < 0.05 \), † \( p < 0.01 \)
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Figure 1.5: Coefficients for year dummies in the Cox estimation after controlling for all observable covariates, firm fixed effect, and information externality. The estimates are reported with 95% confidence interval. The red line marks the commencement of Area Wide Leasing (AWL).

1.6.3 Nonconventional Wisdom

Many standard results in auction theory may not hold in the presence of pre-auction timing and post-auction dynamics, prompting re-examination of conventional beliefs.

Welfare Creation: Are Auctions as Efficient as We Think?

Are auctions as socially efficient as traditionally believed once we consider endogenous timing of auctions? Take private-value cash auctions for example: They are considered efficient in the standard literature but are, in fact, inefficiently delayed. Many other auctions are also less efficient than one believes once strategic timing is taken into consideration. The intuition is that the seller times the auction to maximize her revenue, not the social welfare. For regulators concerned with social efficiency, it is critical to consider auction timing in addition to the seller’s market power.

Auctions versus Negotiations: The More, The Merrier?

It is well-established that in private-value auctions increasing the number of bidders enhances the seller’s revenue. The importance of competition in corporate takeovers is perhaps best articulated in Bulow and Klemperer (1996):
"With independent signals and risk-neutral bidders, an absolute English auction with \( N + 1 \) bidders is more profitable in expectation than any negotiation with \( N \) bidders."

But in reality, sellers restrict the number of bidders even absent entry fees, for example, in sales of private companies and divisions of public companies (see Hansen (2001) and French and McCormick (1984)). Close to half of all corporate takeovers in the 1990s avoided public auctions with more bidders and opted for private negotiations (Boone and Mulherin (2007)). The moral hazard associated with security bids provides a potential explanation.

As competition intensifies, bidders bid more (Proposition 1.5), resulting in greater moral hazard. And it can be shown that the seller does better charging an entry fee or reserve price. Thus negotiations with \( N \) bidders can yield higher revenue than an absolute English auction with \( N + 1 \) bidders. Figure 1.6 gives numerical simulations in the same spirit as in Samuelson (1985) to indicate that revenue and welfare could vary in almost any way with \( N \). There is no contradiction, however, because the monotonicity of revenue in competition is restored with the optimal security. The key lesson is that the impact of competition depends on the security design.

The result generalizes to auctions with standard securities such as friendly debts and call options (see Figure 1.7), and is robust to distributional assumptions and endogenous entries with entry costs. Since the expected social welfare and revenue to the seller need not increase with the number of potential bidders, limiting participation may improve revenue or welfare. Given that many public auctions of companies involve the use of standard securities, this is consistent with the aforementioned empirical observations.

Security Ranking: One Size Fits All?

While DeMarzo, Kremer, and Skrzypacz (2005) derive elegant results on “steepness” ranking security designs, the ranking has to be considered in conjunction with potential misalignment of incentives, timing of auctions, and number of bidders, etc. In Figure 1.7, ranking is sensitive to the number of potential bidders while in Figure 1.3(a), ranking is sensitive to auction timing. In both figures, though equity generally dominates friendly debt, it could either dominate or be dominated by call options despite being less “steep”. These results are robust to the introduction of entry fees or entry costs and are novel in linking security design to level of competition and endogenous auction initiation.
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Figure 1.6: Plots of expected social welfare (a)(b)(c) and seller’s revenue (d)(e)(f) against number of bidders $N$. One million simulations in SPA with equity bids and uniformly distributed $\theta$.

(a) Unif$[20, 50]$, $P_a = 35$, $\beta = 2$, $X = 10$, $Y = 0$
(b) Unif$[20, 50]$, $P_a = 35$, $\beta = 8$, $X = 1$, $Y = 0$
(c) Unif$[30, 60]$, $P_a = 45$, $\beta = 5$, $X = 2.5$, $Y = 0$

(d) Unif$[30, 60]$, $P_a = 45$, $\beta = 5$, $X = 2.5$, $Y = 0$
(e) Unif$[20, 50]$, $P_a = 35$, $\beta = 25$, $X = 0.1$, $Y = 0$
(f) Unif$[20, 50]$, $P_a = 35$, $\beta = 8$, $X = 1$, $Y = 0$

Figure 1.7: Plots of expected social welfare and seller’s revenue against number of bidders $N$ for SPAs with equities, friendly debts and call options. One million simulations with $\theta$ uniformly distributed in $[20, 50]$, $P_a = 35$, $\beta = 10$, and $X + Y = 1$. 

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Cash versus Contingent Securities: Who is the Winner?

Prior studies indicate that security bids usually perform better than cash bids. Rhodes-Kropf and Viswanathan (2000) show that any securities auction generates higher expected revenue to the seller than a cash auction. But since the linkage advantage of security bids lies in the extraction of the winning bidder’s rent, it decreases in expectation when $N$ increases. Yet moral hazard persists with many standard securities. In particular, cash could generate higher revenue than many standard securities such as equities. To illustrate, I focus on pure contingent securities.

First note that any contingent security $S(s, P)$ can be approximated by $\sum_{i \in I} a_i(s) [P - b_i(s)]^+$, where $I$ is a countable set and $\sum_i a_i(s) \leq 1 \forall s$ to ensure $P - S(s, P)$ is weakly increasing in $P$. Suppose $s$ is the security the type $\theta$ bids, without loss of generality $b_i(s) \leq b_j(s)$ if $i \leq j$. I define the following class of securities:

**DEFINITION.** An $M$-regular security is a contingent security for which the above approximation is exact such that for $M > 0$, and $\frac{\beta}{\beta - 1} \theta \in [b_m, b_{m+1}]$,

$$\min \left( \left| \frac{\beta}{\beta - 1} \theta - b_m \right|, \left| \frac{\beta}{\beta - 1} \theta - b_{m+1} \right|, \left| \sum_{i \leq m} a_i b_i - \theta \sum_{i \leq m} a_i \right| \right) > M. \quad (1.6.2)$$

In fact, most common securities are $M$-regular securities. For example, equity corresponds to $a_1 = \alpha(\theta)$, $a_2 = b_1 = 0$, $b_2 = \infty$.

**Proposition 1.13.** For any $M > 0$, cash bids dominate $M$-regular securities in FPAs and SPAs in terms of expected revenue and social welfare, as the number of bidders gets large.

In particular, cash bids dominate equity bids, call option bids, and friendly debt bids. Given that most common securities can be approximated by an $M$-regular securities, the size of the bidders market is an important consideration in security choice. The result also predicts that security bids are seldomly used when the number of bidders is large.

Other considerations also influence security design. For example, the medium of exchange acts as a signal to the market (Eckbo, Giammarino, and Heinkel (1990), Betton, Eckbo, and Thorburn (2009) and Malmendier, Opp, and Saidi (2012)). Gorbenko and Malenko (2011) suggest another scenario where cash dominates equity bids when the number of sellers and the corresponding bidders’ markets are large, because sellers find cash
more effective in attracting more bidders. The battle between cash and contingent securities goes on and there is unlikely to be a clear winner.

1.7 Extensions and Discussions

1.7.1 Regret and Renegotiation

A regret-proof mechanism is easy to implement. Some standard securities allow allocations to be sub-optimal based on the seller’s inference of the bidders’ types, which can cause disqualification of the bid ex post. But people still use them for simplicity of bidding rules, in agreement with the spirit of formal auctions where allocation rules are fait accompli. Moreover, regret is rare in auctions with standard securities.

Renegotiation is another form of ex post regret. In special cases, renegotiation can make everyone better off ex post. I use bonus-bid auctions to illustrate. Winning bidder of type \( \theta \) can renegotiate to invest efficiently and split the additional social surplus in proportion (this includes the Nash bargaining solution); this process is equivalent to contracting to invest at \( P^* (\theta) \) using a new royalty rate \( \hat{\phi} \in \left[ \phi (1 - \phi)^{\beta - 1}, \frac{1 - (1 - \phi)^{\beta}}{\beta} \right] \), where the bounds ensure that both parties are weakly better off. But this leads to \( 1 - \hat{\phi} \beta \geq (1 - \phi)^{\beta} > 1 - \phi \beta \) by Bernoulli’s inequality. Thus \( \hat{\phi} < \phi \). Given that the bidders pay a smaller royalty rate with renegotiation, they bid more cash bonus upfront. This new equilibrium obviously improves welfare. Since the seller receives higher bonus bids, and weakly higher present value of royalty payment, her expected revenue is greater too. What about the winning bidder? By revenue equivalence, in both FPAs and SPAs with bonus bids, the information rent to the winner is proportional to \( (1 - \phi)^{\beta} [\theta^{1 - \beta} - \mathbb{E}[\hat{\theta}^{1 - \beta} | \hat{\theta} \leq \theta]] \), where \( \hat{\theta} \) refers to the second-highest bidder. A reduced royalty rate improves bidders’ payoffs as they are decreasing in \( \phi \). Hence all parties benefit from renegotiation.

The above example hinges on the contractibility of optimal investment. In general, committing to no renegotiation helps the seller to attain the highest payoff and maintain reputation in repeated interactions. Absent a commitment to no renegotiation, the original bidding equilibrium breaks down. There is also the question whether the seller can commit

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49 See, for example, Che and Kim (2010).

50 Another way to prevent post-auction regret is to restrict the range of bids \([s_L, s_H]\). This leads to partial pooling at the top, with all main results are unaltered.
to the initial bidding rule, the absence of which makes the process essentially an informal auction, as discussed in Section 4.

1.7.2 Options with Expirations

So far we have assumed that the investment option is perpetual. This is a reasonable simplification, because investment decisions are often made in relatively short time frames compared to the time scale of investment opportunities. For example, the duration for land leases are typically 30-50 years and constructions are often planned and implemented in a few years. However, it is useful to understand how the results are modified as a real option approaches expiration. Moreover, many investment options lose value due to unforeseen circumstances such as natural disasters and regulatory reforms—circumstances best modeled with stochastic expirations. Neither case would affect the implications of this paper.

Stochastic Expiration: Suppose with arrival intensity $\delta$, the investment option is rendered worthless. This is equivalent to augmenting the discount rate by $\delta$, which makes delaying less desirable. Stochastic expirations can work to the seller’s advantage, for example, when she decides expiration terms in combination with royalty rates.

Deterministic Expiration: With deterministic expiration time $T$, the project is equivalent to an American call option on a stock whose price process follows $P_t$, and pays a stream of dividend $(r - \mu)P_t$. Numerical solutions to optimal exercise and option valuations are well-known in mathematical finance. Figure 1.8 gives an illustration of bonus-bid auctions with a lease duration of 5 years. Although the investment thresholds converge close to the expiration, higher royalty rates still lead to greater investment delays due to post-auction moral hazard. Moreover, auction timing still affects bidding and payoffs in that having deterministic expirations are equivalent to having a different distribution of valuations.

1.7.3 Interdependent Values and Affiliated Signals

With interdependent values and affiliated signals, bidding strategies become more complicated. Moreover, optimal investments post-auction are sensitive to the auction format in addition to the security bid, because the winning bidder obtains different information set for auctions of different formats. Though I leave the detailed analysis for future research, it is reasonable to expect that key results generalize. First, standard security bids still cause inefficiency in investments because the shape of the security is independent of the winning
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Figure 1.8: Investment Thresholds for Real Options with Expirations. The plots are for $\theta = 10$, $\mu = 0.02$, $r = 0.04$, $\sigma = 0.2$, $T = 5$, and fixed royalty rates $\phi = 0, \frac{1}{5}, \frac{1}{6}$ respectively.

bidder’s bid. For example in equity auctions, the security bid inevitably causes investment delay because $E_S < \beta$. Second, even with interdependent values and affiliated signals, the seller’s payoff differs from welfare, and auction timings are generally inefficient.

In the specific case of oil and gas leases, there are both common value and private value components. Even with the traditional “mineral rights” models that emphasize the common value component, the investment threshold is bigger than the efficient one by a factor of $1/(1 - \phi)$, and $P_a$ affects the valuations and bidding strategies. In addition to being analytically tractable, the private-value framework is not unrealistic in the sense that the dispersion of bidder types over the common component such as signals on the amount of oil reserve has decreased in recent years due to technological improvement, but firms often have private retail and transportation contractors which are better captured by private values.

1.7.4 Liquidity Constraints and External Financing

Anecdotal evidence suggests that liquidity constraints may often be a concern. To mitigate such constraints many bidders turn to contingent bids. The results in this paper are
robust to cash-constraints with the use of non-standard securities, which correspond to side contracts, regulatory agreements, etc., in real practice. Moreover, a bidder can raise financing from a third party.

External financing is an interesting extension in itself. Post-auction investment is unaffected because we can view the financier and the seller as a coalition; from the bidders’ perspective, they face the same bidding and investment problem. For example, suppose in bonus-bid auctions for oil leases, the bidders have to externally finance the bonus bids. Further assume the financier only takes shares of the future revenue from the oil production, and the amount of shares increases with the cash demanded by the bidder. To the bidders, this is equivalent to an equity-bid auction on the revenue where the reserve equity share is the royalty rate to the seller. The development of oil will be further delayed, resulting in both loss of welfare and payoff to the bidders. The strategic timing of auctions changes because even though the total payoff to the seller and the financier is the same as in a security-bid auction without external financing, the split depends on the financing terms and form of the auction.

1.7.5 Optimal Dynamic Mechanism

So far we have restricted the mechanism design to formal auctions where the time of ownership transfer to bidder \(i\) conditional on winning is \(t_i = t_A\) \(\forall i\), which is realistic. In the hypothetical situation where pre-auction communication is costless, we can augment the design space to include dynamic mechanisms where the transaction timing \(t_i\) differs across \(i\). In a direct revelation mechanism, Lemma 1.2 still holds, and since \(t_i \leq \tau_i^*\), the seller optimally sets \(t_i = \tau_i^*\) to postpone incurring the cost \(X + Y\) as much as possible. Therefore, optimal selling mechanisms of real options allocate the project to the type with the lowest \(\theta\) at cash flow \(P^*(z(\theta) + X + Y)\) and entails immediate investment upon transfer of ownership. The bidders pay cash from the project such that the IC condition in Lemma 1.2 holds.

In contrast to the optimal auctions in Myerson (1981) and in the previous discussion on

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\(^{51}\) A previous version of the paper solves the case with cash constraints, allowing \(\delta\)-function securities of the form \(S(P) = H(P')1_{\{P-P'\}} + P1_{\{P-P'\}}\).

\(^{52}\) For example, see Rhodes-Kropf and Viswanathan (2005), Povel and Singh (2010), and Liu (2012).

\(^{53}\) Vladimirov (2013) discusses the payoffs to the seller and financier in a static setting.

\(^{54}\) With costless communication, revenue can also be maximized using a dynamic variant of the Vickery-Clarke-Groves mechanism even for \(\theta\)s that evolve over time, as Board (2007b) originally points out.
formal auctions, this dynamic mechanism ensures full participation, because the marginal revenue from each type other than the worst type can become positive if $P$ is high enough. This is a consequence of the flexibility of having a portfolio of options to time the allocation to each bidder, over having an option to time the auction for a portfolio of investment options. Nevertheless the investment and allocation timing are still inefficiently delayed ($\max\{ P_0, \frac{\beta}{\beta-1} [z(\theta) + X] \}$). The seller’s costless communication of the screening contracts to the bidders is crucial here. Otherwise, the seller strategically times that communication and the analysis is identical to that of formal auction design.

1.8 Conclusion

Auctions of real options are ubiquitous, involve tremendous financial resources, and have policy implications. Prior studies have mostly analyzed the sales and exercises in isolation, treating the former as one-shot games while exogenously specifying the agency conflicts in the latter. To better understand these corporate transactions and reconcile theory with empirical observations, this paper introduces endogenous timing and post-sale dynamics into an auction model with investment. I show that common security bids lead to inefficient and often sub-optimal investment; and endogenous auction initiation and seller’s commitment significantly impact bidding equilibrium, auction payoffs, and post-auction investment, and thus are integral to auction design. I further show that optimal auction design corresponds to the popular combination of cash and royalty payments in real life, and entails inefficient sales and investments. Taken together, the results of the paper challenge earlier approaches that analyze auction initiation, security design, and corporate investments separately: the interactions of these factors in dynamic settings give rich interplay that is not accessible otherwise, and as a consequence, many conventional beliefs should be revised.

As a first attempt to capture the salient features of auctions of real options and the underlying economics, this paper adds insights to security bids, endogenous initiation, and agency issues in the real options framework, and gives predictions more in accord with real life observations. Admittedly, the model is not the most general possible and more work is clearly needed for many of the aforementioned applications and extensions: In particular, selling real options with renegotiations and resales is worth exploring further. Incorporating seller’s private information is also important in many applications, especially M&As.
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Moreover, while some of the novel predictions are consistent with stylized facts and available data, others require further empirical studies. The real challenge lies in developing these research options in a timely and responsible manner.

1.9 Appendix

A Derivations and Proofs.

The Arrow-Debreu Security

For sufficiently small $dt$, $D(P_t; P') = e^{-\gamma dt}E[D(P_t + dt; P')]$, an application of Itô’s formula shows $D(P_t; P')$ satisfies $\frac{1}{2}\sigma^2 P^2 D_{PP} + \mu PD_p - r D = 0$, subject to the boundary conditions $D(P'; P') = 1$ and $D(0; P_a) = 0$. This yields $D(P_t; P') = E[e^{-\gamma(\tau-t)}] = (\frac{P_t}{P_a})^\beta$ where $\tau = \inf\{s \geq t : P_s \geq P'\}$.

Proof of Proposition 1.1

Proof. First note $\tilde{V} \in [-X - C, W(P_a; \theta) - X - C]$, thus the valuation is finite. The value function of the optimal stopping is thus the infimum of a class of $C^2$ functions with non-positive drift that majorize $P - S(P) - \theta$, and the stopping time is first-hitting. (Proposition 5.8, and 5.10 in Harrison (2013)). Therefore,

$$\tilde{V}(C, S(\cdot), \theta) = D(P_a; P_L, P_U) [P_L - S(P_L) - \theta] + D(P_a; P_U, P_L) [P_U - S(P_U) - \theta] - C - X,$$

where $D(P_a; P_L, P_U)$ is the Arrow-Debreu security that pays one dollar when $P$ first hits before $P_L$ before hitting $P_U$, and $D(P_a; P_U, P_L)$ is similarly defined. And $P_L \in [0, P_a]$ and $P_U \in [P_a, \infty]$ are the optimal lower and upper thresholds for investment. $D(P_a; P_L, P_H)$ satisfies $\frac{1}{2}\sigma^2 P^2 D_{PP} + \mu PD_p - r D = 0$ with the boundary conditions $D(P_L; P_L, P_U) = 1$ and $D(P_U; P_L, P_U) = 0$. The solution is

$$D(P_a; P_L, P_U) = \left(\frac{P_a^\beta - P_a^\gamma P_U^{\beta-\gamma}}{P_L^\beta - P_L^\gamma P_U^{\beta-\gamma}}\right)^{-1},$$
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and similarly,

\[ D(P_a; P_U, P_L) = \left( P_a^\beta - P_a^\gamma P_L^{\beta - \gamma} \right) \left( P_U^\beta - P_U^\gamma P_L^{\beta - \gamma} \right)^{-1}, \]

where \( \beta \) is given in (1.2.2) and \( \gamma = 1 - 2\mu/\sigma - \beta < 0 \). The optimal \( P_L \) and \( P_H \) are obviously independent of \( X \) and \( C \) and are functions of \( \theta \) and \( P_0 \) in general.

Finally, type \( \theta \) can always do strictly better than \( \tilde{\theta} > \theta \) by using \( \tilde{\theta} \)'s strategy, thus \( \tilde{V}(C, S(\cdot), \theta) \) is decreasing in \( \theta \). For \( P_t \) in the exercise region, \( \tilde{V}(C, S(\cdot), \theta) = P_t - S(P_t) - \theta - C = X \) is obviously continuous in \( \theta \). For \( P_t \) in the continuation region, consider a change of \( \Delta \theta > 0, 0 < \tilde{V}(C, S(\cdot), \theta) - \tilde{V}(C, S(\cdot), \theta + \Delta \theta) \leq \Delta \theta \) because type \( \theta + \Delta \theta \) does weakly better than simply mimicking \( \theta \)'s strategy. As \( \Delta \theta \to 0, \tilde{V}(C, S(\cdot), \theta + \Delta \theta) \to \tilde{V}(C, S(\cdot), \theta) \). The case of \( \Delta \theta < 0 \) is similar. Continuity in \( \theta \) follows.

Proof of Lemma [1.1]

Proof. Since an upper threshold strategy has payoff \( \left( \frac{P_a}{P_U} \right)^\beta [P - S(P) - \theta] \) for \( P \geq P_a \), threshold \( \tilde{P} \) is optimal among all upper threshold strategies. I now verify that it is optimal among all stopping times by showing the expected value following any stopping time is bounded above by the expected value associated with the \( \tilde{P} \)-threshold strategy.

Let \( x_t = e^{-rt}\tilde{W}(P_t) \), where \( \tilde{W}(P_t) = D(P_t; \tilde{P})[\tilde{P} - S(\tilde{P}) - \theta] \) and \( \tilde{P} = \max\{ P_t, \tilde{P} \} \).

For \( P \leq \tilde{P} \), using an extended version of Itô’s formula (as, for example, in Karatzas and Shreve [1988], page 219),

\[ dx_t = e^{-rt}[D\tilde{W}(P_t) - r\tilde{W}(P_t)]dt + e^{-rt}\tilde{W}_P(P_t)\sigma dB_t, \]

where \( D\tilde{W}(P) = \tilde{W}_P(P)\mu P + \frac{1}{2}\tilde{W}_{PP}(P)P^2 \). \( \tilde{W}_P \) is bounded as seen by direct computation, thus by Proposition 5B in Duffie [2009] (also found in Protter [2004]), the last term in \( dx_t \) is a martingale under the current measure. The drift is \( D\tilde{W}(P) - r\tilde{W}(P) = 0 \) by the definition of \( \beta \) in (1.2.2). For \( P > \tilde{P} \), apply Tanaka’s Formula (Revuz and Yor [1999], also Karatzas and Shreve [1988]), the drift \( D\tilde{W}(P) - r\tilde{W}(P) = \mu P[1 - S'(P)] - r[P - S(P) - \theta] - \frac{1}{2}\sigma^2 S''(P) \leq [\mu + \frac{1}{2}(\beta - 1)\sigma^2]P[1 - S'(P)] - r[P - S(P) - \theta] < [\mu\beta + \frac{1}{2}\sigma^2(\beta - 1) - r][P - S(P) - \theta] = 0 \), using (Conv) and the definition of \( \beta \).

Since to the discounted occupancy measure, there is a discounted local time \( l \) (Stokey [2009], Theorems 3.6 and 3.7), the additional local time term in \( dx_t \) when \( S'(P) \) jumps is

\[ \frac{1}{2}\sigma^2 \int_{\mathbb{R}_+} l(p, t, r)\nu(dp), \]

where \( \nu((a, b]) \equiv \tilde{W}'(b) - \tilde{W}'(a) \), is non-positive due to (Conv).

Therefore, \( x_t \) is a super-martingale, implying for any stopping time \( \tau, \tilde{W}(P_0) = x_0 \geq 0 \).
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\[ \mathbb{E}[x_t] = \mathbb{E}[e^{-rT} \tilde{W}(P_t)] \geq \mathbb{E}[e^{-rT}(P_t - S(s, P_t) - \theta)] \]. The equality holds for the first-hitting time with threshold \( \tilde{P} \), establishing its optimality. Finally, when \( P - S(P) \) is non-decreasing in \( P \), \( \tilde{V} \) is non-decreasing in \( P_a \) since the optimal exercise involves upper-threshold only. ■

Proof of Proposition 1.3

Proof. For \( s_1 < s_2 \) and \( \theta_1 < \theta_2 \), because \( V(s, \theta) \) is absolutely continuous with derivative in \( s \) decreasing in \( \theta \),

\[ \ln \left( \frac{V(s_1, \theta_1)V(s_2, \theta_2)}{V(s_1, \theta_2)V(s_2, \theta_1)} \right) = \int_{s_1}^{s_2} \frac{\partial V(s', \theta_2)}{\partial s} ds' - \int_{s_1}^{s_2} \frac{\partial V(s', \theta_1)}{\partial s} ds' < 0 \]  

(1.9.1)
i.e., \( V(s, \theta) \) is log-submodular, and thus strictly submodular. Let \( Q(s) \) be the probability of winning. Because \( s(\theta) \in \arg\max_s Q(s)V(s, \theta) = \arg\max_s \ln(Q(s)V(s, \theta)) \), by Topkis (1978), \( s(\theta) \) is non-increasing in \( \theta \). If \( s(\theta) < s_H \) were constant on an interval, the bidder with the lower \( \theta \) can increase his bid marginally and increase his probability of winning (thus his payoff) by a discrete amount. Therefore \( s(\theta) \) must be decreasing in type for types bidding less than \( s_H \). Therefore, \( Q(s(\theta)) = [1 - F(\theta)]^{N-1} \). Note \( s \) is also continuous in \( \theta \), lest a type right below a discontinuity could lower his bid marginally without affecting the chance of winning.

Next, by direct revelation, \( \theta \in \arg\max_{\theta \in [\underline{\theta}, \overline{\theta}]} Q(s(\theta'))V(s(\theta'), \theta) \). For any \( \theta' < \theta \),

\[ Q(s(\theta))V(s(\theta), \theta) \geq Q(s(\theta'))V(s(\theta'), \theta) = Q(s(\theta'))[V(s(\theta), \theta) + V_1(s^*, \theta)[s(\theta') - s(\theta)]] \]

for some \( s^* \) between \( s(\theta') \) and \( s(\theta) \). Since \( V_1 < 0 \), the above expression can be written as

\[
\frac{Q(s(\theta')) - Q(s(\theta))}{\theta' - \theta} \frac{V(s(\theta), \theta)}{-Q(s(\theta'))V_1(s^*, \theta)} \geq \frac{s(\theta') - s(\theta)}{\theta' - \theta}
\]

Similarly, exchanging \( \theta \) and \( \theta' \), for some \( s^{*\star} \) between \( s(\theta) \) and \( s(\theta') \),

\[
\frac{Q(s(\theta')) - Q(s(\theta))}{\theta' - \theta} \frac{V(s(\theta'), \theta')}{-Q(s(\theta))V_1(s^{*\star}, \theta')} \leq \frac{s(\theta') - s(\theta)}{\theta' - \theta}
\]

Taking the limit we get (1.4.1).

As \( V(s, \theta) \) is continuous in \( s \) over \([s_L, s_H]\) and decreasing in \( \theta \), \( \max_s V(s, \theta) \) exists and
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\[
\sup\{ \theta \leq \hat{\theta}[1 - F(\hat{\theta})]^{N-1} \max_s V(s, \hat{\theta}) \geq 0 \} \]
gives the cutoff type. In equilibrium, \( s(\hat{\theta}) = \sup\{ s \in [s_L, s_H] | V(s, \hat{\theta}) \geq 0 \} \), otherwise bidding slightly more increases the winning probability discretely from zero while still breaking even upon winning. As \( V(s(\hat{\theta}), \hat{\theta}) \leq W(P_a; \hat{\theta}) - X \) and \( W(P_a; \hat{\theta}) - X = 0 \) in cash auctions, the cutoff type for security bids is in general weakly smaller than that in cash auctions. With the absolute continuity assumption in the proposition, the cutoffs are the same as in cash auctions.

This establishes uniqueness of the equilibrium, whose existence follows from the sufficiency of bidders’ F.O.C. - the quasiconcavity of \( \ln(Q(s) V(s, \theta)) \). For any \( s' \in (s(0), s(\theta)) \), \( \exists \theta' \in (0, \theta) \) such that \( s(\theta') = s' \). Submodularity of \( V \) implies \( \frac{d^2}{ds^2} \ln[Q(s') V(s', \theta)] > \frac{d^2}{ds^2} \ln[Q(s') V(s', \theta')] = 0 \). Similarly, \( \frac{d^2}{ds} \ln[Q(s') V(s', \theta)] < 0 \) for \( s' \in (s(\theta), s(\hat{\theta})) \). Therefore for every \( \theta \), there exists a unique \( s \) maximizing \( Q(s) V(s, \theta) \).

Proof of Proposition 1.4

Proof. Since \( \Pi \) is a left-continuous map, \( V(s, \theta) \) is left-continuous in \( s \) by an argument similar to the one in Proposition 1.1 for \( V(s, \theta) \) to be continuous in \( \theta \). Therefore \( s(\theta) \) is well-defined. Suppose a participating bidder of type \( \theta \) bids \( s > s(\theta) \), he benefits from decreasing \( s \) to reduce the states of the world in which he wins but receives negative payoff. Similarly, he wants to increase \( s \) when \( s < s(\theta) \), assuming any indifference in bidding is resolved by bidding higher. As \( V(s, \theta) \) is decreasing in \( \theta \), for \( \theta' > \theta \), \( V(s(\theta'), \theta') < V(s(\theta), \theta) = 0 = V(s(\theta'), \theta') \). Thus \( s(\theta) > s(\theta') \), leading to \( s(\theta) \) being decreasing. The cut-off type is the same as in FPAs by an argument similar to that in the proof of Proposition 1.3.

Proof of Proposition 1.5

Proof. When \( P_a \) increases or \( X \) decreases, \( \hat{\theta} \) weakly increases, thus potentially a positive measure of originally non-participating bidders are bidding. Given the bidding strategy in SPAs and the fact that non-negative \( V \) is non-increasing in \( s \), decreasing in \( X \) and increasing in \( P_a \), for each original participant increasing \( P_a \) or decreasing \( X \) results in a bigger \( s \). In FPAs, let \( \tilde{s}(\theta) \) denote the bidding strategy after \( P_a \) increases or \( X \) decreases. Then \( \tilde{s}(\theta) \geq s(\theta) \) at least for the original cutoff type \( \hat{\theta}_{old} \). If \( \tilde{s}(\theta) = s(\theta) \) for any \( \theta \in [\hat{\theta}, \hat{\theta}_{old}] \), \( s' \) by Proposition 1.3, thus \( \tilde{s}(\theta) \) stays above \( s(\theta) \) for a positive measure of types. Overall we have a positive measure of types bidding bigger \( s \), and all types bidding weakly
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bigger. For the same reason, the result holds when \( N \) increases in FPAs. ■

Proof of Lemma 1.2

Proof. The IC constraint can be written as \( \theta_i \in \arg \max_{\tilde{\theta}_i \in [\underline{\theta}, \bar{\theta}]} U(\theta_i, \tilde{\theta}_i) \forall i \). As \( U(\theta_i, \tilde{\theta}_i) \) is not necessarily differentiable in \( \tilde{\theta}_i \), rewrite this as \( \tilde{\theta}_i \in \arg \max_{\tilde{\theta}_i \in [\underline{\theta}, \bar{\theta}]} [U(\theta_i, \tilde{\theta}_i) - U(\theta_i)] \).

Let \( a = (\tau, \tilde{\theta}) \) denote the action pair of reporting \( \tilde{\theta} \) and rationally exercise following the stopping time \( \tau \). Let

\[
g(a, \theta) = Q(\tilde{\theta}, \theta_{-i}) \mathbb{E}_P \left[ e^{-r\tau} (P_\tau - \theta) - \int_{t_\tau}^{P_\tau} e^{-r t} S(\tilde{\theta}, \theta_{-i}, \mathcal{I}_t) dt - e^{-r\tau} X \right]
\]

Then following the argument in Milgrom and Segal (2002), for any \( \theta', \theta'' \in [\underline{\theta}, \bar{\theta}] \) with \( \theta' < \theta'' \),

\[
|U(\theta') - U(\theta'')| = \mathbb{E}_{\theta_{-i}} \left[ \sup_{a'} g(a', \theta') - \sup_{a''} g(a'', \theta'') \right] \\
\leq \mathbb{E}_{\theta_{-i}} \left[ \sup_a |g(a, \theta') - g(a, \theta'')| \right] = \mathbb{E}_{\theta_{-i}} \left[ \sup_a \int_{\theta'}^{\theta''} g_\theta(a, \theta) d\theta \right] \\
\leq \mathbb{E}_{\theta_{-i}} \left[ \int_{\theta'}^{\theta''} \sup_a |g_\theta(a, \theta)| d\theta \right] \leq |\theta'' - \theta'|
\]

This implies \( U(\theta) \) is absolutely continuous, and thus differentiable everywhere. \( U(\theta) = U(\bar{\theta}) - \int_0^\theta U'(\theta') d\theta' \). By Theorem 1 in Milgrom and Segal (2002), \( U'(\theta) = g_\theta(a^*, \theta) \), and the Lemma follows.

Note that when \( S \) is such that \( U(\theta) \) is locally smooth at the optimal stopping (where is not true for the general contracting space we have), we can directly apply the bounded convergence theorem and envelope theorem, \( U(\theta_i, \tilde{\theta}_i) \) is differentiable in \( \theta_i \), with

\[
U'(\tilde{\theta}_i) = \frac{\partial U(\theta_i, \tilde{\theta}_i)}{\partial \theta_i} \bigg|_{\theta_i = \tilde{\theta}_i} = \mathbb{E}_{\theta_{-i}} \left[ Q(\tilde{\theta}_i, \theta_{-i}) \mathbb{E}_P \left[ -e^{-r\tau} \tilde{\theta}_i, \tilde{\theta}_i, \theta_{-i} \right] \right] (1.9.2)
\]

which is uniformly bounded by 1. Then \( U(\theta_i) \) is Lipschitz continuous because it is the upper envelope of a family of Lipschitz continuous functions \( U(\theta_i, \tilde{\theta}_i) \) indexed by \( \tilde{\theta}_i \) with the same Lipschitz constant (Proposition 6.3 in Choquet (1966)). \( U(\theta_i) \) is thus absolutely continuous and differentiable almost everywhere (Corollary 6.3.7, Cohn (1980)). Writing it in the integral form concludes the proof.
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Proof of Proposition 1.6

Proof. The ex-ante social welfare is

\[ N \mathbb{E}_\theta \left[ Q(\theta_i, \theta_{-i}) \left( \mathbb{E}_P \left[ e^{-r r^*_i (P^*_i - \theta_i)} \right] - e^{-rt_a (X + Y)} \right) \right], \]

and the seller’s ex ante revenue is the social welfare less the agents’ ex-ante utilities:

\[ N \mathbb{E}_\theta \left[ Q(\theta_i, \theta_{-i}) \left( \mathbb{E}_P \left[ e^{-r r^*_i (P^*_i - \theta_i)} \right] - e^{-rt_a (X + Y)} \right) \right] - N \mathbb{E}_\theta [U(\theta_i)]. \]

Using (1.4.3) and taking expectations over the winning bidder’s type, it becomes

\[ N \mathbb{E}_\theta \left[ Q(\theta_i, \theta_{-i}) \left( \mathbb{E}_P \left[ e^{-r r^*_i (P^*_i - \theta_i)} \right] - e^{-rt_a (X + Y)} \right) \right] - NU(\hat{\theta}). \] (1.9.3)

With standard securities, a participant with the least cost wins, the proposition follows.

Proof of Proposition 1.7

Proof. To maximize expression (1.9.3), for every realization of the types and any allocation rule, the seller wants winner \( \theta_i \) to invest when \( P \) first hits \( P^*(z_i) \). The proposed contingent payment achieves this outcome. Moreover, \( U(\hat{\theta}) = 0 \) and the project is only allocated to types that contribute positively to the revenue. With Uniform or Generalized Pareto, \( z \) is increasing in \( \theta \), leading to the unique cutoff type \( \hat{\theta} \) proposed and allocation to a participant with the smallest \( \theta \).

That \( U(\theta_i) \) is decreasing in \( \theta_i \) implies any mechanism satisfying the above meets IR of all types. Suppose \( \theta_i < \hat{\theta_i} \), Lemma 1.2 leads to

\[
U(\theta_i, \hat{\theta}_i) = U(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} U_1(\theta, \hat{\theta}_i, \tau^*(\theta, z_i(\hat{\theta}_i, \theta_{-i}, \cdot))) d\theta \\
\leq U(\hat{\theta}_i) - \int_{\theta_i}^{\hat{\theta}_i} U_1(\theta, \theta, \tau^*(\theta, z_i(\theta, \theta_{-i}, \cdot))) d\theta \\
= U(\theta_i),
\]

where the inequality follows from (1.9.2) and the fact that reporting a higher investment.
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cost leads to a lower probability of winning and a later investment. Similarly, \( U(\theta_i, \hat{\theta}_i) \leq U(\theta_i) \) for \( \theta_i > \hat{\theta}_i \). Thus incentive compatibility holds.

Finally, \( C(\theta_i, \theta_{-i}) \) and \( S(\hat{\theta}_i, \theta_{-i}, \mathcal{T}, \tau^*) \) are such that Lemma 1.2 holds. \( \blacksquare \)

Proof of Proposition 1.8

Proof. The seller’s expected utility for holding auction when \( P_a \) is first reached can be written as

\[
D(P_0; P_a) \int_{\theta}^{\theta_g} N f(\theta)[1 - F(\theta)]^{N-1}[W(P_a; z(\theta)) - X - Y]d\theta. \tag{1.9.4}
\]

The derivative w.r.t. \( P_a \) has the same sign as the LHS of Equation (1.4.5), which is continuous in \( P \). It is non-negative for \( P_a \leq \max\{\frac{\beta}{\beta-1} \theta, P_0\} \), where \( P_0 \) solves \( W(P_0; \theta) = X + Y \) is the minimum cash flow at which there is non-trivial participation. Then for either Uniform or Generalized Pareto distribution, it becomes positive and changes sign only once as \( P_a \) increases. By the same argument as in the proof of Proposition 1.1, there is an optimal threshold strategy for holding the auction. FOC in Equation (1.4.5) gives the solution as \( (1.9.4) \) is quasi-concave. Notice optimal \( P_a \geq \frac{\beta}{\beta-1} \theta \), lest \( P_a \) increases further, making delaying profitable. Thus the seller never holds the auction when she expects no bidder to invest right away. \( \blacksquare \)

Proof of Proposition 1.9

Proof. It can be shown that for an efficient formal auction, there is also an optimal threshold strategy. I show that at the socially efficient timing, the LHS of (1.4.5) is positive for the seller, which implies the threshold for timing an optimal auction is higher.

Let \( \hat{\theta} \geq \hat{\theta}_z \) denote cutoff type of participation with efficient security designs (such as cash). If \( \hat{\theta}_z = \overline{\theta} \), the integrand is weakly bigger with optimal security. It is strictly increasing for a positive measure of \( \theta \) as we increase \( z(\theta) \) because otherwise, the LHS of (1.4.5) is positive at \( z(\theta) = \overline{\theta} \), contradicting the fact that it is zero at the efficient timing. Hence, (1.4.5) is positive with optimal security design at the efficient timing.

If \( \hat{\theta}_z < \overline{\theta} \), consider the following Lemma:
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Lemma 1.5. Suppose \( A(\theta) \) and \( \hat{A}(\theta) \) are positive with their ratio \( g(\theta) \) decreasing in \( \theta \), and \( B(\theta) \) is weakly increasing in \( \theta \) and strictly increasing over some interval in \([\theta_1, \theta_2]\). Then \( \int_{\theta_1}^{\theta_2} A(\theta) B(\theta) d\theta = 0 \) implies \( \int_{\theta_1}^{\theta_2} \hat{A}(\theta) B(\theta) d\theta > 0 \).

To prove this, rescale \( A \) and \( \hat{A} \) so that they are first-order stochastic dominance over \( A \), the result then follows given \( B \) is weakly increasing in \( \theta \). Q.E.D.

Now at the socially efficient timing \( P_a \),

\[
0 = \int_{\theta}^{\hat{\theta}} \frac{\beta(X + Y + \theta') - (\beta - 1)P^*(\theta')}{[N f(\theta') [1 - F(\theta')]^{N-1}]^{-1}} d\theta' = \int_{\theta}^{\hat{\theta}} \frac{\beta(X + Y + \theta') - (\beta - 1)P^*(\theta')}{[N f(\theta') [1 - F(\theta')]^{N-1}]^{-1}} \frac{d\theta'}{d\theta} d\theta
\]

where we have used change of variable \( \theta = z^{-1}(\theta') \). Let \( A(\theta) = N f(\theta') [1 - F(\theta')]^{N-1} \frac{d\theta'}{d\theta} \) and \( \hat{A}(\theta) = N f(\theta) [1 - F(\theta)]^{N-1} \), and \( B(\theta) = \beta(X + Y + z(\theta)) - (\beta - 1)P^*(z(\theta)) \).

Generalized pareto and uniform distributions have \( g(\theta) = \frac{dG(\theta)}{d\theta} f(z(\theta)) [1 - F(z(\theta))]^{N-1} \) being decreasing in \( \theta \). For example, exponential distribution with parameter \( \lambda_o \) has \( g'(\theta) = -\lambda_o^3 N e^{\lambda_o \theta} - \lambda_o N e^{\lambda_o \theta} - 1 < 0 \), and uniform distribution has \( g'(\theta) = -2\frac{N-1}{\theta-\theta} [1 - F(\theta)]^{N-2} < 0 \). Thus lemma [1.5] gives,

\[
\int_{\theta}^{\hat{\theta}} N f(\theta) [1 - F(\theta)]^{N-1} [\beta(X + Y + z(\theta)) - (\beta - 1)P^*(z(\theta))]d\theta > 0. \quad (1.9.5)
\]

This implies \( B(z^{-1}(\hat{\theta})) > 0 \) as \( B(\theta) \) is increasing. Together with the fact \( \hat{\theta}_x \geq z^{-1}(\hat{\theta}) \), \( (1.4.5) \) at \( P_a \) has the same sign as \( \int_{\theta}^{\hat{\theta}} N f(\theta) [1 - F(\theta)]^{N-1} [\beta(X + Y + z(\theta)) - (\beta - 1)P^*(z(\theta))]d\theta > 0 \).

\[\blacksquare\]

Proof of Lemma [1.3]

Proof. I first show that \( R(\Pi^i) = C^i + R_{\theta_i}(S^i) \). This is obviously true if only one type uses \( \Pi^i \). If more than one type use this bid, either it holds or one of the types \( \theta_1 \) has \( R_{\theta_1}(S^i) + C^i \neq R(\Pi^i) \). Then \( \exists \theta_2 \) (potentially \( \theta_1 \)) s.t. \( R(\Pi^i) < C^i + R_{\theta_2}(S^i) \). Consider the deviation for bidder 2 to a cash bid equal to \( R(\Pi^i) \) and invest efficiently. This deviation
is profitable because he creates weakly greater social surplus, pays less, and has the same marginal probability of winning. Thus by contradiction \( R(\Pi^i) = C^i + R_{\theta_i}(S^i) \) always.

Now suppose \( \tau_{\theta_i}^i \neq \tau_{\theta_i}^* \), consider deviating to a cash bid \( C = R(\Pi^i) \). The payoff from deviation \( \mathbb{E}[e^{-r_{\theta_i}^i (P_{\theta_i}^i - \theta_i)}] - R(\Pi^i) \) dominates the original payoff \( \mathbb{E}[e^{-r_{\theta_i}^i (P_{\theta_i}^i - \theta_i)}] - R_{\theta_i}(S^i) - C^i \). Thus the deviation is profitable and the claim follows.

**Proof of Lemma 1.4**

*Proof.* Suppose a non-singleton set \( \Theta_p \) of types pool to bid \( \Pi \). The claim follows if there is always a profitable deviation by a type in this set.

From Lemma 1.3 a type \( \theta \) in expectation pays \( C + D(P_a; P^*(\theta))S(P^*(\theta)) \). Let \( \theta_k = \arg\max_{\theta \in \Theta_p} R_{\theta}(S) \) where \( R_{\theta}(S) = D(P_a; P^*(\theta'))S(P^*(\theta')) \). Then \( R(\Pi) \leq C + D(P_a; P^*(\theta_k))S(P^*(\theta_k)) \). If the inequality is strict, type \( k \) can profitably deviate to cash bid \( R(\Pi) \). Otherwise, \( R_{\theta_i}(S) = R_{\theta_j}(S) = R(\Pi) - C \), for some \( \theta_i < \theta_j \) both in \( \Theta_p \), but there is still a profitable deviation:

We first argue that \( \Theta_p \) contains a positive measure of types. For any \( \theta_n \in (\theta_i, \theta_j) \cap \Theta_p^c \), call his bid \( \bar{\Pi} \). Let \( Q \) and \( \bar{Q} \) be the probability of winning when bidding \( \Pi \) and \( \bar{\Pi} \). Since \( \theta_i \) does not want to deviate to cash bid \( R(\bar{\Pi}) \), \( Q[W(P_a; \theta_i) - R(\Pi) - X] \geq \bar{Q}[W(P_a; \theta_i) - R(\bar{\Pi}) - X] \). Similarly, \( Q[W(P_a; \theta_j) - R(\Pi) - X] \geq \bar{Q}[W(P_a; \theta_j) - R(\bar{\Pi}) - X] \). As \( \theta_i \neq \theta_j \), the equality signs cannot hold simultaneously. Thus for \( \theta_n \in (\theta_i, \theta_j) \), \( Q[W(P_a; \theta_n) - R(\Pi) - X] > \bar{Q}[W(P_a; \theta_n) - R(\bar{\Pi}) - X] \). This means \( \theta_n \) can profitably deviate to cash bid \( R(\Pi) \). Therefore, it has to be that \( [\theta_i, \theta_j] \in \Theta_p \).

Next, note \( W(P_a; \theta_i) - X - R_{\theta_i}(S) - C > W(P_a; \theta_j) - X - R_{\theta_j}(S) - C \geq 0 \). Type \( \theta_i \) can deviate profitably to cash bid \( \epsilon + R(\Pi) \) which reduces his payoff by \( \epsilon \) upon winning but increases his marginal chance of winning by a discrete amount (because he separates from a positive measure of types).

**Proof of Proposition 1.10**

*Proof.* Consider the bidding strategy from a PFA in cash. The valuations for the bids are simply the cash amounts. I show there exists a belief that supports an equilibrium with this bidding strategy in the informal auction. First, there would not be any deviation to another cash amount since the bidding strategy comes from the equilibrium in FPA cash auction.

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Next, for beliefs such that upon seeing an out-of-equilibrium bid $\Pi^i$, the auctioneer believes it comes from $\tilde{\theta}_i = \arg \min_{\theta \in [0, \bar{\theta}]} \mathbb{E}[\frac{1}{2} R_\theta(S^i) + C^i]$ and gives it a valuation $\tilde{R}$. If bidder $i$ finds this deviation attractive (yielding an expected payoff more than the original cash amount he is paying), then he also finds deviating to cash bid $\tilde{R}$ weakly more attractive, contradicting the fact that no deviation to another cash amount is profitable. Thus the equilibrium from a first-price cash auction is an equilibrium in the informal auction. The argument also applies to cash-like bid $\Pi$ such that $R_\Pi$ is independent of the seller’s belief on the bidders’ types.

Next I show any equilibrium in the informal auction has the same expected payoffs as cash auctions. The seller forms correct beliefs about types since Lemma 1.4 rules out pooling. Bidder $i$’s bid $S^i$ can be replaced by an equivalent cash bid. Note $\tau_i = \tau^*_i$ from Lemma 1.3. This would not change the marginal probability of winning, neither does it change the payoff upon winning. Since the bidders face the same maximization problem as in a FPA with cash, almost every bid is cash-like in terms of its expected payoff.

Proof of Proposition 1.11

Proof. The derivative of the seller’s payoff has the same sign as

$$\int_0^{\theta_c} f(\theta) F(\theta) [1 - F(\theta)]^{N-2} [\beta(X + Y + \theta) - (\beta - 1) P(\theta)]d\theta, \quad (1.9.6)$$

where $\theta_c$ is the cutoff type in cash auction with reserve price $Y$. The derivative changes sign only once, implying a unique optimal threshold strategy for timing the auction. At the efficient timing, apply the lemma in the proof of Proposition 1.9 and note $g(\theta) = \frac{1-F(\theta)}{(N-1)F(\theta)}$ is decreasing in $\theta$, we get that the sign of the derivative is positive. Given the uniqueness of the optimal timing, the threshold for having the auction is higher - the seller waits inefficiently longer in timing a cash auction.

Proof of Proposition 1.12

Proof. Conjecture that in equilibrium bidder $\theta$ initiates the auction with threshold $P_1(\theta)$, which is potentially non-monotone, and the seller initiates with a threshold $P_S$. Let the probability of auction initiation before $P$ is reached be $\tilde{F}(P)$ in equilibrium. Consider
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SPAs first. The expected payoff to the bidder \( \theta \) following initiation threshold \( P_a \leq P_S \) is

\[
\int_{P_0}^{P_s} d\tilde{F}(P') \int_{\Theta_H(P')} d\theta' \left( \frac{P_0}{P'} \right) \beta \left[ [W(P'; \theta) - X - Y]^+ - [W(P'; \theta') - X - Y]^+ \right]^+ \\
+ \int_{P_s}^{\infty} d\tilde{F}(P') \int_{\Theta_H(P_a)} d\theta' \left( \frac{P_0}{P_a} \right) \beta \left[ [W(P_a; \theta) - X - Y]^+ - [W(P_a; \theta') - X - Y]^+ \right]^+ 
\]

where \( \Theta_H(P) \) is the set of types that initiate with thresholds higher than \( P \); payoff when \( P > P_S \) is

\[
\int_{P_0}^{P_s} d\tilde{F}(P') \int_{\Theta_H(P')} d\theta' \left( \frac{P_0}{P'} \right) \beta \left[ [W(P'; \theta) - X - Y]^+ - [W(P'; \theta') - X - Y]^+ \right]^+ \\
+ \int_{P_s}^{\infty} d\tilde{F}(P') \int_{\Theta_H(P_s)} d\theta' \left( \frac{P_0}{P_s} \right) \beta \left[ [W(P_s; \theta) - X - Y]^+ - [W(P_s; \theta') - X - Y]^+ \right]^+. 
\]

Denote the solution to \( W(P_a, \theta) = X + Y \) by \( \hat{P}(\theta) \). Note when \( P_a \leq P_s \), \( (P_0/P_a) \beta \left( [W(P_a; \theta) - X - Y]^+ - [W(P_a; \theta') - X - Y]^+ \right) \), if positive, is decreasing when \( P_a > P^s(X + Y + \theta) \), increasing at \( \hat{P} \), and constant for \( P_a < \hat{P} \). Differentiating (1.9.7) and applying Leibniz’s formula gives that in equilibrium \( \hat{P} \leq P_s(\theta) \leq P^s(\theta + X + Y) \). Now for the seller, if she uses threshold \( P_a \), the expected payoff is,

\[
\int_{P_0}^{P_a} d\tilde{F}(P') \int_{\Theta_H(P')} d\theta' \left( \frac{P_0}{P'} \right) \beta \left[ W(P'; \theta') - X - Y \right]^+ \frac{f(\theta')F(\theta')}{[1 - F(\theta')]^{2-N}} N(N - 1) \\
+ \int_{P_a}^{\infty} d\tilde{F}(P') \int_{\Theta_H(P_a)} d\theta' \left( \frac{P_0}{P_a} \right) \beta \left[ W(P_a; \theta') - X - Y \right]^+ \frac{f(\theta')F(\theta')}{[1 - F(\theta')]^{2-N}} N(N - 1). 
\]

(1.9.8)

Suppose \( \Theta_H(P_a) \) contains positive measure of types. For any \( \theta' \in \Theta_H(P_a) \), the earlier argument leads to \( P_a < P^s(\theta' + X + Y) \), for otherwise \( \theta' \) would initiate earlier than \( P_a \) - a contradiction. The derivative of (1.9.8) is thus positive for any \( P_S \) unless \( \Theta_H(P_S) \) is measure-zero. Thus almost surely the seller never initiates.
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Now the bidder’s problem is reduced to expression \((1.9.7)\). The derivative at \(P_l(\theta)\) is

\[
1 - \tilde{F}(P_l) \int_{\Theta_\mu(P_l)} d\theta \frac{d}{dP} \left[ \frac{[W(P; \theta) - X - Y]^+}{P^\beta} - \frac{[W(P; \theta') - X - Y]^+}{P^\beta} \right]_{P=P_l},
\]

which is positive at \(\hat{P}(\theta)\) and non-positive at \(P^*(\theta + X + Y)\). The integrand is weakly monotone in \(P_l\) path-by-path, thus \((1.9.9)\) changes sign at a unique \(P_l\).

Given \((1.9.7)\) is concave in \(P_a\) with non-negative cross-partial in \(P_a\) and \(\theta\), and there exists unique maximizer \(P_l(\theta)\), Implicit Function Theorem gives that \(P_l(\theta)\) is non-decreasing. This ensures \((1.9.9)\) is continuous, establishing the optimality of \(P_l\) and the FOC in the proposition for SPAs. There could be multiple equilibria with different initiation thresholds below \(P_0\), but in terms of initiation outcome and payoffs, they are all equivalent, making the proposed equilibrium essentially unique.

Now consider the FPA equilibria with increasing initiation thresholds. Having seen no initiation up to \(P\) allows the bidders to truncate the support of the distribution of types to \([P_l^{-1}(P), \tilde{\theta}]\). Fortunately, the bidding strategy of type \(\theta\) only depends on his belief of types that are worse than her, i.e., types distributed in \((\theta, \tilde{\theta})\), thus the payoff in symmetric bidding strategy is equivalent to SPA in terms of revenue and allocation conditional on the auction time \(P_a\). The analysis carries through and the same initiation equilibrium results.

Given that a bidder’s threshold for holding the auction is lower than his threshold if he were maximizing social welfare, the initiation is accelerated in the ex post sense. Moreover, he would invest in the project right away, making the exercise of the real option faster than in seller-initiated auctions where the realized winning type might still wait after the auction.

**Proof of Proposition 1.13**

**Proof.** For simplicity, consider pure contingent securities. Extension to include cash is straightforward. Conditional on an auction timing, cash auctions lead to efficient investments and obviously dominate in terms of welfare. For the seller’s revenue, first consider
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SPAs. The revenue is

$$\mathbb{E}[e^{-r\hat{\tau}}S(s(\theta(2)), P_{\hat{\tau}})1_{\{	heta(2) \leq \theta\}}] = \mathbb{E}[e^{-r\hat{\tau}}(P_{\hat{\tau}} - \theta(1)) - U(\hat{\tau}, s(\theta(2)), \theta(1))] 1_{\{	heta(2) \leq \theta\}} \leq \mathbb{E}[e^{-r\hat{\tau}}(P_{\hat{\tau}} - \theta(1)) - X) 1_{\{	heta(1) \leq \theta\}} \equiv R_0,$$

where $\hat{\tau} = \arg\max_\tau U(\tau, s(\theta(2)), \theta(1))$ and $U(\tau, s, \theta) = \mathbb{E}[e^{-r\tau}(P_{\tau} - S(s, P_{\tau}) - \theta)]$. Similarly in FPAs, the revenue is bounded above by $R_0$ with $\hat{\tau} = \arg\max_\tau U(\tau, s(\theta(1)), \theta(1))$. Let $s_w$ denote the index the winning bidder pays in general. Then in FPAs and SPAs, the revenue is bounded above by $R_0$ with $\hat{\tau} = \arg\max_\tau U(\tau, s_w, \theta(1))$

The revenue from cash auction would be the expected second highest valuation $R_2 \equiv \mathbb{E}[(W(P_a; \theta(2)) - X) 1_{\{	heta(2) \leq \theta\}}]$. When $N \to \infty$, $\theta(2) - \theta(1) \overset{a.s.}{\to} 0$. Thus $W(P_a; \theta(2)) - W(P_a; \theta(1)) \overset{a.s.}{\to} 0$. Now $1_{\{	heta(2) \leq \theta\}}$ and the above are bounded, by bounded convergence, $R_2$ converges a.s. to $R_1 \equiv \mathbb{E}[(W(P_a; \theta(1)) - X) 1_{\{	heta(1) \leq \theta\}}].$

If $R_1 - R_0$ converges to a quantity bounded below by a positive constant, the claims follow. First note $U(\tau, s_w, \theta(1))$ admits an optimal stopping solution involving threshold strategies. To see this, write $U(\tau, s_w, \theta(1)) = D(P_a; P)[P - \theta(1) - \sum_{i \in I} a_i(s_w)[P - b_i(s_w)]^+]$, which admits a maximizer $\tilde{P}(\theta(1))$. Then use that as an investment trigger and apply the standard verification argument. Next, as $\theta(1) - \theta \overset{a.s.}{\to} 0$, the investment trigger in cash auctions converges to $P^* = \frac{\beta}{\beta - 1} \theta$, and $\tilde{P}(\theta(1))$ to $\tilde{P}^* = \tilde{P}(\theta)$. Whether $\tilde{P}^* \in [b_m, b_{m+1}]$ or not, $|\tilde{P}^* - P^*| \geq M$. Since $P^*$ is the optimal trigger for $\mathbb{E}[e^{-r\tau}(P_{\tau} - \theta)]$, $R_1 - R_0 \overset{a.s.}{\to} \epsilon$ for some $\epsilon > f(M)$, where $f(M)$ is a function of $M$ that is positive and independent of $N$. Therefore as $N$ becomes big, $R_2$ converges to $R_1$ which dominates $R_0$ in the limit. Thus cash auctions yield higher revenue than the security-bid auctions. ■

B Technical Requirements for “Well-behaved” Distributions

To make the design problem “regular” and avoid discussing “ironing” techniques, I require $z(\theta) = \theta + F(\theta)/f(\theta)$ to be increasing in $\theta$. A standard assumption in the economics literature is that the “Inverse Hazard Rate” being increasing. In our case, it is $F(\theta)/f(\theta)$ being increasing.

To have single threshold strategies for optimal auction timing, I require $z(\theta)$ to be differentiable and invertible. Moreover, for $\theta$s that solve $W(P; \theta) = X + Y$ and $W(P; z(\theta)) =$
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\[ X + Y, \text{ where } W(P; \theta) \text{ is given in (2.2.3), the marginal increase in participation expected from waiting for higher cash flow level } f(\theta)d\theta/dP \text{ is singly-peaked as a function of } P. \]

Finally, I require \( \frac{dz(\theta) f(z(\theta))}{f(\theta)} \left[ \frac{1-F(z(\theta))}{1-P(\theta)} \right]^{N-1} \) being decreasing in \( \theta \).

Note these requirements are satisfied by continuous distributions commonly used in the economics and finance literature, such as Uniform Distribution, Exponential Distribution, and Pareto Distribution.

C Institutional Details and Empirical Evidence

Oil and Gas Auction and Drilling in the Gulf of Mexico

Offshore drilling activities in the gulf of Mexico date back to the 1940s. The US Congress passed the Outer Continental Shelf Lands Act (OCSLA) in 1953 to grant the Department of the Interior the authority for conducting lease auctions, collecting royalties, and overseeing all activities associated with the drilling in federal waters. The Minerals Management Service (MMS) traditionally conducted the lease auctions, but due to a reorganization in response to the DeepWater Horizon oil spill in 2010, it was replaced by the Bureau of Ocean Energy Management (BOEM) and the Bureau of Safety and Environmental Enforcement (BSEE). Most leases were sold in “Bonus-bid” auctions, where the royalty rate on future revenue is fixed and the bidders bid upfront cash. The current royalty rate is standardized at 18.75\%, but has historically taken on different values at various times and for different leases.

The empirical tests in this paper utilize the introduction of Area Wide Leasing (AWL) in May 1983 marks an important break in the lease auction and drilling environment for offshore tracts. Prior to AWL, potential bidders could nominate most nearshore tracts (less than 200 meters of water depth) and certain deepwater tracts (exceeding 200 meters of water depth) to be auctioned. Following comments by other interested parties, such as fishery and environmental interests, BOEM carried out lease sales which were typically on the order of a few hundred tracts. AWL eliminated the nomination process and made most of the offshore blocks in a region available in each sale, including thousands of tracts in deep water areas. Moreover, some lease tenures were increased from 5 or 8 years to 10 years, and the royalty rates on tracts with water depth of more than 400 meters were lowered from 1/6 to 1/8. Opaluch, Grigalunas, Anderson, Trandafir, and Jin (2010) give more details.
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Data Construction

Data on the lease auctions, drilling, and mineral production are from the Minerals Management Service of the Department of Interior. I observe detailed lease-level variables, bolehole-level variables, lease sale data, ownership data, and production data. For the cost of drilling and equipping a borehole that varies by year, region, well type, and well depth, I use John Beshears’s inflation-adjusted estimates based on annual surveys by the American Petroleum Institute (API) and GDP implicit price deflator index from the Bureau of Economic Analysis. The detailed description of various variables are in Beshears (2011). The estimation related to AWL uses leases auctioned in 1978-1989 with a total of 5399 leases. The estimation related to DWRRA uses leases auctioned in 1991-2000 with a total of 7858 leases.

Monthly prices for oil and natural gas are obtained from the World Bank Commodity Price Data (Index Mundi Data Set and the Energy Information Administration (EIA) also contain futures prices, but are not monthly). The prices are inflation-adjusted using monthly CPI data from the Federal Reserve Bank in St. Louis. The discussion in this paper uses average spot prices and the results are robust to using different categories of crude oil and natural gas (such as West Texas Intermediate, Brent Crude Oil, etc.).

The key variables in the empirical exercises are listed below:

**Event**: The event is first exploratory drill. For this, I take the spud date for the first exploratory bolehole drilled in each lease tract.

**AWL**: Dummy for the implementation of AWL policy. It takes the value 0 for leases auctioned before May 1983 and 1 afterwards.

**DEEP**: Dummy for leases with water depth greater than 200 meters in areas west of the 87 degrees 30 minutes West longitudinal line.

**RELIEF**: Dummy for the implementation of DWRRA. It takes the value 0 for leases auctioned before August 1995 and 1 afterwards.

**DEEP*RELIEF**: Interaction term for the DEEP and RELIEF dummies. The coefficient for this variable is of interests in the diff-in-diff test.

**RTY**: The royalty rate specified in the lease agreement. It is typically 1/6 or 1/8 in this data set.

**DUR**: The number of days from lease auction to expiration. Most lease terms are 5-10

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55 http://research.stlouisfed.org/fred2/graph?sid=CPIAUCSL
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years.

**DEPTH**: The water depth of the leased tract. The results reported use minimum water depth, and are robust to alternative specifications using maximum water depth or average water depth.

**SIZE**: The area covered under each lease. Most leases had an area of approximately 5,000 acres, though some were smaller.

**MKT**: The number of leases sold in the same sale as a proxy for the market demand for oil and gas leases. When the market demand is high, the quality of the marginal lease sold may be low in the sense that the reserve quantity is small or there is huge uncertainty, which in turn affect drilling decisions.

**P(t)**: Average spot price for oil and gas. The results are robust to the inclusion of oil and gas prices separately, or prices lagged by 1-5 months.

**VOL(t)**: Average trailing-12-month volatilities for oil and gas spot prices. The results are robust to the inclusion of oil and gas price volatilities separately and to variations within one year of the trailing window.

**COST(t)**: The industry average drilling cost for dry, oil, and gas boreholes. The drilling cost is the initial cost plus the equipping cost.

**Firm f.e.**: I control for firm-specific characteristics by including the firm ID as a factor. Over the years, it tends to be the same group of firms that bid for leases. For jointly-owned leases, I use the largest shareholder. The results are robust to using the second-largest shareholder.

**Info.Ext.**: I control for information externality by using subsamples with lease areas exceeding 4,000 acres and 5,000 acres (full control) respectively. Prior studies (Lin (2009, 2012)) have shown that information externality is not significant for exploratory drills, and decreases with the size of the tract. Even for development drills, information externality is negligible for tracts larger than 5,000 acres.

**Empirical Models and Tests**

I employ a Cox proportional hazard model to test the following hypothesis.

**Null Hypothesis**: Auction initiation has no impact on how quickly tracts are explored.

Cox hazard models probably represent the state of the art in survival analysis with reduced-form models. They make the assumption that the hazard rate $\kappa(t)$ of exploratory drill
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at time $t$ conditional on lack of drill until time $t$ is $\kappa(t) = \psi(t) \exp(\bar{X}(t)^T \bar{\gamma})$, where

$\psi(t)$ is the baseline hazard rate that is completely unrestricted, and $X(t)$ is a vector of independent variables listed earlier (AWL is only for the first hypothesis where as DEEP, RELIEF, DEEP*RELIEF are only for the second). This specification handles censoring of observations and allows time-varying covariates. There is no survivorship bias or response bias because the leases are sampled at birth (the auction), and all leases are recorded by the Department of Interior.

The Hypothesis is rejected and the results are reported in Section 5 of the paper and in Table 1.2 and Figure 1.5.
Chapter 2

Costly Learning and Agency Conflicts in Investments under Uncertainty

2.1 Introduction

An important aspect of corporate investment is the strategic timing of market, as manifested in managers’ flexibility to postpone irreversible expenses and learn about market conditions. Models of real options typically concern the option to wait. In reality, the option of active learning is equally important. This paper studies endogenous learning in a real options model of investment and the associated agency conflicts, which have profound implications on market timing, managerial contracts, relationship between uncertainty and investment, and empirical tests of real options models.

Costly learning is real, for example, in the exploration of the Pebble deposit, one of the largest copper-gold porphyry systems and stores of mineral wealth ever discovered. Although as early as 1992 Cominco Alaska Exploration has estimated that the ore contains 3 million tonnes of copper metal and 11 million ounces of gold, little further work to ascertain the amount of deposit was done for nearly a decade. It is only till 2006-2007, Northern

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2 For example, uncertainty over the permanency of past shocks are central in [Gorbenko and Strebulaev (2010) and Grenadier and Malenko (2010)]. Sundaresan (2000) underscores the need for enriching the real options framework with incomplete information. Trigeorgis (2002) also lists “investments that can generate information and learning”, “extending real options in an agency context” and “using real options to explain empirical phenomena” as three pending issues that future research must focus on.
Dynasty Minerals Ltd, the firm that acquired the project back in 2001, formed a partnership and started spending hundreds of millions to explore the deposit further and resolve existing uncertainties and conflicts before full operations. There are three observations: first, the business environment are highly volatile, but uncertainties such as the amount of deposit, the environmental cost of mining, and the social-economic impact on local workers, change little over time; second, they can be resolved to a large extent by explorations and studies, a form of learning that is very costly - in 2008 alone, 140 million dollars was budgeted to complete a prefeasibility study; third, the firm seems to exhibit market timing behavior in explorations and investments based on the fluctuating copper and gold prices (Figure 2.1).

Figure 2.1: Historical Prices of Copper and Gold (Jan 2, 1998 - Dec 28, 2007) from InfoMine.com. The intensive exploration seems to coincide with the run-up in 2006-2007.

In fact, these are general themes in situations with learning options. Evidence of such costly learning is abundant: firms regularly conduct market surveys; IT companies test new softwares within the firm; pharmaceutical companies carry out clinical trials before full scale production; social websites launch beta versions to gauge general interests; venture capitalists provide seed funding to assess the potential of candidate projects.\(^3\)

\(^3\)Gibson and Hamilton (1994), and Wang and Gibson (2010) demonstrate that such pre-project planning is crucial to optimal investment timing and investment choices.
CHAPTER 2. Costly Learning and Agency Conflicts

The presence of learning options has two immediate consequences: while they are useful tools to the managers, learning options are often unobserved by outsiders, and investment timing may appear distorted based on observable investment costs alone. For empirical tests of real options models, or evaluations of managerial decisions in practice, it is important to take into considerations the learning options in conjunction with the investment options. More importantly, learning generates new information, thus often leads to agency conflicts that plague decentralized organizations and public firms where investment options abound. In particular, for firms hiring specialists to acquire information, additional information asymmetry can arise post-contract. This is definitely the case for oil fields where the contracted developer acquires more information post-auction through exploratory drills. While a number of studies have shown how agency conflicts affect capital allocation for investment (e.g., Holmstrom and i Costa (1986), Rappaport (1978), Lambert (1986), and Stein (2003)), the timing of investment by managers has received scant attention, despite being fundamental to firms’ investment behaviors as predicted in Grenadier and Wang (2005) and empirically documented in Levitt and Syverson (2008). This paper characterizes the optimal contracts in the presence of learning options, and show that learning-induced agency conflicts are costly to the option owner and social welfare, and how contracting on learning could ameliorate the situation.

This paper is the first to consider both costly learning and its associated agency conflicts in a real options framework. In particular, I derive optimal strategies for learning and investing, and characterize optimal contracts with adverse selection and moral hazard incorporating costly learning. At the core of the model is the following intuitive tradeoff: there is option value to postpone costly learning because the value of information evolves over time; but learning too late risks missing optimal times for investments in high value projects. This tradeoff determines the optimal strategy uniquely. Subsequently, only high value projects are delayed.

Agency conflicts add additional layers of complexity. Managers may have moral hazard to suboptimally learn, and learning induces post-contracting information asymmetry that distorts investment timing. The optimal contract is really dictated by three underlying forces: the individual rationality for participation (IR-P) to ensure the agent accepts the contract, the incentive compatibility for learning after signing the contract (IC-L) and the incentive compatibility for truth-telling upon learning (IC-T). Learning and investments in high value projects are weakly accelerated whereas investments in low value projects are
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delayed compared to the benchmark case without agency issues. The contractibility on learning action, i.e., whether the agent has learned or not, is crucial. When learning is non-contractible, (IC-L) implies (IC-T) and (IR-P); timings for learning and investments are always distorted. When learning is contractible, either (IR-P) or (IC-T) dominates. If the learning cost is sufficiently high, (IR-P) implies truth telling, and efficiency of learning and investments is restored. Fundamental market conditions have significant impacts on the timing distortions, in particular, the accelerations and delays are more severe in more volatile markets.

Model implications are manifold. In addition to the aforementioned, moral hazard of learning also differs from other moral hazards studied previously in that it can both accelerate and delay investments, and cause investment timing distortion even to the best type. Further implications include investments at stable or decreasing cash flows as exemplified in the wide-spread office constructions in Denver and Houston during the 1970s and 1980s when the demand and housing value were declining. Learning and investing as a joint signal for private information, and adverse consequences of regulating a monopolist information seller on price structure.

This paper is related to learning in a real options framework. While earlier research such as McDonald and Siegel (1986) discusses the “option to wait” in investments under uncertainty, recent contributions focus on various “options to learn”, nevertheless they all involve costless passive learning that occurs either with the passage of time or the actions of other players. But passive learning is futile in situations such as the development of oil fields or copper mines, where the quantities of deposits do not change in the relevant time-scale for investment, but costly exploratory drills could enlighten the developer. In contrast, this paper develops a tractable dynamic model of active learning.

This paper is also related to information asymmetry in investments under uncertainty. Grenadier and Malenko (2011) study real option signaling games, and conclude that projects could either be delayed or accelerated, depending on how the outsiders’ belief about the option value influence the decision-maker’s payoff. Maeland (2002) studies adverse selection in real investments and specializes to exogenous information asymmetry in investment cost.

\[\text{In the terminology of Math and Statistics, these are “downcrossing” investments.}\]

CHAPTER 2. Costly Learning and Agency Conflicts

In contrast, this paper studies costly learning, and the moral hazard of both learning and investment. The analysis of optimal contracts is similar to [Grenadier and Wang (2005)] who also consider both adverse selection and moral hazard. However, this paper differs primarily in the consideration of endogenous information asymmetry, and the dynamics of learning. In particular, when learning is non-contractible, information rent is never sufficient for inducing effort, and both acceleration and delays can occur depending on the realized type of the project. These are novel implications that warrant further empirical tests. Moreover, the agent’s effort affects the payoff only through the information channel - a realistic assumption for many real-life scenarios. For example, an oil firm could ascertain the amount of oil reserves on a particular site, but could not alter that amount. The model also corroborates the empirical observation that information asymmetry frequently arises after managerial compensations are determined because managerial contracts are drafted less frequently than the changes in investment opportunities.

Delegated information acquisition is another related topic. [Lewis and Sappington (1997), Szalay (2009), and Eso and Szentes (2007)] study agent’s acquisition of verifiable information; [Chade and Kovrijnykh (2011)] study the case where the principal decides on information acquisition; [Krähmer and Strausz (2011)] study the optimal contract problem for endogenous learning with moral hazard. This paper differs in that it considers a dynamic setting with post-contract learning, and focuses on the timing implications.

Finally, this paper adds to the theoretical foundation for any empirical tests of real options models. [Paddock, Siegel, and Smith (1988) and Quigg (1995)] empirically test option-based models in the markets for natural resources and real estate. [Henriques and Sadorsky (2011)] and [Mohn and Misund (2009)] discuss the U-shaped relationship between oil price volatility and firm investment. [Kellogg (2010)] is another study on the relationship between uncertainty and investment using implied volatility. This paper cautions that in such empirical tests, learning options and their associated agency costs need to be taken into consideration.

The remainder of the paper is organized as follows. Section 2.2 presents the basic model of costly learning. Section 2.3 introduces agency issues and derives the optimal contract. Section 2.4 generalizes the model. Section 2.5 discusses further implications. Section 2.6 concludes. The appendix contains all proofs.
2.2 Investment with Costly Learning

This section considers a model in which the manager’s interests are completely aligned with the firm. This simplification allows us to understand the fundamental impacts of costly learning and establish a benchmark, before discussing any agency conflicts.

2.2.1 The Investment Problem

A firm owns a project with lump sum revenue $P_t$ upon investment at time $t$. Revenue process $P_t$ is observable and stochastic, evolving according to a geometric Brownian motion (GBM)

$$dP_t = \alpha P_t dt + \sigma P_t dB_t,$$

(2.2.1)

where $B_t$ is a standard Brownian Motion, $\alpha$ is the instantaneous conditional expected percentage change per unit time in $P_t$, and $\sigma$ is the instantaneous conditional standard deviation per unit time.\footnote{Everyone is risk-neutral with discount rate $r$, where $r > \alpha$.\footnote{The irreversible investment cost is $K$ and the time-zero revenue level is $P_0$. If $K$ were known, the investment problem is standard in the real options literature and involves a threshold strategy for optimal stopping.\footnote{See Dixit and Pindyck (1994).} The investment threshold $P^*(K)$ and the option value $W(P_0; K)$ are:}

$$P^*(K) = \max\left\{P_0, \frac{\beta}{\beta - 1} K\right\},$$

(2.2.2)

$$W(P_0; K) = D(P_0; P^*(K)) (P^*(K) - K),$$

(2.2.3)

where

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1,$$

(2.2.4)

and

$$D(P; P') = \left(\frac{P}{P'}\right)^\beta, \quad P \leq P'$$

(2.2.5)
When revenue level is $P$, $D(P; P')$ is the price of an Arrow-Debreu security that pays one dollar the first moment threshold $P'$ is reached. It is essentially an “expected discount factor”, and the option value $W(P_0; K)$ has an intuitive interpretation: the payoff when exercised $P^*(K) - K$, properly discounted to the present.

However, $K$ is uncertain: $K = K_H$ (low value project) with probability $q$, and $K = K_L$ (high value project) with probability $1 - q$, where $K_H > K_L$. Subsequent analysis always assumes $P_0 < \frac{\beta}{\beta - 1}K_L$ to ensure some positive option value inherent in the project.

**Proposition 2.1.** With uncertain investment cost, the investment threshold and option value are $P^*(K_M)$ and $W(P_0; K_M)$ respectively, where $K_M = qK_H + (1 - q)K_L$ is the mean investment cost.

Since $P^*(K_L) < P^*(K_M) < P^*(K_H)$, high value projects (low investment cost) appear accelerated whereas low value projects (high investment cost) appear delayed if one only observes the realized investment cost.

### Costly Learning

There is clearly value of knowing the investment cost accurately, thus the manager may want to expend effort to learn $K$. Learning is instantaneous and incurs a fixed cost $c$.

The value of information on $K$ is defined as the difference between the expected option value of the project when the manager knows $K$, and that when she does not:

$$I(P_t) = qW(P_t; K_H) + (1 - q)W(P_t; K_L) - W(P_t; K_M)$$

(2.2.6)

Since $W(P_t; K_H), W(P_t; K_L)$ and $W(P_t; K_M)$ are all continuously differentiable in $P_t$, so is $I(P_t)$. Knowing $K$ does not affect the form of payoff directly, but simply enables the manager to time investment optimally. It can also be shown that $I(P_t)$ is non-negative, i.e., information has value.

If learning were a one-shot decision, one should learn if $I(P_0) > c$. Investments for

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9 The results are later generalized to general prior distributions of $K$ and to learning about quantities other than investment cost, such as the quantity of production. Here $K$ could the posterior mean of some more fundamental parameter. Learning is valuable as long as it resolves the uncertainty to some extent.

10 The duration for learning in many cases is much shorter than the duration of waiting or investment. I relax this assumption when discussing further implications. $c$ could be pecuniary or non-pecuniary.
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projects of both high and low option values occur at the first-best timing. But learning is dynamic and intuitively, the agent owns a portfolio of learning option and investment option. She decides at every instant either to learn, to invest or to take no action. Learning only occurs if the value of information exceeds the cost; subsequently the agent times investment optimally with the new information. An optimal strategy for learning and investing exists and is characterized by the next proposition:

**Proposition 2.2.** For $c \leq c_1 \equiv (1 - q)[K_H (1-q)(K_H/k_M)^{1-q}]^{1/2} - K_L$, the threshold trigger for learning is $P_e^{bm} = \frac{\beta}{\beta-1}(K_L + \frac{c}{1-q})$, and upon learning,

(a) if $K = K_L$, investment trigger is $P_L^{bm} = P_e^{bm}$,

(b) if $K = K_H$, investment trigger is $P_H^{bm} = \frac{\beta}{\beta-1}K_H$.

For $c > c_1$, investment occurs at threshold trigger $P^*(K_M)$ without learning.

The superscript $bm$ indicates benchmark, as this solution serves as a no-agency benchmark when discussing agency conflicts later. The main trade-off for the manager is delaying the incidence of learning cost versus missing the optimal trigger for investment if the project happens to have low investment cost. Note $P_e^{bm}$ is increasing in $c, q$ and $K_L$, but independent of $K_H$. The more costly it is to learn, the later the learning takes place, if at all. The higher $q$ is, the less it matters to miss the lower optimal trigger. The higher $K_L$, the higher the cash flow level it has to reach before missing the lower optimal trigger. While here $P_e^{bm}$ depends linearly on $c$, in general its second derivative with respect to $c$ is non-negative: higher cost delays learning with weakly diminishing marginal effect until learning is eventually forgone.

The presence of learning introduces three interesting features. First, the optimal strategy involves threshold triggers for both learning and investing, as described in the proposition. Second, since $P_e^{bm}(K_h) = P^*(K_H)$ and $P_e^{bm}(K_L) \in (P^*(K_L), P^*(K_M))$, learning mitigates the apparent delays in high value projects and obliterates the apparent accelerations in low value projects. Third, unlike the value of the “option to wait” in traditional real options models, the option value to learn could be decreasing in market volatility, as the next proposition subsumes.

**Proposition 2.3.** When $c \leq c_1$ and $\frac{\beta K_H}{\beta-1} e^{-\frac{q}{\beta-1}} < P_0 < \frac{\beta K_L}{\beta-1} e^{-\frac{1}{\beta-1}}$, the option value of learning is decreasing in the fundamental uncertainty $\sigma$ and cash flow drift $\alpha$, and increasing in the discount rate $r$. 

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All of $W(P_0; K_H)$, $W(P_0; K_L)$ and $W(P_0; K_M)$ are increasing in $\sigma$, thus in general it is unclear how $I(P_0)$ changes with $\sigma$. But when $P_0 > \frac{\beta K_H}{\beta - 1} e^{-\frac{\beta - 1}{\beta - 1}}$ gives sufficient condition for $I(P_0)$ to be decreasing in $\sigma$. The market is so volatile that the impact of future uncertainties significantly outweighs that of uncertainties over $K$. The condition $P_0 < \frac{K_L}{\beta - 1} e^{-\frac{1}{\beta - 1}}$ ensures that the option value to learn is more sensitive to $\sigma$ than $I(P_0)$. Thus greater uncertainty could reduce the option value of learning when investing at suboptimal times matters less in the face of the increased exogenous uncertainty. Similarly, option value of learning could be decreasing in $\alpha$ and increasing in $r$. Therefore, the “option to learn” is fundamentally different from the traditional “option to wait”.

Costly learning also encompasses intermediate stages of project development that generate valuable information for later stages. For example, the costly learning could be the R&D stage of a product, which is necessary for any further development. Because such intermediate stages are often mandatory for developing the project, as opposed to the optional learning discussed earlier, optimal strategies differ. The manager first decides when to learn, bearing in mind the subsequent optimal investment strategy, making it essentially a compound option.

**Proposition 2.4.** An optimal strategy for learning and investing exists for mandatory learning. If $c \leq \bar{c}_2 = (1 - q) \Delta K$, optimal trigger is given by $P_{e}^{bm} = \frac{\beta}{\beta - 1} (K_L + \frac{c}{1 - q})$. Upon learning, investment occurs at $P_{L}^{bm} = P_{e}^{bm}$ if $K = K_L$, or at $P_{H}^{bm} = \frac{\beta}{\beta - 1} K_H$ if $K = K_H$. If $c > \bar{c}_2$, both learning and investment occur at $P_{e}^{bm} = \frac{\beta}{\beta - 1} (K_M + c)$.

Note $\bar{c}_2 > \bar{c}_1$ as $\left( \frac{1 - q}{(K_H/K_M)\pi - 1 - q} \right)^{\frac{1}{\pi - 1}} < 1$, which makes sense because if a learning time is optimal when learning is optional, it must be so when learning is mandatory too. The comparative statics for $c \leq c_2$ are thus the same as in optional learning.

When cost is high ($c > c_2$) so that the time value of learning cost $c$ outweighs the information value, the owner delays the learning beyond the true optimal triggers for both $K = K_L$ and $K = K_H$. The trade-off is the time value of the additional cost $c$ versus investing suboptimally for both project types. Since $c > c_2$ implies that endogenous resolution of uncertainty does not affect investment strategy, subsequent sections assume $c \leq c_2$. 69
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Impact of Learning on Investment Timing

Without learning, low value projects are accelerated by an expected advance of
\[
\ln K_H - \ln K_M \alpha - \sigma^2 / 2
\]
relative to the case with learning, and high value projects are delayed by an expected lag of
\[
\ln \left( \frac{P^*(K_M) / P_{m0}}{\alpha - \sigma^2 / 2} \right)
\]
This is illustrated in Figure 2.2. These advances and lags are decreasing in market outlook \( \alpha \), increasing in market volatility \( \sigma \), and independent of discount rate \( r \).

Figure 2.2: Investment Thresholds and Times. Plotted with \( \alpha = 0.01, \sigma = 0.2, r = 0.06, K_H = 40, K_L = 20, q = 0.5, c = 1, \) and \( P_0 = 18. \)

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\(^{11}\)To see this, the first-hitting time for Brownian Motion is an inverse Guassian or Wald distribution. (See Karlin and Taylor (1975) pg.363, or Chhikara and Folks (1989).) Adapted to the GBM case, the density of the stopping time \( \tau \) starting from \( P_1 \) to first hit \( P_2 > P_1 \) is given by
\[
f_{GBM}(\tau; P_1, P_2) = \frac{\ln(P_2/P_1)}{\sqrt{2\pi\sigma^2\tau}} \exp \left[-\frac{[\ln(P_2/P_1) - (\alpha - \sigma^2/2)\tau]^2}{2\sigma^2\tau}\right],
\]
with the mean and shape parameter given by \( m = \frac{\ln(P_2/P_1)}{\alpha - \sigma^2/2} \) and \( y = [\ln(P_2/P_1)/\sigma]^2 \), where I assume \( \alpha - \sigma^2/2 > 0 \) for the mean to exist. When \( \alpha < \sigma^2/2 \), other statistics such as the median lag yield same comparative statics under a wide range of parameters based on numerical simulations. Note \( y \rightarrow \infty \), the distribution approaches a Gaussian with the same mean and variance.

\(^{12}\)Use \( \frac{\partial}{\partial \alpha} = \frac{2\beta(\beta-1)\sigma}{\sigma^2(2\beta-1)+2\alpha} < 0, \frac{\partial}{\partial\sigma} = -\frac{2\beta}{\sigma^2(2\beta-1)+2\alpha} < 0, \) and \( \frac{\partial}{\partial r} = \frac{2}{\sigma^2(2\beta-1)+2\alpha} > 0 \) from differentiating (2.2.4).
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Apparent delays due to past uncertainty and learning

Oftentimes one only observes the realized investment decisions and costs, and a number of prior studies have argued how competition or signaling incentives could lead to investments delayed beyond the first-best timing\(^\text{13}\) Cost learning also lead to such apparent delays and must be accounted for in any empirical analysis.

The key difference between the endogenous dynamic learning and a one-shot learning decision is that when learning occurs, high value projects appear to be delayed beyond the first-best timing in the former situation. This is so because time is irreversible, and once an investment trigger for the project is past, investment is irreversibly delayed. The delay is increasing in \(\sigma\) and \(c\), decreasing in \(\alpha\) and \((1 - q)K_L\), and independent of \(r\). Thus higher learning cost, bearish market outlook, and greater market volatility all exacerbate such apparent delays.

Even when learning option is not available, the uncertainty on cost causes an expected lag of \(\frac{\ln(K_M K_H^{-1} K_L^n)}{\alpha - \sigma^2/2} > 0\) by the inequality of weighted arithmetic and geometric mean. Based on ex post observations, expected investments lags are particularly severe when the market outlook is bearish and volatile. Yet investment timings are actually optimal once costly learning is taken into consideration.

Distinguishing Past and Future Uncertainties

One key prediction from the real options literature is that firms delay investment if uncertainty increases, and several empirical studies have been directed at verifying it\(^\text{14}\) In macroeconomics, studies linking the changes in economy-wide uncertainty to the level of aggregate investment are also numerous\(^\text{15}\) However, most papers do not emphasize the distinction of past and future uncertainties. But as the current models shows, increasing past uncertainties can lead to opposite effect to increasing future uncertainties.

To see this, note past uncertainty here is over \(K\) whereas future uncertainty is in the market volatility \(\sigma\). Suppose \(q\) decreases from 0.8 to 0.6 while \(K_H\) and \(K_L\) remain constant. The past uncertainty \((K_H - K_L)^2(1 - q)q\) is greater, but the learning threshold \(\beta \frac{p}{p + e}\(K_L + \frac{e}{1-q})\), and the expected waiting time to investment which is proportional to

\(^{13}\)See, for example, Chamley and Gale (1994) and Grenadier and Malenko (2011).
\(^{14}\)For example, see Moel and Tufano (2002) and Kellogg (2010).
\(^{15}\)Bernanke (1983), Bloom (2009) and Bloom, Bond, and Van Reenen (2007) are some examples.
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$q \ln K_H + (1 - q) \ln (K_L + \frac{c}{1-q})$ are both smaller, implying accelerated investment with higher uncertainty. While this is contrary to the traditional prediction, it is easy to understand that when learning is very useful, one may want to learn early and thus would be less likely to miss out the lower investment trigger if the investment cost turns out to be low.

2.3 Costly Learning and Agency Conflicts

Costly learning generates information. When the option owner does not have the expertise to exercise the option, she has to delegate it to others. This is particularly true in large public firms and de-centralized organizations. The primary ramification of costly learning is therefore the endogenous emergence of information asymmetry, and the principal has to design proper incentives for desirable investment outcomes - the topic of this section.

Both the principal and the agent are risk-neutral with the same discount rate $r$. The agent is sufficiently liquidity constrained that she could not purchase the investment option from the principal upfront, but can afford the learning cost $c$ that could be non-pecuniary and small enough that learning is desirable absent agency conflicts. $q$ is common knowledge. It is extremely hard for the principal to learn $K$ and invest - this is true, for example, when the principal is a venture capitalist not having the technical expertise to develop an entrepreneurial project. Without learning, there is no agency conflict and the agent earns no rent. The principal does not observe the content of the agent’s learning, but can contract on the investment timing and potentially the threshold for learning.

Subsequent discussions apply the revelation principle, and focus on optimal contracts conditional on learning’s occurring; otherwise the optimal contract simply requires the agent investing at $P^*(K_M)$. Following Grenadier and Wang (2005), I assume that an optimal contract induces the agent to deliver to the principal the payoff of the option corresponding to the exercise price and no value diversion takes place in equilibrium. This is achieved by a nonpecuniary penalty in the event that the agent fails to abide by the terms specified in the contract (Diamond (1984)). I also rule out renegotiation and justify it on the grounds that a reputation of committing to the original contract is needed if the principal receives new potential projects continually, or it is simply mandated by law and regulations.

\footnote{Previous versions of the paper also consider differential discount rates as a way to model managerial impatience. The main results hold with moderate managerial impatience and learning cost, and the analysis is available upon request.}
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Optimal Contracts

The principal designs a contract with payment to the agent contingent on \( P_t \) at the time of investment, such that the agent expends effort to learn and invests in accordance with the principal’s rational expectations, and delivers the exact amount of project payoff to the principal. Learning action is often unobservable to the principal or simply not contracted upon in practice. For example, most firms do not disclose much information on their pre-investment activities, and monitoring is costly. If the principal are the dispersed shareholders of a firm, they may not even have the power to inspect the CEO and free-riding would likely frustrate such an attempt. This section thus focuses on optional learning where the principal does not contract on the learning time\(^7\).

Let \( P_L \) and \( P_H \) be the triggers specified in the contract for the privately observed \( K_L \) and \( K_H \), and let \( w_L \) and \( w_H \) be the corresponding wages. In an optimal contract, wages are paid out only if the investment option is exercised when the revenue level first hits \( P_L \) or \( P_H \). Suppose otherwise, then either the investment option is exercised at a revenue level different from \( P_L \) or \( P_H \), or it is exercised at \( P_L \) or \( P_H \) when the revenue has crossed these levels before. The truth-telling requirement rules out the first scenario. Now, if the agent has learned the value of \( K \) before revenue reaches either \( P_L \) or \( P_H \), then she would exercise the option the first time revenue hits \( P_L \) or \( P_H \). Thus the second case could only happen if the learning occurs after the revenue first hits \( P_L \) or \( P_H \). But the principal could be better off by making the agent learn the first time revenue hits \( P_L \) or \( P_H \), capitalizing on the time value of money. In other words, if the learning is to take place and the investment option is to be exercised at a particular revenue level with a profit, it is better to realize this profit earlier than later. Therefore, the principal specifies in the contract that wages are paid out only if the investment option is exercised when the revenue hits \( P_L \) or \( P_H \) for the first time and rule out all other actions by nonpecuniary punishments.

Given this, the agent finds it optimal to learn at the mininum of \( P_L \) and \( P_H \), if she ever learns. No matter who is paying the cost of learning, learning before revenue level reaches \( P_L \) or \( P_H \) would be dominated because of time value of the learning cost. Now suppose the learning occurs strictly beyond the minimum of \( P_L \) and \( P_H \), then the agent is no better off learning the information because she could only exercise at the larger of the two values to

\(^7\)Mandatory learning usually involve observable initial development of the project, such as R&D research, downpayment agreement for an oil lease, etc, and is therefore contractible most of the time. Contractible learning is the subject of the next section.
avoid nonpecuniary punishment. Seeing this, the principal can assume learning occurs at 
\( \min\{P_L, P_H\} \) when designing the contract. Subsequently, the agent indeed finds it optimal
to learn at \( \min\{P_L, P_H\} \), if she ever opts to learn.

Suppose the principal decides to cover \( 1 - \lambda \) portion of the learning cost \( c \) where \( \lambda \leq 1 \)\(^{18}\).
The Principal gives the agent \( (1 - \lambda)c \) at \( \min\{P_L, P_H\} \) and receives \( P - K - w(P) \) at exercise revenue level \( P \in \{P_L, P_H\} \), where \( w(P) \) is the wage schedule specified in the contract. The agent receives \( w(P) \) at the time of exercise, conditional on her exerting effort
to learn at \( \min\{P_L, P_H\} \) at a personal cost \( \lambda c \).

At \( P = P_0 \) and the terms specified in the contract \( w_H, w_L, P_H, P_L, \lambda \), the value of the contract to the owner is

\[
V^p(P_0; w_H, w_L, P_H, P_L, \lambda) = (1 - q) D(P_0; P_L)(P_L - K_L - w_L) \\
q D(P_0; P_H)(P_H - K_H - w_H) - (1 - \lambda)c D(P_0; \min\{P_L, P_H\})
\]

This is the expected payoff from exercising the option less the wage, discounted back to the present time, minus the portion of learning cost discounted back to the present time. Similarly, the value of the contract to the agent is

\[
V^a(P_0; w_H, w_L, P_H, P_L, \lambda) = q D(P_0; P_H) w_H + (1 - q) D(P_0; P_L) w_L - \lambda c D(P_0; \min\{P_L, P_H\})
\]

The principal maximizes her value subject to the constraints:

\[\text{(IR-P)} \quad q D(P_0; P_H) w_H + (1 - q) D(P_0; P_L) w_L - \lambda c D(P_0; \min\{P_L, P_H\}) \geq 0 \quad (2.3.3)\]

\[\text{(IC-L)} \quad q D(P_0; P_H) w_H + (1 - q) D(P_0; P_L) w_L - \lambda c D(P_0; \min\{P_L, P_H\}) \geq \max\{(K_H - K_M + w_H) D(P_0; P_H) + (1 - \lambda)c D(P_0; \min\{P_L, P_H\}),
(K_L - K_M + w_L) D(P_0; P_L) + (1 - \lambda)c D(P_0; \min\{P_L, P_H\})\} \quad (2.3.4)\]

\[\text{(IC-T)} \quad D(P_0; P_H) w_H \geq D(P_0; P_L) (K_L - K_H + w_L) \quad (2.3.5)\]

\[\text{(IC-T)} \quad D(P_0; P_L) w_L \geq D(P_0; P_H) (K_H - K_L + w_H) \quad (2.3.6)\]

\[\text{(LL)} \quad w_H \geq 0, \quad w_L \geq 0. \quad (2.3.7)\]

\(^{18}\)The section on contractible learning discusses the consequences when \( \lambda > 1 \). This does not occur
when the agent is liquidity-constrained and as mentioned in the proof of Lemma (2.1), with non-contractible
learning, the principal finds it sub-optimal to set \( \lambda > 1 \).
(IR-P) \((2.3.3)\) is individual rationality constraint for participation where the agent’s reservation wage is set to zero for simplicity; (IC-L) \((2.3.4)\) is the incentive compatibility constraint for learning; (IC-T) \((2.3.5)\) and \((2.3.6)\) are the incentive compatibility constraints for truth-telling; finally, (LL) \((2.3.7)\) are the limited liability constraints to prevent the agent from walking away after learning \(K\), assuming the principal loses the investment option when this happens. One subtle point is that when the agent deviates to forego learning, a mixed strategy for investing at \(P_H\) and \(P_L\) is dominated under an optimal contract. The following lemma simplifies the problem significantly:

**Lemma 2.1.** In an optimal contract, \(w_H = 0\), \(w_L = \Delta K\left(\frac{P_L}{P_H}\right)^\beta + \frac{c}{1-q}\) and \(\lambda = 1\). As a result, the principal’s optimization problem can be reduced to,

\[
\max_{P_H, P_L} D(P_0; P_H)(qP_H - K_H + (1 - q)K_L) + (1 - q)D(P_0; P_L)\left(P_L - K_L - \frac{c}{1-q}\right)
\]

subject to \(P_L \leq P_H\left(1 - \frac{c}{q(1-q)\Delta K}\right)^{\frac{1}{\beta}}\). \((2.3.8)\)

Since the principal designs the contract, she does not compensate the agent if \(K\) turns out to be high - the “worst type” earns no rent. The agent will accept the contract for sure because she can always choose not to learn and pretend the cost is high to get non-negative payoff. Thus the principal gains no additional benefit to offer a positive wage for high cost \(K_H\). Moreover, it is optimal for the principal to cover no learning cost directly because covering the learning cost does not bolster the provision of incentives: \(\lambda\) only enters through (IR-P) that is non-binding, thus for any contract that involves the principal covering some learning cost, she can choose not to cover it and still maintain the right incentives. \(w_L\) consists of two components: \(\Delta K\left(\frac{P_L}{P_H}\right)^\beta\) is enough to induce truth-telling after learning; but an additional premium \(\frac{c}{1-q}\) is needed to induce learning effort, effectively making the principal internalize the cost of learning. Some general properties are apparent from this Lemma:

**Proposition 2.5.** Learning and investments in high value projects are weakly accelerated whereas investments in low value projects are weakly delayed relative to the no-agency benchmark.

This is so because if the investment trigger for project with low option value is lower than
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the no-agency benchmark, one could raise it slightly to improve the principal’s payoff without violating the constraint. The argument is similar for learning and investment triggers for high value projects. It is in the principal’s best interest to have such terms because the agent tends to learn early. To see this, if \( K = K_L \), the agent receives a positive wage. Thus she would not want to miss the lower trigger too much, causing a disposition toward early learning. The principal has to set \( P_H \) high enough so that when \( K = K_H \), the agent would expend effort to learn. Otherwise the agent can pretend \( K = K_H \) without learning and get a profit of \( (K_H - K_M)D(P_0; P_H) \) which could be attractive if \( P_H \) is low. The optimal contract in the class of contracts that ensure learning and truth-telling can be solved in closed form:

**Proposition 2.6 (Non-contractible Learning).** Define \( \hat{c} \) to be the unique solution to

\[
c = q(1 - q)\Delta K \left[ 1 - \left( \frac{qK_L + \frac{qc}{1 - q}}{K_H - (1 - q)K_L} \right)^\beta \right].
\]

In the moderate-cost region \( (c \leq \hat{c}) \),

\[
P_{H_{pa}} = \frac{\beta}{q(\beta - 1)}(K_H - (1 - q)K_L) \quad P_{L_{pa}} = \frac{\beta}{\beta - 1}(K_L + \frac{c}{1 - q})
\]

\[
w_{H_{pa}} = 0 \quad w_{L_{pa}} = \frac{c}{1 - q} + \Delta K \left( \frac{qK_L + \frac{qc}{1 - q}}{K_H - (1 - q)K_L} \right)^\beta \quad \lambda_{pa} = 1;
\]

otherwise in the high-cost region \( (c > \hat{c}) \),

\[
P_{H_{pa}} = \frac{\beta}{\beta - 1} \frac{K_H + \frac{(1 - q)cK_M}{q(1 - q)\Delta K - c}}{q(1 - q)\Delta K - c} \quad P_{L_{pa}} = \left( 1 - \frac{c}{q(1 - q)\Delta K} \right)^{\frac{1}{\beta}} P_{H_{pa}}
\]

\[
w_{H_{pa}} = 0 \quad w_{L_{pa}} = \Delta K - \frac{c}{q} \quad \lambda_{pa} = 1.
\]

In this optimal contract, both (IC-L) \((2.3.4)\) and (IC-T) \((2.3.5,2.3.6)\) are at work because learning is non-contractible. In the moderate-cost region \( c \leq \hat{c} \), unlike Grenadier and Wang (2005), information rent alone is insufficient to induce learning effort, so distortion of investment timing only factors in for \( K = K_H \) in order to reduce the information rent. In the high-cost region \( c > \hat{c} \), the ratio of \( P_L \) to \( P_H \) cannot be too big even though we want them to be as close as possible to the constraint-free optimizers of the principal’s
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objective. The cost $c$ is so high that this ratio has to be small enough to provide the right incentives (for the agent not to pretend $K = K_H$ when $K = K_L$.) The investment triggers are sensitive to $c$. This makes sense as in this region the cost of effort affects the agent’s incentives to a greater extent. Note also that the agent’s option value could be decreasing in $\sigma$, for example, when $P_0 > \beta (\beta - 1)^{-1} q^{-1} [K_H - (1 - q) K_L] e^{-\frac{1}{\beta - 1}}$, thus the agent does not always have a more valuable option in volatile markets.

Contracting on Learning

Oftentimes the principal has the discretion to choose the optimal time of learning because she can observe the learning action and give nonpecuniary punishment if the agent does not learn optimally. When a division conducts R&D, the headquarter knows by observing whether the lab has been set up; early merger talks are routinely reported in the news; big public firms are constantly under the watchful eyes of analysts, the media, and regulators. Contracting on learning in such cases is a useful tool to the principal. The optimal contract derived below with contractible learning works for both optional learning and mandatory learning because the optimization programs are the same.

Similar to the earlier argument, whether the principal is paying for the cost of learning or not, it is optimal to learn at $\min\{P_L, P_H\}$. The optimization program is no longer subject to (IC-L) with contractible learning because if the agent does not exert effort to learn, the principal does not pay the agent any wage and potentially gives a nonpecuniary punishment.

Lemma 2.2. With contractible learning, the principal’s optimization problem can be reduced to:

$$\max_{P_H, P_L, \lambda} qD(P_0; P_H)(P_H - K_H) + (1 - q) D(P_0; P_L)(P_L - K_L - w_L) - (1 - \lambda)cD(P_0; P_L)$$

subject to

$$\max \left\{ \frac{c\lambda}{1 - q}, \Delta K \left( \frac{P_L}{P_H} \right)^{\beta} \right\} \leq w_L \leq \Delta K$$

Prop. 2.5 still holds by the same argument, and the optimal contract is characterized by the following proposition.

---

\textsuperscript{19} Even under such circumstances, the learning outcome may be hard to verify. Moreover, auditing technologies that reveal the exact learning outcome can be incorporated in the manner of Shibata (2009), who partially extends the model in Grenadier and Wang (2005) to allow costly auditing.
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Proposition 2.7 (Contractible Learning). In the moderate-cost region \( (c < \tilde{c}) \equiv (1 - q) \Delta K \left( \frac{qK_L}{K_H - (1 - q)K_L} \right)^{\beta} \), an optimal contract has the following terms:

\[
\begin{align*}
P^p_H &= \frac{\beta}{q(\beta - 1)}(K_H - (1 - q)K_L) \\
\lambda^p &= 1 \\
w^p_H &= 0 \\
P^p_L &= P^e = \frac{\beta}{\beta - 1} K_L \\
w^p_L &= \Delta K \left( \frac{qK_L}{K_H - (1 - q)K_L} \right)^{\beta} \\
\end{align*}
\]

otherwise in the high-cost region \( (c > \tilde{c}) \),

\[
\begin{align*}
P^p_H &= \frac{\beta}{\beta - 1} K_H \\
\lambda^p &= \left[ \frac{(1 - q) \Delta K}{c} \left( \frac{K_L + \frac{c}{1 - q}}{K_H} \right)^{\beta}, 1 \right] \\
w^p_H &= 0 \\
P^p_L &= P^e = \frac{\beta}{\beta - 1} \left( K_L + \frac{c}{1 - q} \right) \\
w^p_L &= \frac{c\lambda^p}{1 - q}.
\end{align*}
\]

Note that the first part of the contract has terms completely independent of \( c \), indicating dominance of information asymmetry over learning. Unlike the results in previous studies, the timing distortion could occur for the best type \( K_L \) too. Now if the learning cost is high enough that \( \text{(IR-P)} \) dominates, agency conflicts disappear, and investment efficiency is restored. \( \text{(IR-P)} \) likely dominates if the market is stable enough (small \( \sigma \)), or bearish enough (small \( \alpha \)), or the discount rate is big enough (big \( r \)), so that \( \tilde{c} \) is small.

Moreover, if the agent is not liquidity-constrained, contracting on learning has the following implication:

Proposition 2.8. When learning is contractible, and the agent is not liquidity constrained, an optimal contract ensuring learning and truth-telling always exists and it achieves the benchmark outcome, i.e., “no distortion” for the timing of investments, and the principal’s expected payoff and social welfare are maximized.

Social welfare is defined as the present value of the principal’s and the agent’s benefits from the contract. Commiting payments upon learning reduces the incentives to lie later. In fact, this payment can be specified prior to learning, as long as it corresponds to the same present value at the signing of the contract. It cannot occur after learning because the investment of high value project happens at the learning trigger, after which the agent can simply abscond rather than making this additional payment. The important take-away is that with contractible pre-project activities, if either the cost is high or the principal can charge the agent an upfront payment, the social optimum can always be achieved. Furthermore, this
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differs from the agent’s purchasing the investment option upfront because the contingent payments require much less upfront liquidity.

Moral Hazard of Learning and Distortion at the “Top”

Moral hazard of learning is distinct from other types of moral hazard. Earlier studies have shown that moral hazard typically delayes or reduces investment (Grenadier and Wang (2005), DeMarzo, Fishman, He, and Wang (2012)), but when the agent’s effort is substitute to investment, investments can be accelerated (Hartman-Glaser and Gryglewicz (2014)). But moral hazard is neither complementary nor substitute to investment in this paper, and can both accelerate or delay investment depending on the type of the project. Once information asymmetry endogenously arises, the standard result of “no distortion at the top” does not always hold either, in the sense that the best type’s investment could be accelerated relate to the benchmark.

The key driver for these results is the fact that learning is a dynamic decision that changes the investment opportunity for the good type. If learning occurs very late, high-value project may be already in the money. Consider a situation where the learning decision is make at time zero. Then even with agent’s moral hazard, there still would not be acceleration of projects and timing distortion for the best type. Thus the dynamics play an important role.

Distortions in Learning and Investment Timings

Agency conflicts alter the learning and investment thresholds, which in turn delay or accelerate learning and investment. Using properties of the Wald distribution, I examine the distribution of time lags and advances. The expected time advance in learning and investment in high value projects is \( \frac{\ln(P^{bn}/P^{bn})}{\alpha - \sigma^2} \) whereas the expected time lag in the investment in low value projects is \( \frac{\ln(P^{bn}/P^{bn})}{\alpha - \sigma^2} \). For example, with contractible learning and moderate learning cost, investments in high value projects are made earlier than the benchmark by an expected time \( \frac{\ln(1+c/(K_L(1-q))}{\alpha - \sigma^2} \).

Consider the impact of fundamental market conditions and the learning cost on the delays and accelerations of learning and investments. When learning is non-contractible, it can be explicitly verified that expected advance in learning and investments in high value projects, and the expected lag in investments in low value projects are weakly increasing in
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σ and decreasing in \( r \). They are decreasing in \( \alpha \) and independent of \( c \) in the moderate-cost region, but the effects are ambiguous in the high-cost region.

However, when learning is contractible, the expected advances and delays are eliminated if \( c \) or \( r \) is big enough, or if \( \alpha \) and \( \sigma \) are small enough (so that the solution is in the high-cost region), but are otherwise independent of fundamental market conditions. Unlike the non-contractible case, projects of both high and low option values are invested inefficiently in the moderate-cost region due to agency, as illustrated in Figure 2.2.

Whether learning is contractible or not, the distortions of learning and investment timing due to agency conflicts are particularly severe in volatile markets, or with low discount rate, and warrants greater attention in the current economy where the financial crisis and stimulus measures have resulted in volatile financial markets and low interest rate. Moreover, relative to the benchmark, investments appear delayed.

Taking the limit as \( c \to 0 \), the agent learns as soon as possible, and absent a mechanism that reveals the agent’s private information, adverse selection leads to suboptimal investment timing. In this limit, the optimal contracts for both contractible learning and non-contractible learning converge to:

\[
P^{pa}_L = \frac{\beta}{\beta - 1} K_L
\]
\[
P^{pa}_H = \frac{\beta}{q(\beta - 1)} (K_H - (1 - q)K_L)
\]
\[
w^{pa}_L = \Delta K \left( \frac{qK_L}{K_H - (1 - q)K_L} \right)^{\beta}
\]
\[
w^{pa}_H = 0
\]

The “good type” invests efficiently and gets an “information rent” whereas the “bad type” invests suboptimally and gets no rent. Low value projects are inefficiently delayed.

Cost of Agency Conflicts

Due to the distortions in timing, learning-induced agency conflicts significantly reduce social welfare and potentially the owner’s option value. In general, the optimal contract in the high-cost region is unlikely to be globally optimal as the principal does not want to ensure learning when the learning cost is high; cost for pre-project activities is also rather moderate in real life relative to investment costs. Thus unless stated otherwise, the following discussion concerns the moderate-cost region. The agency cost to social welfare
\[ \Delta SW = c[D(P_0; P_{e}^{bm}) - D(P_0; P_{e}^{pa})] \]
\[ + q \left[D(P_0; P_{H}^{bm})(P_{H}^{bm} - K_H) - D(P_0; P_{H}^{pa})(P_{H}^{pa} - K_H)\right] \]
\[ + (1 - q) \left[D(P_0; P_{L}^{bm})(P_{L}^{bm} - K_L) - D(P_0; P_{L}^{pa})(P_{L}^{pa} - K_L)\right]. \]

The terms are the losses in accelerated learning and delayed investment in low value projects, and the gain in the accelerated investment in high value projects, respectively. Overall, the further away the triggers are from the benchmark, the greater the social loss.

Contracting on learning has significant impact on the cost of agency conflicts. For \( c > \bar{c} \), \( \Delta SW \) is smaller when learning is contractible. For \( c < \min\{\hat{c}, \bar{c}\} \), \( \Delta SW \) is bigger when learning is contractible. For other costs contracting on learning could either lower or augment the social cost of agency conflicts. In industries where this social cost of agency is potentially large, governments and societies tend to force firms to be structured privately to avoid agency conflicts. The results show that the contractibility of learning has to be considered.

The owner’s option value is the difference between the welfare generated and the information rent of the agent, thus the agency cost on the owner’s option value is:

\[ \Delta OV = \frac{\Delta SW}{\text{Welfare Loss}} + D(P_0; P_{L}^{pa})[(1 - q)w_{L}^{pa} - c], \]
\[ \text{Agent’s Rent} \]

which is always smaller for contractible learning.

Further more, when learning is contractible, both \( \Delta SW \) and \( \Delta OV \) is increasing in \( c \). But when learning is non-contractible, \( \Delta SW \) is independent of \( c \). Thus for morderate learning costs, only when contracting on learning does one need to consider the sensitivity of agency costs to learning cost.

Finally note that in the high-cost region with contractible learning, the agency cost is zero for both the option owner and the society. Therefore with extremely high learning cost, contracting on learning is essential and improves social welfare and the owner’s option value dramatically.
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2.4 Generalizations

The analysis thus far extends to more general forms of payoff $V(P_t, K)$, where $P_t$ is an observable process and $K$ is an unknown component, and more general prior distributions of $K$ on the positive support $[K, \overline{K}]$, with cumulative function $F(K)$ and density $f(K)$. So far the model has focused on the additive form $V(P_t, K) = P_t - K$ and binary distribution of $K$ as investment cost. This section formulates a general model of learning and agency conflicts in investments under uncertainty, introducing key assumptions and providing two illustrations.

To ensure tractability, the following regularity conditions are assumed:

(A1) $V_1 > 0$ and $V_{11} = 0$; (A2) $V_2 < 0$ is continuous in $K$; (A3) $\frac{\partial}{\partial P} V_2(P, K) < V_{12}(P, K)$,

(A4) An optimal stopping threshold exists for each $K$ and satisfies $B_P > 0$.

The subscripts of $V$ to denote partial derivatives and the derivatives can be replaced by differences in the case of non-differentiability. $V_1 > 0$ basically says the option payoff depends monotonically on the revenue level and the unknown component. $V_{11} = 0$ is not necessary but simplifies the analysis and allows easy generalization to other linear diffusion processes. Moreover, in most real options models, the payoff is affine in revenue level, whether the uncertainty is about the cost of investment, or the output, or the permanency of past shocks. (A2) and (A3) are needed once I introduce agency. $V_2$ being continuous in $K$ is a sufficient condition for proving integral form of incentive compatibility; it is an innocuous assumption that almost all real options models satisfy. (A3) implies the cross-partial of $D(P_0; P) V(P, K)$ being positive - the “single-crossing” property on the present value of the payoff. (A4) assumes the existence of optimal stopping thresholds and is crucial in the derivation. In addition, the following Lemma is crucial in deriving the optimal contracts:

Lemma 2.3. A revelation contract is incentive compatible with limited liabilities if and only if the monotone property (MP),

$$P(K) \text{ is non-decreasing in } K,$$

and the integral condition of first order conditions (ICFOC),

\[^{20}\text{For example, Dixit and Pindyck (1994) and Grenadier and Malenko (2010).}\]
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\[ D(P_0; P(K))w(K) = - \int_K^\infty D(P_0; P(K'))V_2(P(K'), K')dK' \]

hold.

The proof is in Appendix 2.7. In general, learning follows a threshold strategy. In the benchmark case absent agency conflicts, it occurs beyond the optimal investment trigger of projects of the highest option value, causing delays in projects in high option value. Agency conflicts weakly accelerates learning and investments in high value projects (thus ameliorates the inefficient delays), but weakly delays investments in low value projects, regardless of the contractibility of learning. Appendix G generalizes Propositions 2.2 and 2.4, and related corollaries. Appendix H derives the optimal contracts under general settings and characterizes the resulting investment behaviors.

One special benchmark case involves uncertain production capacity \( Q \). This corresponds to the situation where the copper mining firm is unsure of its mining capacity on a new site, or uncertain about the potential local demand. Suppose \( Q = Q_H \) with probability \( q \), \( Q = Q_L \) with probability \( 1 - q \) and learning is worthwhile at some revenue level. Then \( P_{L}^{bn} = P_{L}^{bn} = \frac{\beta}{\beta - 1} \frac{c + qK}{qQ_H} \) and \( P_{H}^{bn} = \frac{\beta}{\beta - 1} \frac{K}{Q_L} \). Basically \( Q_H \) corresponds to the lower trigger and high value project. \( P_{L}^{bn} \) is decreasing in \( q \) and \( Q_H \), increasing in \( c \) and \( K \). If either the high capacity is more likely or larger in value, one would worry more about missing the optimal trigger and thus wants to learn early. On the other hand, if the investment cost or the learning cost is high, one would not want to learn very early because of the time value of money.

Figure 2.3 gives another illustration of the general model, with contractible learning and uniform investment cost \( K = dU(2, 6) \). The investment thresholds in an optimal contract are plotted against the actual \( K \). When \( c > \frac{\ln 5}{4} - \frac{1}{5} \), the optimal contract has \( P^{pa}(K) = 4(1 + \sqrt{2c})I_{K \leq 2(1 + \sqrt{2c})} + 2K I_{K > 2(1 + \sqrt{2c})} \); when \( c \leq \frac{\ln 5}{4} - \frac{1}{5} \), the optimal contract has \( P^{pa}(K) = 4(K - 1) \). Incomplete information delays high value projects \( K \leq 3.2649 \) when \( c = 0.2 \), \( K \leq 4 \) when \( c = 0.5 \), as indicated by the blue line. Agency either has no effect on the investment timing (black line \( c = 0.5 \)) or accelerates high value projects while delaying low value projects (green line \( c = 0.2 \)). If the project has high option value, agency makes the investment more efficient which could dominate other agency costs.

In this particular example, optimal contract for non-contractible learning can also be solved explicitly. The investment triggers are shown in Figure 2.4. When \( c \leq 0.0382 \), (IC-T) dominates and \( P^{pa}(K) = 4(K - 1) \); when \( c > 0.0382 \), (IC-L) becomes more important.
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Figure 2.3: Investment Timing Trigger in terms of $K$ with contractible learning. Plotted with $\alpha = 0.01$, $\sigma = 0.2$, $r = 0.06$, $K_H = 6$, $K_L = 2$ and $P_0 = 3$.

Figure 2.4: Investment Timing Trigger in terms of $K$ with non-contractible learning. Plotted with $\alpha = 0.01$, $\sigma = 0.2$, $r = 0.06$, $K_H = 6$, $K_L = 2$ and $P_0 = 3$.

\[
P^{pa}(K) = 4(1 + \sqrt{2c})I_{K \leq 2(1+\sqrt{2c})} + 2K I_{2(1+\sqrt{2c}) < K \leq 4} + 2(K + 4)I_{K > 4}.
\]
The jump
occurs at $K = 4$ when $c = 0.2$ is necessary for (IC-L) because otherwise the best “lying” strategy (pretending $K = 4$) can be better for the agent than exerting learning effort. Even with such a simple distribution, the solution is significantly more complicated than the case with contractible learning.

Since in general $K$ can be any resolvable uncertainty and $V(P, K)$ can be of very broad specifications, the results are robust to the assumptions on the revenue process, the prior belief of the unknown, and the payoff function of the investment option. This convenient feature thus enables the general model to be applied to many situations involving contingent-claim analysis.

### 2.5 Further Implications

This section presents further implications of the model. First, the observed investment efficiency, and the observed impact of incomplete information and agency are state-dependent and need careful interpretation - the presence of costly learning could produce opposite conclusions. Costly learning can also help explain the empirical puzzle that investments occur at stable or decreasing cash flows. Moreover, the sequential actions of learning and investment act as a joint signal for private information, and leads to novel market reactions and strategic behaviors in investment games. Finally, the model also sheds lights on how the price of information is endogenized and how pricing structure affects the division of information rent and social welfare.

#### 2.5.1 State-dependent Cost of Incomplete Information and Agency

Section 2 and 3 have already shown that investments could appear accelerated or delayed even with optimal strategies. This section takes the state-dependent nature of apparent investment distortions and warns against interpretations of the cost of incomplete information and agency based on ex post observations.

First consider how learning cost affects option values absent agency issues. Figure 2.5 contains some plots of total option value of the investment against learning cost. The red line is the expected value that determines whether learning occurs or not. The green line is the actual present value of the investment option when $K = K_L$ and the blue line is the value when $K = K_H$. First, there are a couple observations worth discussing. There are
discontinuities in the actual present values, to the right of which learning does not occur. For high value projects ($K = K_L$), the trade-off is the learning cost versus the extent of missing the optimal investment trigger without learning. The jump could be either upwards or downwards as seen in (a)-(d). When the probability that the project has high option value is low (high $q$), the switching point is smaller since in expectation learning is not that valuable. But if the project has high option value at this switching cost, the loss due to missing the optimal investment trigger outweighs the saving on the learning cost. For low value projects, the trade-off is the learning cost and investing too early. Again for high $q$, the switching cost is smaller because if it is more likely to be low option value, the learning trigger is closer to the optimal investment trigger (because $P_{e}^{bm}$ is increasing in $q$), so the loss due to early investment is less significant compared to the saving on learning cost. Another observation is that higher diffusion increases the option value, especially when $q$ is high, as seen in (b) and (d). Higher drift also increases the option value, as seen in (e) and (f). Finally, when learning does take place, it is not necessarily monotone in the learning cost for low value projects, as seen in (d) and (e). The reason for this is that while increasing learning cost reduces option value, it also brings the learning trigger closer to the optimal trigger for low value projects, which increases the option value. The latter could dominate the former. However this does not apply for high value projects because increasing learning cost only brings the learning trigger further away from the optimal investment trigger.

Therefore, depending on the parameter, increasing the cost of learning could increase the actual option value of the investment with optional learning, or cause a switch from learning to not learning. Even though higher learning cost is always bad in expectation, the option owner could be lucky to have a state of the world where her actual option value increases. Conversely, if investment costs are only observed ex-post, it would appear occasionally that a higher learning cost leads to increased option value. Mandatory learning is similar.

With agency issues, this state-dependent effect manifests itself through two additional channels. First, in the earlier analysis it has been demonstrated that agency could distort investment timing. In some states of the world, the distortion makes investment more efficient. If this effect dominates, there would be an improvement in individual and social welfares. For example, in the illustration with uniform distribution in Section 4, the ex-ante total welfare is less than the benchmark case, yet for $K \leq 2.3076$, social welfare is strictly improved with agency, as shown in Figure 2.6. Note also in Figure 2.7, the option owner
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Figure 2.5: Total Option Value with Optional learning as a Function of learning Cost with $r = 0.06$, $K_H = 6$ and $K_L = 2$
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is better off with the introduction of agency if \( K > 5.3929 \). Second, cost adjustment in
the presence of agency now has the effect of altering the optimal contract. For example,
when learning cost is increased from 0.2 to 0.5, social welfare is strictly improved for
\( K > 2.7507 \) in Figure 2.6. In fact, the option owner’s option value is also increased for
\( K > 4.7078 \) in Figure 2.7.

These results caution us that interpretations of investment efficiency and cost of agency
conflicts must take into consideration costly learning that is often unobserved. Policies
aimed at controlling the learning cost may have state-dependent effects and it is not always
true that agency and costly learning are deleterious to the ex post social welfare. For ex-
ample, suppose the government knows the state of the world is bad \( (K = K_H) \) and also
understands that the investors are overly optimistic \( (K = K_H \text{ with probability } q < 1) \).
To protect the investors, the government may occasionally need to increase learning cost
rather than reducing it even though intuitively, one would imagine reducing learning cost
makes the investors better off.

2.5.2 Investments at Stable or Decreasing Cash Flows

It has long puzzled researchers why investments occur at decreasing cash flows (which
is the revenue in the current model), as exemplified in the wide-spread office constructions
in Denver and Houston during the 1970s and 1980s when the demand and housing value
were declining. While traditional models predict that investments only occur at increasing
cash flows, learning adds in another dimension. The intuition is that when information is
acquired, the belief about the profitability of investment changes discontinuously. While
the learning process is triggered at increasing cash flows, there could be multiple channels
that lead to investments at non-increasing cash flows. Here I consider two of them: “time-
to-learn” and multiple firms with correlated investment costs.

Time-to-Learn

So far learning has been treated as instantaneous to simplify computation. In reality,
learning takes time and commitment of resources. Suppose after paying the learning cost \( c \),
it takes a period of \( \delta \) to learn before the firm can invest. The value of the information is then
\( I^\delta(P_t) = \mathbb{E}_t[e^{-\gamma \delta} I(P_{t+\delta})] \), where \( I(P_t) \) is the value of information without time lag. Since

\[^{27}\text{Grenadier (1996) and Grenadier and Malenko (2010) are some alternative explanations.}\]
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Figure 2.6: Realized Social Welfare in terms of K with contractible learning. Plotted with $\alpha = 0.01$, $\sigma = 0.2$, $r = 0.06$, $K_H = 6$, $K_L = 2$ and $P_0 = 3$.

Figure 2.7: Option Value for the Principal in terms of K with contractible learning. Plotted with $\alpha = 0.01$, $\sigma = 0.2$, $r = 0.06$, $K_H = 6$, $K_L = 2$ and $P_0 = 3$. 
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I is continuously differentiable, generalized versions of Ito’s formula still apply and an optimal trigger for learning exists. In general the problem can only be solved numerically. It is possible that the learning is completed at declining cash flow, and investment follows right away if the optimal triggers have been reached already. To drive the intuition home, one simulation of $I^k(P_t)$ and cash flow trigger for learning $P_e$ is shown in Figure 2.8(a). Learning occurs at $P = 4.4$ but the true optimal lower trigger under full information is $P = 2$. Figure 2.8(b) illustrates why this leads to investment at decreasing cash flows. Learning is triggered at $t = 1.451$ and learning is completed at $t = 2.451$ when $P_t = 3.22$. Since 3.22 is above the lower trigger, if $K$ is found out to be low, investment should start right away even though at this moment the cash flow is decreasing.

Correlated Investment Costs

Suppose two investment options have correlated fixed investment costs, and the cash flow processes are correlated. For example, the options can be owned by contractors for office building and residential buildings respectively on the same real estate. The rental prices for offices and residential apartments $\tilde{P}_{1t}$ and $\tilde{P}_{2t}$ are given by,

$$d\tilde{P}_{1t} = \alpha_1 \tilde{P}_{1t} dt + \sigma_1 \tilde{P}_{1t} dB_{1t}, \quad d\tilde{P}_{2t} = \alpha_2 \tilde{P}_{2t} dt + \sigma_2 \tilde{P}_{2t} dB_{2t},$$

where $dB_{1t}$ and $dB_{2t}$ are Brownian motions (potentially correlated). The investment cost are highly correlated since they are in the same area. For illustration, suppose the cost is either $K_H$ or $K_L$ and the correlation is perfect. The two contractors can have different private priors on the distribution of the investment cost, and only one possesses the technology to learn the exact value with an effort cost $c$. Without loss of generality let it be the office contractor. Because the priors are private information, the office contractor’s optimal learning trigger is $\tilde{P}_{1t} = \frac{\beta_1 (r - \alpha_1)}{\beta_1 - 1} (K_L + \frac{c}{1-q_1})$. If this is lower than the investment trigger of the residential contractor based on her prior information, the office contractor will learn first, and the other will know the investment cost by observing whether the office contractor invests right after learning. If she decides that $K = K_L$, the residential contractor could invest even though the cashflow might be decreasing. Figure 2.9 gives an illustration of investment at decreasing cash flow in residential construction. As long as the Brownian shocks to the cash flows of the participating agents are not perfectly correlated, an “industrial leader” who possesses learning technology can potentially cause pronounced bursts of
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(a) Value of Information against Cash Flow Level under Time-to-learn.

(b) One realization of investment at decreasing cash flow.

Figure 2.8: Investment at decreasing Cash Flow. Simulated at $q = 0.5$, $\alpha = 0$, $\sigma = 0.2$, $r = 0.04$, $c = 0.6$, $K_H = 8$, $K_L = 1$, and $\delta = 1$
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investments which in the literature has been attributed to information cascades (for example, in Grenadier (1999).)

![Figure 2.9: Investment at decreasing cash flow under perfectly correlated investment costs. Simulated at $r = 0.06$, $\alpha = 0.01$, $\sigma = 0.2$, $c = 0.6, K_H = 6$, $K_L = 2$, $q = 0.5$ with correlation coefficient for the Brownian shocks being 0.7.]

With either time-to-learn or correlated investment costs, investments at non-increasing cash flows only occur for high value projects. It would be interesting to empirically examine whether investments at non-increasing cash flows generate above average returns as they are more likely associated with high value projects.

2.5.3 Learning and Investing as a Joint Signal

Since the investment dynamics involve sequential actions of learning and investing, the immediate investment after learning or the lack thereof could serve as signals in economically meaningful ways. First, they can act as a signal for firm value, and the market reaction can be analyzed in the fashion of Grenadier and Wang (2005); second, they can act as a signal of the private information gained through learning, and has interesting consequences in investment games, for example, under duopolistic settings.
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Signal for Firm Value and Market Reactions

Imagine a public firm owning an investment option. The firm does not know the exact cost of the investment but can investigate it. From our earlier analysis, if the cost happens to be low, investment option is immediately exercised after learning and the firm is worth $P_L - K_L$, otherwise it is worth $(\frac{P_L}{P_H})^\beta (P_H - K_H)$. Recall that without agency issues, $P_L = \frac{\beta}{\beta - 1} (K_L + \frac{c}{1-q})$ and $P_H = \frac{\beta}{\beta - 1} K_H$. Thus,

$$(\frac{P_L}{P_H})^\beta (P_H - K_H) = \left( \frac{K_L + \frac{c}{1-q}}{K_H} \right)^\beta \frac{K_H}{\beta - 1} < K_L + \frac{c}{1-q} < \frac{K_L + \frac{\beta c}{1-q}}{\beta - 1} = P_L - K_L$$

Given this, the combination of learning and investing could serve as a signal for firm value. Immediate investment after learning conveys good news: the firm is worth more than expected. Stock price of the firm jumps upwards (assuming the learning cost is small enough) if investment follows immediately after learning, and downwards otherwise. The case with agency is similar and Fig (2.10) illustrates the stock price responses with and without agency issues. In prior studies, investments are interpreted as positive signals and not investing as bad signals, with the implicit albeit unrealistic assumption that market participants know the exact time investment decisions should take place. However in the current model, even if the outside investor does not know the domain and prior distribution of $K$, as long as pre-project activities are observable (not necessarily contractible); no investment after learning is bad news because it implies the project has low option value. Whether investment takes place right after learning thus conveys the learning outcome to the markets. One caveat is that the firm may strategically use this channel to mislead the market participants. This prediction is potentially testable by examining the stock price response to announcements/news/observations of pre-project activities and subsequent investment decisions. Certainly the price reaction would not be so abrupt because learning and investment decisions are not instantaneous, nonetheless it would be interesting to see whether qualitatively the results hold.

Signal for Profitability in Investment Games

Learning and investing can serve as a joint signal in both non-competitive and competitive settings. The real estate investments with correlated investment costs discussed earlier is an illustration of a scenario where developers are not in direct competition. However, it
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Figure 2.10: Stock Price Jumps with and without Agency Issues. Simulated at $r = 0.04$, $\alpha = 0.01$, $\sigma = 0.2$, $c = 0.6$, $K_H = 6$, $K_L = 2$, $q = 0.5$

has long been demonstrated that investment timings are influenced by market competition; Grenadier (1996), Weeds (2002), and Lambrecht and Perraudin (2003) are some examples. There has also been some emerging research on signaling games in a real options framework, notably Grenadier and Malenko (2011) and Morelec and Schürhoff (2011). But signaling games under competitive environments have not been discussed much, not to mention the crucial role of learning and endogenous emergence of information asymmetry under such settings. I give a preliminary account how the framework in this paper naturally leads to signaling in investment games.

Consider two mobile network companies contemplating investing in a new product. Firm A is the industry leader with investment cost $K_A$, and firm B has investment cost $K_B \gg K_A$ so that firm B is the follower. Firm A gets a market share of $A = A_H$ with a probability $q$, and $A_L$ otherwise, where $A_L < A_H$. When firm B enters the market, it gets the remainder of the market and a fraction $B$ of A’s users. Due to network effect, $B = B_H$ for $A = A_L$ and $B_L$ for $A = A_H$ with $B_H > B_L$. Firm A enjoys a monopolistic profit flow $AP$ where $P$ is given by a GBM until firm B invests, when a fraction $B$ of market shares is lost to firm B. Firm A has a learning technology whose cost is small enough that it is worthwhile to ascertain $A$ and then optimally time the investment. Firm B can potentially
utilize this learning-investment signal. Appendix I outlines a situation where the following candidate constitutes a Perfect Bayesian equilibrium.

Firm B believes $B = B_l$ if firm A invests right after learning, and $B = B_H$ otherwise. Firm A learns at $P_e = \frac{\beta}{\beta - 1} \frac{c + q K_A}{q A_H}$, and invests at $P_{AH} = P_e$ if $A = A_H$ and $P_{AL} = \frac{\beta}{\beta - 1} \frac{K_A}{A_L}$ if $A = A_L$. Firm B invests at $P_{BL} = \frac{\beta}{\beta - 1} \frac{K_B}{A_H B_L + 1 - A_H}$ if it believes $\lambda = L$ and $P_{BH} = \frac{\beta}{\beta - 1} \frac{K_B}{A_L B_H + 1 - A_L}$ otherwise.

In this separating equilibrium, firm B uses the signal of whether firm A invests right after learning, to decide its optimal investment strategy. This is a “truthful” equilibrium because firm A indeed invests right away if $A = A_H$. There could be other pooling or partial-pooling equilibria where A deliberately postpone investment even when $A = A_H$ so as to conceal this information from B, who would in turn invest later, potentially prolonging the monopolistic profit to A. The solutions in such games and their applications are interesting on their own for future research.

### 2.5.4 Market for Information

Finally, consider the market for information on the resolvable uncertainty. If the market is competitive, the price for information simply equals the cost of generating the information. This section focuses on monopoly markets for information to understand what dictates the price for information and how the pricing structure affects the division of information rent and social welfare. It also provides a way to endogenize the learning cost in the basic model.

It has been shown the value of information $I(P_t)$ is continuously differentiable and singly-peaked w.r.t. $P_t$. The cost of information in a non-perfectly-competitive market thus changes with time. In the extreme case of information monopoly, the seller can charge up to the total value of information. In the region of interests $[P^*(K_L), P^*(K_H)]$, the present value to the seller is thus bounded by

$$
\max_{P_e} D(P^*(K_L); P_e) I(P_e) = \max_{P_e} (P^*(K_L))^\beta \left[ \frac{(\beta - 1)^{\beta - 1}}{\beta^\beta} \left( \frac{q}{K_H^{\beta - 1}} - \frac{1}{K_M^{\beta - 1}} \right) + (1 - q) (P_e^{1 - \beta} - K_L^{-\beta}) \right]
$$
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By charging $I(P_e)$ for all $P_e > P^*(K_L)$ and $I(P^*(K_L)) - \epsilon$ for $P_e = P^*(K_L)$ where $\epsilon$ is infinitely small (zero if we assume the buyer always buy early when indifferent), one can get infinitely close to this bound (or even achieve it). When this happens, the subsequent investment is always efficient because there is no benefit to the buyer for delaying information acquisition. The information seller’s revenue and social welfare are both maximized. The revenue to the buyer is $D(P_0; P^*(K_M)) [P^*(K_M) - K_M]$.

What if the information seller is restricted when setting the price? For example, if the government forbids price fluctuations and requires the vendor to set a fixed price $c$. In this fixed-price case, the seller has to trade off between setting a high price and receiving that payment early. The buyer optimally acquires information and invests as described in Proposition 2.2. Thus the seller maximizes $D(P_0; P^{bm}) c$. The optimal price is $c^* = \frac{(1-q)K_L}{\beta - 1}$ in mandatory learning and $c^* = \max\{\frac{(1-q)K_L}{\beta - 1}, \bar{c}_1\}$ in the optional learning case. If information is sold, the revenue to the seller is $P_0^\beta (1-q)(\beta - 1)^{2\beta - 1} K_L^{-\beta}$. The social welfare is lower because when $K = K_L$, the investment happens at $\beta^2/(\beta - 1)^2 K_L > P^*(K_L)$ and is delayed relative to the variable-price case. Finally the revenue to the buyer is $D(P_0; P^{bm}) [(1-q)(P^{bm}_e - K_L) - c] + qD(P_0; P^*(K_H))(P^*(K_H) - K_H)$, which differs from the variable-price case by $D(P_0; P^{bm}_e)[I(P^{bm}_e) - c]$ that is non-negative. Thus this rigid pricing enhances the buyer’s payoff at the expense of the seller and the social welfare. Despite sounding counter-intuitive, regulators should give the information monopolist the flexibility to dynamically adjust the price, in order to achieve socially efficient outcomes. The above analysis also endogenizes the learning cost in the basic model under information monopoly to be $(1-q)K_L/(\beta - 1)$, which is increasing in $K_L$, $\sigma$, $\alpha$ and decreasing in $r$ and $q$. In particular, equilibrium learning cost is higher in volatile and bearish markets.

2.6 Conclusion

This paper highlights the importance of endogenous learning and its associated agency conflicts in the real options models. In particular, ignoring learning options leads to apparent delays and accelerations of investments even with the optimal strategy. An optimal contract under principal-agent settings entails weakly accelerated learning and investments in high value projects but weakly delayed investments in low value projects, relative to the no-agency benchmark. In particular, the moral hazard of learning could lead to accelerated investment and timing distortion can happen to the best type. Moreover, greater
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uncertainties could accelerate investments too, and contracting on learning potentially re-
stores investment efficiency in de-centralized firms. The model also explains, among other
implications, how investment could happen at decreasing cash flow and how learning and
investing can act as a joint signal for firm value.

The model admits a few extensions. First, the analysis applies to abandonment and
exit decisions. One particularly intriguing case is the sale of houses by real estate agents:
[Levitt and Syverson (2008)] document that homes owned by real estate agents sell not only
for 3.7% more than other houses, but also stay on the market 9.5 days longer, providing
support for the significant impacts of agency conflicts on market timing. Second, it is useful
to know how costly learning and agency issues are modified in a competitive environment,
including that of standard auctions. Finally, empirical analysis to corroborate or refute the
model would be a worthwhile effort too. For example, newspaper articles and reports on
pre-project activities and subsequent investment decisions can be juxtaposed with the stock
price changes over the same period of time to see if learning and investing indeed serve as
a joint signal of firm value.

2.7 Appendix

A. Proof for Proposition 2.1

Proof. I first derive an optimal strategy in the space of threshold strategies, then verify it is
optimal in the space of all investment strategies. The expected payoff when using threshold
P is \( \mathbb{E}[e^{-rt}(P - \tilde{K})] \), where \( \tau \) is the stochastic first hitting time and \( \tilde{K} \) is the unknown
investment cost. Note \( \tilde{K} \) is independent of cash flow process \( P_t \). Thus the payoff reduces to
\( \mathbb{E}[e^{-rt}]\mathbb{E}[P - \tilde{K}] = D(P_0; P)[P - qK_H - (1 - q)K_L] \) F.O.C. gives the optimal threshold
is \( P^*(K_M) \), and the option value is \( W(P_0; K_M) \), defined in (2.2.2) and (2.2.3) respectively.

To verify the optimality of the threshold strategy, let \( x_t = e^{-rt}W(P_t) \). Using an ex-
tended version of Itô’s formula (as, for example, in [Karatzas and Shreve (1988), page 219],
\( dx_t = e^{-rt}[DW(P_t) - rW(P_t)]dt + e^{-rt}W_P(P_t)\sigma P_t dB_t \), where \( DW(P) = W_P(P)\alpha P + \frac{1}{2}W_{PP}(P)\sigma^2 P^2 \) except at \( P = P^* \), where \( P^* = P^*(K_M) \) for short and we may replace
\( W_{PP}(P^*) \) with zero. \( W_P \) is bounded as seen by direct computation, thus by Proposition 5B
in [Duffie (2009)] (also found in [Protter (2004)]), the last term in \( dx_t \) is a martingale under
the current measure. For \( P \leq P^* \), \( DW(P) - rW(P) = 0 \) by the definition of \( \beta \) in (2.2.4).
For \( P > P^* \), \( DW(P) - rW(P) = rK_M - (r - \alpha)P < rK_M - (r - \alpha)P^* = \frac{K_M(\alpha\beta - r)}{\beta - 1} = -\frac{1}{2}K_M\beta\sigma^2 < 0 \). The drift of \( x \) is thus never positive, implying for any stopping time \( \tau \), \( W(P_0) \geq \mathbb{E}[e^{-r\tau}W(P_\tau)] \geq \mathbb{E}[e^{-r\tau}(P_\tau - K_M)] \) with equality holding for the first-hitting time of \( P^* \). This implies the optimality of the threshold strategy for investment.

### A. Proof for Proposition 2.2

**Proof.** I first derive an optimal threshold strategy for learning. Differentiating (2.2.6) gives \( I'(P) \leq 0 \) for \( P > P^*(K_M) \), thus if learning ever occurs at some revenue \( P_e \), it must be weakly below \( P^*(K_M) \), otherwise the option owner can do better by learning at \( P^*(K_M) \) because \( I(P^*(K_M)) - c \geq D(P^*(K_M); P_e)[I(P_e) - c] \). Learning before cash flow first reaches \( P^*(K_L) \) is dominated by learning upon first hitting \( P^*(K_L) \). Conditional on optimal investment after learning at \( P_e \), the total value of the whole operation at \( P_0 \) is given by,

\[
F(P_0, P_e) = D(P_0; P_e)[qW(P_e; K_H) + (1 - q)W(P_e; K_L) - c] = qD(P_0; P^*(K_H))(P^*(K_H) - K_H) + D(P_0; P_e)[(1 - q)(P_e - K_L) - c]
\]

Maximizing \( F \) over \( P_e \) gives the optimal learning trigger \( P_e^{bm} = \frac{\beta}{\beta - 1}(K_L + \frac{c}{1 - q}) \). Hence the option owner learns at \( P_e^{bm} \) if \( I(P_e^{bm}) \geq c \), which requires \( c \leq \bar{c}_1 \). If \( c > \bar{c}_1 \), this becomes a standard real option with investment cost \( K_M \), and the results follow.

To establish the optimality of the strategy, I work backwards to verify the optimality of the threshold strategy for investment assuming learning has taken place or is not going to occur, then I verify the optimality of the threshold strategy for learning. The optimality of the threshold strategy for investment upon learning is the same as in the proof of Proposition 2.1, it thus remains to show that if learning ever occurs (\( c \leq \bar{c}_1 \)), the threshold strategy proposed is optimal. The argument is similar to that for the optimality of threshold strategy of investment, except for the following steps. First, \( W \) is replaced by \( F(P) = D(P; P_e^{bm})[qW(P_e^{bm}; K_H) + (1 - q)W(P_e^{bm}; K_L) - c] \) when \( P \leq P_e^{bm} \) and \( F(P) = [qW(P; K_H) + (1 - q)W(P; K_L) - c] \) when \( P > P_e^{bm} \), which is still convex and \( C^1 \), and is \( C^2 \) except at \( P_e^{bm} \). Second, when \( P \leq P_e^{bm}(c) \), \( DF(P) - rF(P) = 0 \) by definition of \( \beta \); when \( P_e^{bm}(c) < P \leq P^*(K_H) \), \( DF(P) - rF(P) = \alpha P - rP + r(K_L + \frac{c}{1 - q}) = \frac{1}{2}(K_L + \frac{c}{1 - q}) \beta\sigma^2 < 0 \); and when \( P > P^*(K_H) \), \( DF(P) - rF(P) < -\frac{1}{2}(K_M + c) \beta\sigma^2 < 0 \).
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Third, \( F(P_0) > \mathbb{E}[e^{-rt}F(P_r)] > \mathbb{E}[e^{-rt}(qW(P_r; K_H) + (1-q)W(P_r; K_L) - c)] \). Fourth, when learning doesn’t occur, this is reduced to a simple investment real option. This validates the optimality of the proposed strategy for learning and investing. ■

B. Proof for Proposition 2.3

Proof. Given that the threshold of learning is \( P_e^{bm} \) for \( c < c_1 \), the option value of learning is \( F_{Learning} = D(P_0; P_e^{bm})[I(P_e^{bm}) - c] \), whose derivative w.r.t. \( \beta \) equals,

\[
\frac{P_0^\beta (\beta - 1)^{\beta - 1}}{\beta^\beta} \left[ \frac{1 - q}{(K_L + \frac{c}{1-q})^\beta} \ln \left( \frac{P_0(\beta - 1)}{\beta K_L} \right) + \frac{q}{K_H^\beta} \ln \left( \frac{P_0(\beta - 1)}{\beta K_H} \right) \right]
\]

\[
- \frac{1}{K_M^\beta} \ln \left( \frac{P_0(\beta - 1)}{\beta K_M} \right)
\]

\[
> \frac{P_0^\beta (\beta - 1)^{\beta - 1}}{\beta^\beta} \left[ \frac{1 - q}{K_L^\beta} \ln \left( \frac{P_0(\beta - 1)}{\beta K_L} \right) + \frac{q}{K_H^\beta} \ln \left( \frac{P_0(\beta - 1)}{\beta K_H} \right) \right]
\]

\[
- \frac{1}{K_M^\beta} \ln \left( \frac{P_0(\beta - 1)}{\beta K_M} \right) > 0,
\]

where the fact \( \ln(P_0(\beta-1)\beta^{-1}y)\beta^{-\beta} \) is increasing in \( y \) when \( c < c_1 \) and \( P_0 > \frac{\beta K_H}{\beta - 1} e^{-\frac{\beta - 1}{\beta - 1}} \) gives the first inequality and it is convex in \( y \) when \( P_0 < \frac{\beta K_L}{\beta - 1} e^{-\frac{\beta - 1}{\beta - 1}} \) gives the second inequality. As \( \beta \) is decreasing in \( \sigma, \alpha \) and increasing \( r \), the proposition follows. ■

C. Proof for Proposition 2.4

Proof. If mandatory learning occurs between \( P_e^*(K_L) \) and \( P_e^*(K_H) \), then a similar argument as in the proof of Proposition 2.2 leads to learning trigger at \( P_e^{bm} = \frac{\beta}{\beta - 1} (K_L + \frac{c}{1-q}) \). Note \( c < (1-q)\Delta K \) to be consistent with \( P_e^{bm} \leq P_e^*(K_H) \), which implies \( F(P_0, P_e^{bm}) > 0 \), so it is indeed optimal to invest in the project. Different from the optional learning case, it is possible that mandatory learning occurs beyond \( P_e^*(K_H) \) when \( c > (1-q)\Delta K \), in which case, \( F(P_0, P_e) = D(P_0; P_e)(P_e - K_M - c) \) yielding optional learning trigger \( P_e^{bm} = \frac{\beta}{\beta - 1} (K_M + c) \). Note \( F(P_0, P_e^{bm}) > 0 \), thus it is never optimal to walk away. The same arguments as in Appendix A establishes the optimality of the strategy. ■
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D. Proof of Lemma 2.1

Proof. The proof is broken into several steps:

Step 1: Limited liability of \( w_L \) does not bind. From constraint (2.3.6), \( w_L \geq \frac{D(P_0;P_H)}{D(P_0;P_L)}(w_H + \Delta K) \equiv \frac{D(P_0;P_H)}{D(P_0;P_L)} \Delta K > 0 \), where \( \Delta K = K_H - K_L \).

Step 2: IR constraint for exerting effort does not bind. We know that \( K_H - K_M + w_H \geq K_H - K_M > 0 \) and \( (1 - \lambda)cD(P_0; \min\{P_L, P_H\}) \geq 0 \), thus IC-L implies IR-P. Hence \( w_0 \) on the RHS of constraint (2.3.4) can be ignored. The constraint becomes,

\[
qD(P_0; P_H)w_H + (1 - q)D(P_0; P_L)w_L - cD(P_0; \min\{P_L, P_H\})
\geq \max\{(K_H - K_M + w_H)D(P_0; P_H), (K_L - K_M + w_L)D(P_0; P_L)\}
\]

Step 3: Limited liability on \( w_H \) binds. Intuitively from IC-T (2.3.6), increasing \( w_H \) will likely increase \( w_L \). Thus to maximize the owner’s value of the contract, \( w_H \) should be as small as possible yet satisfying all other constraints. Suppose in an optimal contract, \( w_H \) is positive. Then there exists \( \epsilon \) such that \( 0 < \epsilon < \min\{w_H, w_LD(P_0; P_L)/D(P_0; P_H)\} \). Reduce \( w_H \) by \( \epsilon \), and reduce \( w_L \) by \( D(P_0; P_H)\epsilon/D(P_0; P_L) \), all the constraints are still satisfied because both sides of the constraint inequalities are reduced by the same amount. However, this would make the principal strictly better off because the wages she pays are less, contradicting the premise that this contract is optimal. This is also the reason why \( \lambda \leq 1 \). Otherwise, suppose the principal requires the agent to pay in addition to bearing the learning cost at the time of learning, the agent can deviate to learning slightly earlier (learning is non-contractible) by paying \( c \), and simply walk away if \( K \) turns out to be \( K_H \), as opposed to paying more than \( c \) to learn, and getting nothing if \( K = K_H \). Arguably raising \( w_H \) gets around this, but this in expectation simply offsets the amount the principal charges the agent for learning \( (\lambda - 1)c \), so that any additional benefit of having \( \lambda \leq 1 \) is exactly eroded by raising the wages.

Step 4: \( P_L \leq P_H \). Step 3 and constraint (2.3.5) imply \( w_L \leq \Delta K \). Then constraint (2.3.6) and the fact \( D(P_0; P_1) \geq D(P_0; P_2) \) for \( P_0 \leq P_1 \leq P_2 \) imply \( P_L \leq P_H \).

Step 5: In an optimal contract, truth-telling IC (2.3.5) is not binding but (2.3.6) is potentially binding. Because \( w_H = 0 \) (derived in Step 4), \( (\frac{P_L}{P_H})^\beta \Delta K \leq w_L \leq \Delta K \). Choosing a bigger \( w_L \) leads to losses in the principal’s profit.

Step 6: Constraint (2.3.4) reduces to \( w_L \geq \Delta K(\frac{P_L}{P_H})^\beta + \frac{c}{1 - q} \). Given Steps 3-5,
it remains to show \((K_H - K_M)D(P_0; P_H) \geq (K_L - K_M + w_L)D(P_0; P_L)\) in an optimal contract. Suppose otherwise, let \((w_{pa}^P = 0, w_L^H, P_H^pa, P_L^pa)\) be the optimal contract, then \((K_H - K_M)D(P_0; P_H^pa) < (K_L - K_M + w_L^pa)D(P_0; P_L^pa)\) implies \(w_L^pa > [q + (1 - q)(\frac{P_L}{P_H})^\beta]\Delta K\), which in turn implies (2.3.6) and (2.3.7) are not binding. There exists an \(\epsilon > 0\) satisfying \((K_H - K_M)D(P_0; P_H^pa) \leq (K_L - K_M + w_L^pa - \epsilon)D(P_0; P_L^pa)\) such that decreasing \(w_L^pa\) by \(\epsilon\) increases the objective of the principal while still satisfying all the constraints. Thus \((0, w_L^pa - \epsilon, P_H^pa, P_L^pa)\) is better than the optimal contract, giving a contradiction. To see this, note \(V^P\) is greater with \(w_L^pa - \epsilon\). (2.3.6) and (2.3.7) can still hold when \(w_L\) decreases. (2.3.5) is not affected. The LHS of (2.3.4) decreases by \((1 - q)D(P_0; P_L)\epsilon\), which is less than the decrease in the RHS-\(D(P_0; P_L)\epsilon\), thus (2.3.4) still holds. This also implies that mixed strategy on “lying” is dominated.

Step 7: \(w_L = \Delta K(\frac{P_L}{P_H})^\beta + \frac{\epsilon}{1-q} \) and \(\lambda = 1\) in an optimal contract. Now after Steps 1-6, the optimization problem becomes,

\[
\max_{w_L, P_H, P_L, \lambda} qD(P_0; P_H)(P_H - K_H) + (1 - q)D(P_0; P_L)(P_L - K_L - w_L) - (1 - \lambda)cD(P_0; P_L) \\
\text{subject to} \quad \Delta K(\frac{P_L}{P_H})^\beta + \frac{c}{1-q} \leq w_L \leq [q + (1 - q)(\frac{P_L}{P_H})^\beta]\Delta K
\]

The objective has negative partial derivative w.r.t. \(w_L\) and positive partial derivative w.r.t. \(\lambda\). Thus in the solution to the maximization problem, \(w_L = \Delta K(\frac{P_L}{P_H})^\beta + \frac{c}{1-q}\) and \(\lambda = 1\).

The above arguments reduce the optimization problem to the claimed form, the constraint comes from the Step 6 that \((K_H - K_M)D(P_0; P_H^pa) \geq (K_L - K_M + w_L^pa)D(P_0; P_L^pa)\) in an optimal contract, and \(w_L^pa = \Delta K(\frac{P_L}{P_H})^\beta + \frac{\epsilon}{1-q}\).

E. Proof of Proposition 2.6

**Proof.** For \(c \leq \hat{c}\), since (2.3.1) is decreasing in \(w_L\) so we set \(w_L = \Delta K(\frac{P_L}{P_H})^\beta + \frac{c}{1-q}\). The F.O.C. of (2.3.1) w.r.t. \(P_L\) and \(P_H\) give \((1 - q)(\frac{P_L}{P_H})^\beta[1 - \beta(\frac{P_L}{P_H})(P_L - K_L - \frac{c}{1-q})] = 0\) and \((\frac{P_L}{P_H})^\beta[q - \beta(P_H + (1 - q)K_L - K_H)] = 0\). Solving for the optimal \(P_H\) and \(P_L\) gives \(P_L = \frac{\beta - 1}{\beta}K_L + \frac{\epsilon}{1-q}\) and \(P_H = \frac{\beta}{\beta - 1}(K_H - (1 - q)K_L)\). It can be shown that \(\frac{\partial^2 V^P}{\partial P_L^pa} < 0, \frac{\partial^2 V^P}{\partial P_H^pa} < 0\) and \(\frac{\partial^2 V^P}{\partial P_L^pa}\frac{\partial^2 V^P}{\partial P_H^pa} = 0\), thus second order conditions are satisfied. Moreover, it can be easily verified that \(P_H^pa > P_L^pa\), \(P_e^pa = P_L^pa\) and \(w_L = \frac{c}{1-q} + \Delta K(\frac{qK_L - \frac{\epsilon}{1-q}K_L}{K_H - (1 - q)K_L})^\beta\). But all these calculations are done under the
premise that the constraint \((2.3.3)\) is not binding, this translates into \(c - q(1 - q)\Delta K [1 - (\frac{qK_L + \frac{qc}{K_H - (1 - q)K_L}}{K_H - (1 - q)K_L})^\beta] \leq 0\), whose LHS is monotone in \(c\), implying a unique cutoff value of \(c\). The first part of the proposition thus follows. These contract terms together with \(w_H = 0\) satisfy all the constraints.

For the second part, if \(c > q(1 - q)\Delta K [1 - (\frac{qK_L + \frac{qc}{K_H - (1 - q)K_L}}{K_H - (1 - q)K_L})^\beta]\), i.e. \(c \geq \hat{c}\), the constraint is binding and \(P_{L}^{pa} = (1 - \frac{c}{q(1 - q)\Delta K})^{\frac{1}{\beta}} P_{L}^{pa}\). Substitute this into \((2.3.1)\), and solving for optimal \(P_H\) gives the expression in the proposition. At this value, the second derivative is negative, implying a maximum. Moreover, \(w_H = 0\) and \(w_L = \Delta K - \frac{\epsilon}{q}\) satisfy all the constraints.

**F. Proofs of Lemma 2.2, Propositions 2.7 and 2.8 for Contractible Learning**

**Proof.** Similar to the non-contractible case, it can be shown \(w_L > 0\) and \(P_L < P_H\), limited liability on \(w_H\) may not bind. However, in an optimal contract, \((2.3.3)\) is either binding or non-binding. If it is non-binding and \(w_H > 0\), there exists \(\epsilon > 0\) such that \(w_H\) and \(w_L\) can be recuced by \(\epsilon / D(P_0; P_H)\) and \(\epsilon / D(P_0; P_L)\) respectively without violating any constraints. This would make the principal strictly better off, contradicting the optimality. If \((2.3.3)\) is binding and \(w_H > 0\), \((2.3.6)\) implies \(\lambda c D(P_0; P_L) > D(P_0; P_H) w_H\). Reducing \(w_H, w_L\) and \(\lambda\) by \(w_H D(P_0; P_H)/D(P_0; P_L)\) and \(w_H D(P_0; P_H)/c D(P_0; P_L)\) respectively, all the constraints are still satisfied and the principal gets the same profit. Therefore, for every candidate optimal contract, there is a weakly better candidate with \(w_H = 0\). Note that the agent’s profitable deviation from step 3 in the proof of \((2.1)\) is not valid when learning is contractible. This proves the lemma.

First suppose \(\frac{c\lambda}{1 - q} \geq \Delta K (\frac{P_L}{P_H})^\beta\), then \(w_L = \frac{c\lambda}{1 - q}\) and the objective is independant of \(\lambda\). F.O.C.s and S.O.C.s w.r.t. \(P_H\) and \(P_L\) gives the optimal terms \(P_{H}^{pa} = \frac{\beta}{\beta - 1} K_H, P_{L}^{pa} = P_{L}^{pa} = \frac{\beta}{\beta - 1} (K_L + \frac{e}{1 - q})\), \(w_H = 0\) and \(w_L = \frac{c\lambda}{1 - q}\). \(\frac{c\lambda}{1 - q} \geq \Delta K (\frac{P_L}{P_H})^\beta\) implies \(c \geq (1 - q)\Delta K (\frac{P_L}{P_H})^\beta\). Next suppose \(\frac{c\lambda}{1 - q} \leq \Delta K (\frac{P_L}{P_H})^\beta\), the objective is now increasing in \(\lambda\) for \(\lambda \leq \frac{(1 - q)\Delta K (\frac{P_L}{P_H})^\beta}{c}\). If \(\frac{(1 - q)\Delta K (\frac{P_L}{P_H})^\beta}{c} \leq 1\), the solution is the same as the earlier case. Otherwise, \(\lambda = 1\) and the optimal terms are found to be \(P_H = \frac{\beta}{\beta - 1} (K_H - (1 - q)K_L)\), \(P_L = P_L = \frac{\beta}{\beta - 1} K_L, w_H = 0\) and \(w_L = \Delta K (\frac{qK_L}{K_H - (1 - q)K_L})\). The condition for \(\frac{(1 - q)\Delta K (\frac{P_L}{P_H})^\beta}{c} > 1\) is \(c < (1 - q)\Delta K (\frac{qK_L}{K_H - (1 - q)K_L})\) which automatically satisfies \(\frac{c\lambda}{1 - q} \leq \Delta K (\frac{P_L}{P_H})^\beta\). In summary, if \(c < (1 - q)\Delta K (\frac{qK_L}{K_H - (1 - q)K_L})\), the solution is as given in
the first part of the proposition, otherwise, it is as given in the second part of the proposition.

Finally note that if \( \lambda \) is unrestricted, the first solution above is optimal and can always be achieved as long as \( c > 0 \). Props 2.8 follows.

**G. Optimal Learning and Investments in a General Setting**

The value of the investment option is of the following general form now,

\[
W(P_0; K) = \begin{cases} 
D(P_0; P^*(K))V(P^*(K), K) & \text{for } P_0 < P^*(K) \\
V(P_0, K) & \text{for } P_0 \geq P^*(K)
\end{cases}
\]

For optional learning, \( P_e \in [P^*(K), P^*(\bar{K})] \) by the same argument. The value of the investment option at the point of learning is

\[
D(P_0; P_e) \left[ \int_{P^*(K)_{P_e}}^{P^*(K)} W(P_e; K) f(K) dK + \int_{P^*(K)_{P_e}}^{\bar{K}} W(P_e; K) f(K) dK - c \right]
\]

\[
= \int_{P^*(K)_{P_e}}^{\bar{K}} D(P_0; P^*(K))V(P^*(K), K) f(K) dK + D(P_0; P_e) \left[ \int_{P^*(K)_{P_e}}^{\bar{K}} V(P_e, K) f(K) dK - c \right]
\]

F.O.C. w.r.t. \( P_e \) gives \( \int_{P^*(K)_{P_e}}^{\bar{K}} (V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K)) f(K) dK = c \), whose LHS is increasing in \( P_e \) but is zero when \( P_e = P^*(\bar{K}) \). Thus for \( c \) small enough, there exists \( P_{e_{\text{bm}}} \) s.t. the F.O.C. is satisfied and \( I(P_{e_{\text{bm}}}) \geq c \). S.O.C. does not hold generally, but the sufficiency of the F.O.C. is implied by quasiconcavity, i.e., the value is increasing for \( P_e < P_{e_{\text{bm}}} \) and decreasing for \( P_e > P_{e_{\text{bm}}} \). To see this, suppose \( P_e > P_{e_{\text{bm}}} \), the first derivative has the same sign as \( D'(P_0; P_e) \left[ \int_{P^*(K)_{P_e}}^{\bar{K}} (V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K)) f(K) dK - c \right] \). But
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\( P^{*\text{-}1}(\cdot) \) is increasing, so

\[
\int_{K}^{P^{*\text{-}1}(P_e)} \left( V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K) \right) f(K) dK - c
\]

\[
= \int_{K}^{P^{*\text{-}1}(P^{\text{bom}}_e)} \left( V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K) \right) f(K) dK - c
\]

\[
+ \int_{P^{*\text{-}1}(P^{\text{bom}}_e)}^{P^{*\text{-}1}(P_e)} \left( V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K) \right) f(K) dK
\]

\[
> \int_{K}^{P^{*\text{-}1}(P^{\text{bom}}_e)} \left( V(P^{\text{bom}}_e, K) - \frac{P^{\text{bom}}_e}{\beta} V_1(P^{\text{bom}}_e, K) \right) f(K) dK - c
\]

\[
+ \int_{P^{*\text{-}1}(P^{\text{bom}}_e)}^{P^{*\text{-}1}(P_e)} \left( V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K) \right) f(K) dK
\]

\[
= \int_{P^{*\text{-}1}(P^{\text{bom}}_e)}^{P^{*\text{-}1}(P_e)} \left( V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K) \right) f(K) dK > 0
\]

where we’ve used (A1), the F.O.C. and the smooth-pasting property of \( P^{*}(K) \). As \( D'(P_0; P_e) < 0 \), the value is indeed decreasing for \( P_e > P^{\text{bom}}_e \). Similar argument leads to the value being increasing for \( P_e < P^{\text{bom}}_e \). Thus when \( I(P^{\text{bom}}_e) \geq c \), learning occurs beyond the optimal triggers for some high value projects, but the low value projects are subsequently exercised optimally; when \( c > I(P^{\text{bom}}_e) \), option is exercised without learning.

For mandatory learning, when \( c \leq \int_{K}^{P^{*}(K)} \left( V(P^{*}(K), K) - \frac{P^{*}(K)}{\beta} V_1(P^{*}(K), K) \right) f(K) dK \), the above analysis applies; when \( c > \int_{K}^{P^{*}(K)} \left( V(P^{*}(K), K) - \frac{P^{*}(K)}{\beta} V_1(P^{*}(K), K) \right) f(K) dK \), the option value becomes \( D(P_0; P_e) \left[ \int_{K}^{P^{*}(K)} V(P_e, K) f(K) dK - c \right] \) and the F.O.C. gives \( V(P_e, K) - \frac{\beta}{P_e} V_1(P_e, K) = c \) whose LHS is increasing in \( P_e \) without upper bound, implying there always exists a trigger \( P^{\text{bom}}_e \) that solves the F.O.C. Learning and investment take place at the same time. S.O.C. is easily verified. The optimality of the strategy is similarly established as in the proofs for Proposition 2.2 and 2.4.

In a nutshell, in optional learning when learning occurs, or in mandantory learning with \( c \leq \hat{c} \equiv \int_{K}^{P^{*}(K)} \left( V(P^{*}(K), K) - \frac{P^{*}(K)}{\beta} V_1(P^{*}(K), K) \right) f(K) dK \), there is a threshold strategy for learning with the trigger \( P_e \) being the solution to

\[
\int_{K}^{P^{*\text{-}1}(P_e)} \left( V(P_e, K) - \frac{P_e}{\beta} V_1(P_e, K) \right) f(K) dK = c. \tag{2.7.2}
\]

Only high value projects with \( K \leq (\beta - 1) P^{\text{bom}}_e / \beta \) are delayed. For mandatory learning
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with \( c > \hat{c} \), the trigger is the solution to

\[
\int_{K}^{\hat{K}} \left( V(P_e, K) - \frac{\beta}{P_e} V_1(P_e, K) \right) f(K) dK = c. \tag{2.7.3}
\]

H. Optimal Contract in a General Setting

To prove Lemma 2.3, note that once the contract terms \( \{ P(\cdot), w(\cdot), \lambda \} \) have been specified, the agent’s compensation conditional on costly learning is

\[
V^a(\hat{K}, K, P_e) = D(P_0; P(\hat{K})) [w(\hat{K}) - V(P(\hat{K}), \hat{K}) + V(P(\hat{K}), K)] - \lambda c D(P_0; P_e) \tag{2.7.4}
\]

\( \hat{K} \) is the agent’s “reported” value of \( K \), meaning the agent exercises the option at \( P(\hat{K}) \). Let \( V^a(\hat{K}, K) \) denote \( D(P_0; P(\hat{K})) V(P(\hat{K}), K) - D(P_0; P(\hat{K})) [V(P(\hat{K}), \hat{K}) - w(\hat{K})] \). This is precisely the “surplus” to the agent when the agent is type \( K \) and reports \( \hat{K} \).

The IC constraints for truth-telling are the same whether learning is contractible or not, i.e. \( V^a(K, K) \geq V^a(\hat{K}, K) \) for all \( \hat{K} \) and \( K \). The “single-crossing” assumption (A3) and the IC constraints of truth-telling give the “monotone property” (MP) (standard results proved in Edlin and Shannon (1998), Milgrom and Shannon (1994) and Topkis (1998)): in an optimal contract \( P(K) \) is a non-decreasing function of \( K \). It would be strictly increasing if \( D(P_0; P) [V(P(K), K) - w(K)] \) were differentiable, but there could be “kinks” that result in a (partial-)pooling equilibrium. It is straightforward to mimic the arguments in section 3.2 to show \( P_e = \min_K P(K) = P(\hat{K}) \), and limited liability of \( w(K) \), \( K < \hat{K} \) does not bind.

The last term in (2.7.4) does not depend on \( \hat{K} \) or \( K \). Thus the IC constraint can be written as \( K \in \arg \max_{K \in [\hat{K}, \bar{K}]} V^a(\hat{K}, K) \). As \( V^a(\hat{K}, K) \) is not necessarily differentiable in \( \hat{K} \), define \( U(K) = V^a(\hat{K}, K) \), and rewrite the IC constraint as \( \hat{K} \in \arg \max_{K \in [\hat{K}, \bar{K}]} [V^a(\hat{K}, K) - U(K)] \). Now \( V^a(\hat{K}, K) \) is differentiable in \( K \) (because \( V(P, K) \) is), thus at any point \( K \in [\hat{K}, \bar{K}] \) at which \( U'(K) \) exists, differentiating the objective gives the first-order condition.

\[
U'(\hat{K}) = \left. \frac{\partial V^a(\hat{K}, K)}{\partial K} \right|_{K=\hat{K}} = D(P_0; P(\hat{K})) V_2(P(\hat{K}), \hat{K})
\]

Under the assumption (A2) and the fact \( \hat{K} \) has bounded support on the real line (thus compact), \( V_2(P, K) \) is uniformly bounded, implying \( U(K) \) being Lipschitz continuous because it is the upper envelop of a family of Lipschitz continuous functions \( V^a(\hat{K}, K) \) indexed by
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\( \hat{K} \) and it has the same Lipschitz constant (uniform boundedness gives the same constant, proposition 6.3 Choquet (1966)). \( U(K) \) is thus absolutely continuous, and thus differentiable a.e. (Corollary 6.3.7. Cohn (1980)). Written in the integral form it is

\[
U(K) = U(\bar{K}) + \int_{\bar{K}}^{K} D(P_0; P(\hat{K})) V_2(P(\hat{K}), \hat{K}) d\hat{K}
\]

With \( w(\bar{K}) = 0 \), and after simplification, we get

\[
D(P_0; P(K))w(K) = -\int_{\bar{K}}^{K} D(P_0; P(K')) V_2(P(K'), K') dK'
\]

This is the Incentive Compatibility First Order Condition (ICFOC) in the contract literature. In fact, (MP) and (ICFOC) are equivalent to the IC constraints for truth-telling under (A3). The latter implying the former has already been established, it suffices to show with (MP) and (ICFOC), \( V^a(\hat{K}, K) - U(K) \) is maximized at \( K = \hat{K} \). To see this, \( \frac{\delta V^a(K, K)}{\delta K} - U'(K) = D(P_0; P(\hat{K})) V_2(P(\hat{K}), K) - D(P_0; P(K)) V_2(P(K), K) \) has the same sign as \( P(\hat{K}) - P(K) \) under (A3). Thus the derivative of the objective is non-positive for a.e. \( K > \hat{K} \) and non-negative for a.e. \( K < \hat{K} \). Since by (ICFOC) the objective is absolutely continuous and can be written as the integral of its derivative, it is indeed maximized at \( K = \hat{K} \), at which point its value is zero by the definition of \( U(K) \). This proves Proposition 2.3.

With this, learning trigger \( P_{pa}^a = P_{pa}(\bar{K}) \) and the principal’s program is reduced to maximizing over \( \lambda \), \( P(\cdot) \) and \( w(\cdot) \) the present value of the contract,

\[
V^p(P(\cdot), w(\cdot)) = \int_{\bar{K}}^{K} D(P_0; P(K)) [V(P(K), K) - w(K)] dF(K) - (1 - \lambda)cD(P_0; P(\bar{K}))
\]

\[
= \int_{\bar{K}}^{K} D(P_0; P(K)) [V(P(K), K) + V_2(P(K), K) \frac{F(K)}{f(K')} f(K) dK - (1 - \lambda)cD(P_0; P(\bar{K}))
\]

subject to (MP) and, when learning is non-contractible,

\[
\min_{\bar{K}} \int_{\bar{K}}^{K} [V^a(K, K) - V^a(\hat{K}, K')] dF(K) \geq cD(P_0; P(\bar{K})); \quad (2.7.5)
\]
or when learning is contractible,

\[ \lambda c D(P_0; P(K)) + \int_{K} D(P_0; P(K)) V_2(P(K), K) F(K) dK \leq 0. \tag{2.7.6} \]

when learning is contractible.

When learning is non-contractible and \( \lambda \leq 1 \), IR constraint for effort cannot bind because the RHS of (2.7.5) is positive. To see this, it suffices to show \( \int_{K} V^a(K, K') dF(K) \geq 0 \). Suppose otherwise, then \( w(K) - V(P(K), K') + \int_{K} V(P(K), K') dF(K) < 0 \) which cannot possibly hold given the limited liability \( w(K) \geq 0 \) and the fact \( V_2 < 0 \), yielding a contradiction. It follows that limited liability on \( w_H \) binds. Given this, the principal’s problem when learning is non-contractible reduces to

\[
\max_{P(\cdot), w(\cdot)} \int_{K} D(P_0; P(K)) [V(P(K), K) - w(K)] dF(K) - (1 - \lambda) c D(P_0; P(K)) 
\]

subject to

\[
D(P_0; P(K)) w(K) = - \int_{K} D(P_0; P(K')) V_2(P(K'), K') dK' \tag{2.7.8}
\]

\[
w(K) = 0 \tag{2.7.9}
\]

\[
\min_{K} \int_{K} [V^a(K, K) - V^a(K, K')] dF(K) \geq c D(P_0; P(K)) \tag{2.7.10}
\]

Using integration by parts and (2.7.8), and assume the density of \( F(K) \) exists,

\[
\int_{K} D(P_0; P(K)) w(K) dF(K) = D(P_0; P(K)) w(K)
\]

\[= - \int_{K} D(P_0; P(K)) V_2(P(K), K) F(K) dK \]

\[= - \int_{K} D(P_0; P(K)) V_2(P(K), K) \frac{F(K)}{f(K)} f(K) dK \]

In addition, the optimal contract needs to have \( P(K) \) non-decreasing in \( K \) to be consistent with (MP). An optimal contract may not exist, and can only be explicitly solved case by case by imposing further structure on \( F(K) \) and \( V(P, K) \).
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The problem is simplified when learning is contractible. For every optimal contract, there is a potentially better optimal contract candidate with \( w(K) = 0 \), by similar reasoning as in the binary distribution case. For sufficiently large \( c \) or if \( \lambda \) is unrestricted, the principal always increases \( \lambda \) until (2.7.6) binds, and the problem becomes

\[
\max_{P(U)} \int_{K}^{K} D(P_0; P(K)) V(P(K), K) f(K) dK - cD(P_0; P(K))
\]

subject to \( P(K) \) being non-decreasing in \( K \). This leads to the following proposition

**Proposition 2.9.** For contractible learning, if the learning cost is high or the agent is not liquidity-constrained, the optimal contract achieves “no distortion” in learning and investment timing, and maximizes social welfare. The optimal contract involves learning trigger \( P^*_e \) that solves,

\[
\max_{P_e} \left\{ \int_{(P^*)^{-1}(P_e)}^{(P^*_e)^{-1}(P_e)} D(P_0; P(K)) V(P_e, K) f(K) dK + \int_{(P^*)^{-1}(P_e)}^{K} D(P_0; P^*(K))[V(P^*(K), K)] f(K) dK - cD(P_0; P_e) \right\} \tag{2.7.11}
\]

and the investment triggers are

\[
P^*_e(K) = \begin{cases} 
    P^*_e & \text{for } K \in [K, (P^*_e)^{-1}(P^*_e)] \\
    P^*(K) & \text{for } K \in [(P^*)^{-1}(P^*_e), K]
\end{cases}
\]

where \( P^*(\cdot) \) is the first-best investment triggers with full information.

This conclusion highlights the importance of contracting on learning and reducing cash-constraints when the agent learns, in achieving socially optimal level of investments. The learning and investments would be the same as in the benchmark case absent agency issues. Note (MP) is also satisfied. It thus is possible to achieve “no distortion” when \( \lambda \) is unrestricted. The optimal contract corresponds to a partial-pooling equilibrium.

Now if \( \lambda \) is restricted and \( c \) is small, (2.7.6) does not bind and the “no distortion” cannot be achieved. Set \( \lambda = 1 \) in such scenarios, and solve by point-wise maximization of virtual surplus,

\[
V_1(P, K) + V_2(P, K) \frac{F(K)}{f(K)} - \frac{\beta}{P} \left( V(P, K) + V_2(P(K), K) \frac{F(K)}{f(K)} \right) = 0 \tag{2.7.12}
\]
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\[ D(P_0; P) D''(P_0; P) + 2D'(P_0; P) \leq 0 \] gives S.O.C. Note the term for “virtual surplus”

\[ VS(P, K) = \left( \frac{P_0}{P(K)} \right)^\beta [V(P(K), K) + V_2(P(K), K)] \frac{F(K)}{f(K)} \]

Due to asymmetric information, the principal has to pay a rent of \(-\frac{F(K)}{f(K)}\) upon investment. A sufficient condition for (MP) is that the cross-partial of the “virtual surplus” being strictly positive. \(D(P_0; P)V(P, K)\) has strictly positive cross-partial by the “single-crossing” assumption (A3), derivative of \(D(P_0; P)V_2(P, K)\) w.r.t. \(P\) is,

\[ [D(P_0; P)V_{12}(P, K) + D'(P_0; P)V_2] \frac{F(K)}{f(K)} = D(P_0; P)[V_{12}(P, K) - \frac{\beta}{P} V_2(P, K)] \frac{F(K)}{f(K)} \]

Thus a set of sufficient conditions are: 1. \(V_{12}(P, K) - \frac{\beta}{P} V_2(P, K)\) is non-decreasing in \(K\) and 2. the inverse hazard rate \(F(K)/f(K)\) is non-decreasing. The first condition is satisfied for the case of uncertainty in production capacity for example. The second condition is satisfied by many common distributions.

**Proposition 2.10.** For contractible learning, if the agent is liquidity-constrained and learning cost is small, learning and investment timings are distorted. Assume \(V_{12}(P, K) - \frac{\beta}{P} V_2(P, K)\) and the inverse hazard rate \(F(K)/f(K)\) are non-decreasing in \(K\). The optimal contract involves \(\lambda = 1\) and solution is characterized by

\[ V_1(P, K) + V_{12}(P, K) \frac{F(K)}{f(K)} - \frac{\beta}{P} (V(P, K) + V_2 \frac{F(K)}{f(K)}) = 0 \quad (2.7.13) \]

where \(C\) is defined by the set of \(c\) such that (2.7.6) does not bind.

If (MP) is violated, this becomes an optimal control problem with bounded control, and one can use the standard “ironing” technique to obtain a solution, but this is beyond the scope of this study. In general the solution is either a fully separating equilibrium or a partial-pooling equilibrium.

Even for the non-contractible learning, the following characterization holds.

**Proposition 2.11.** When learning is non-contractible (\(\lambda\) restricted), the optimal contract in Proposition 2.10 applies if (2.7.5) is not binding. Otherwise, the optimal contract involves
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\( \lambda = 1 \) and the following optimization problem.

\[
\max_{P(\cdot)} \left\{ \int_{K}^{\hat{K}} D(P_0; P(K)) V(P(K), K) f(K) dK - c D(P_0; P(\hat{K})) \\
- D(P_0; P(\hat{K})) \left[ V(P(\hat{K}), K) - V(P(\hat{K}), \hat{K}) \right] - \int_{K}^{\hat{K}} D(P_0; P(\hat{K})) d\hat{K} \right\}
\]

subject to (MP), where \( \hat{K} \) solves \( \arg\max_{P(\cdot)} P(p) \).

Going back to contractible learning, when there is distortion of investment timing and suppose (MP) is satisfied (no “ironing” needed), \( P_{pa}(\hat{K}) = P^*(\hat{K}) < P^{blm}(\hat{K}) \). Implicit differentiation of \( (2.7.12) \) gives \( \frac{\delta P_{pa}(\hat{K})}{\delta K}|_{K=\hat{K}} = \frac{\delta P^*(\hat{K})}{\delta K}|_{K=\hat{K}} \) and \( \frac{\delta P_{pa}(\hat{K})}{\delta K}|_{K>\hat{K}} \), implying \( P_{pa}(\hat{K}) > P^*(\hat{K}) = P^{blm}(\hat{K}) \). Since \( P^{blm}(\cdot) \) and \( P_{pa}(\cdot) \) are continuous schedules and monotone in \( K \), there exists a \( \tilde{K} \) such that \( P_{pa}(K) < P^{blm}(K) \) for all \( K < \tilde{K} \) and \( P_{pa}(K) > P^{blm}(K) \) for all \( K > \tilde{K} \). This establishes the general result that learning is weakly accelerated and investments are weakly delayed for low value projects \( (K > \tilde{K}) \) but weakly accelerated for high value projects \( (K < \tilde{K}) \), all relative to the benchmark case of incomplete information absent agency. In particular, the delays relative to the full information case are ameliorated. A similar argument would work for non-contractible learning but there could be discontinuities that have to be dealt with case by case, and is thus left out of the discussion, except for the numerical illustration.

**Corollary 2.1.** In the benchmark case with incomplete information, learning occurs beyond the optimal investment trigger of the project of the highest option value, causing delays in high value projects in general. Agency weakly accelerates learning and investments in high value projects (thus ameliorates the inefficient delays), but weakly delays investments in low value projects, regardless of the contractibility of learning.

Taking the limit as \( c \to 0 \) gives complete characterization of delegated real options with pure adverse selection. Investments other than the “best” type \( K \) are delayed compared to the case of full information, in agreement with [Maeland (2002)].

The above results can be generalized to many other linear diffusion processes \( dP_t = \alpha(P_t) dt + \sigma(P_t) dB_t \) with the corresponding expected discount factor \( D(P_0; P) \) as the solution \( W \) to the ODE \( \frac{1}{2} \sigma(P_t)^2 W_{PP} + \alpha(P_t) W_P - r W = 0 \) with the boundary conditions \( D(P_0; P_0) = 1 \) and \( \lim_{P \to \infty} D(P_0; P) = 0 \), and technical assumptions including
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\[ D'(P_0; P) < 0, \quad D(P_0; P)D'(P_0; P) + 2D'(P_0; P) \leq 0, \quad \text{and} \quad D(P_0; P)V(P, K) \text{ having positive cross-partial.} \]

I. Derivation of a Separating Equilibrium in the Signaling Game

Using the results in the general model, I outline the argument why the candidate equilibrium is valid when the following condition holds:

\[ 0 < B_H(A_L B_H + 1 - A_L) \beta - 1 - B_L(A_H B_L + 1 - A_H) \beta - 1 < \frac{K_A}{\beta - 1} \left[ \frac{(K_B)}{(K_A)} - \left( \frac{qK_B}{c + qK_A} \right)^{\beta} \right] \]

Note firm A would only learn if the cash flow is below \( \frac{\beta}{\beta - 1} K_H A_L \) because otherwise learning has no value. It would not learn if the cash flow is below \( \frac{\beta}{\beta - 1} K_H A_H \) due to the time value of money. Given firm B’s investment strategy in equilibrium, suppose A finds out that \( A = A_L \). If A invests right away, B would believe \( \lambda = L \) and the payoff to A is \( A_LP_e - K_A - LD(P_e; P_BL)P_BL \), however if A waits to invest at some \( P_AL \), then the payoff to A is \( D(P_e; P_AL)(A_L P_AL - K_A) - HD(P_e; P_BH)P_BH \). We maximize over \( P_AL \) and concludes \( P_AL = \frac{\beta}{\beta - 1} K_A A_L > P_e \). The parameter assumptions imply waiting until \( P_AL \) is better. Similarly, if A finds out that \( A = A_H \), investing right away yields \( A_HP_e - K_A - LD(P_e; P_BL)P_BL \), and waiting to invest at some \( P_AH \), then the payoff to A is \( D(P_e; P_AH)(A_H P_AH - K_A) - HD(P_e; P_BH)P_BH \), we get \( P_AH = \frac{\beta}{\beta - 1} K_A A_H \leq P_e \), thus firm A invests at \( P_e \) for some infinitely small \( \epsilon \), the parameter assumptions imply that investing right away is better. Going back one step, we get the optimal learning strategy is at \( P_e = \frac{\beta}{\beta - 1} \frac{c + qK_A}{qA_H} \). Given firm A is going to behave in the above manner, one can verify that B’s investment strategy is indeed optimal. Finally, since this is a pure strategy equilibrium, B’s belief upon observing the joint signal of learning and investing (or the lack of) satisfies Bayesian updating.
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