ALGEBRAIZATION THEOREMS FOR COHERENT SHEAVES ON STACKS

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

David Benjamin Weixuan Lim
August 2020

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Ravi Vakil, Primary Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Jarod Alper, Co-Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Richard Taylor

Approved for the Stanford University Committee on Graduate Studies.

Stacey F. Bent, Vice Provost for Graduate Education

This signature page was generated electronically upon submission of this dissertation in electronic format. An original signed hard copy of the signature page is on file in University Archives.

Abstract

Given a moduli problem, two important questions one can ask are the following: Is there an algebraic stack representing the moduli problem? If so, does it have a coarse space? A necessary step to answering these questions in the post-GIT era is to prove an algebraization result akin to formal GAGA for proper schemes. In this dissertation, we will present two such results. The first result (Chapter 2) is Grothendieck's existence theorem for relatively perfect complexes on an algebraic stack. This generalizes work of Lieblich in the setting of algebraic spaces. The second result (Chapter 3) is coherent completeness of BG (with G a split reductive group) and $[\mathrm{SL}_d \backslash \mathbf{A}^d]$. The results of this chapter are joint work with Jack Hall and are an important step to extending the étale slice theorem of Alper-Hall-Rydh to positive characteristic. Finally, in future work with Jack Hall and Jarod Alper, we prove that an arbitrary $[G \backslash \operatorname{Spec} A]$ with G reductive is coherently complete.

Acknowledgments

My dream in life of doing math like Grothendieck would never have been possible without five incredible people: My advisors Ravi Vakil and Jarod Alper, my teacher and collaborator Jack Hall, my undergraduate advisor David Smyth, and my mentor Dr. Chong Hon Yew.

Ravi, when I first came to grad school I thought that being a disciple of Grothendieck meant being able to say lots of technical jargon. However, as time went on I learned from you that it is just as important to be able to think of math in terms of broad, overarching philosophies. I have come to the realization that Grothendieck was as much of a philosopher and visionary as a mathematician with supreme technical ability. Thanks for all your guidance during my time here at Stanford.

Jarod, growing up I was told by society that I should strive to own a Tesla, a mansion in Beverly Hills, or a million dollars in FAANG stocks. However, as I grew older, I realized that this was completely orthogonal to my view of what the world should be. I wanted to do math like Grothendieck - or even better - *with* Grothendieck. But Grothendieck is not here anymore, he is up there with enlightened beings such as Gauss, Hilbert and Noether. So the only way I could awaken to the Dhamma was to do math with Grothendieck's disciples. I am very fortunate to have been given the opportunity to extend the boundary of the moduli space of knowledge with you and Jack. Thanks for everything.

Jack, you have always been a source of great inspiration to me. From the day I first learned about schemes from you, I was blown away by your perspective and philosophy of doing math, many of which are a continuation of Grothendieck's: The relative point of view, the idea of dévissage, the soaking the walnut analogy, to name a few. The Dhamma still lives today because of technical masters such as yourself.

David, I am thankful for the enormous amount of time and effort that you put into developing my mathematical maturity as an undergrad. You and Jarod brought me to the other shore.

Dr. Chong, I'll never forget the countless nights after stargazing where we'd sit and talk in a mamak restaurant about math, physics and astronomy *ad infinitum*. You were the first person to ignite my fire and passion for math all those years ago. Just one thing - I don't understand why I still address you by your academic title, especially since I now have a PhD too!

I am very grateful to Brian Conrad for many exciting mathematical discussions during my time here at Stanford. I would never have arrived at the main descent theorems of Chapter 3 without his paper on formal GAGA for Artin stacks. Brian, your contributions to algebraic geometry and number theory, as well as style of mathematical writing that leaves no stone unturned, will always be invaluable to generations of mathematicians to come.

I would like to thank Michael Kemeny for his support during my time at graduate school, and also Richard Taylor, Shamit Kachru and Eric Larson for serving on my committee and taking part in my thesis defense.

The Stanford math community has been amazing. I learned an incredible amount of abstract highfalutin algebraic geometry from Aaron Landesman, Arpon Raksit, Nikolas Kuhn, Libby Taylor, Daniel Dore and Jesse Silliman. Special thanks to Gretchen Lantz, Debbie Lemos and Mandy Ferguson for all their help - especially during COVID-19.

There are so many people I'd like to thank that have made my time at Stanford awesome. Bogdan Zavyalov, Zev Rosengarten, Brian Lawrence, Mark Sellke, Yang Liu, Dat Pham Nguyen, Juan Carlos Ortiz Rhoton, Daylan Siemann, Anela Wong, Kurt Russell and Ross Conroy.

Bogdan, you're probably the closest incarnation of Grothendieck I know of personally. Zev, I'll never forget how you'd do all these crazy spectral sequence calculations in your head while peeling the crust off those burger buns at Arrillaga. Brian, you always make fun of me (especially the day I got confused that 69 was prime!). Nonetheless, I got my revenge when I smoked you running up that slope in the dish. Mark and Yang, I would never have looked beyond pure math without you guys. Dat, you're really funny and I learned a lot from you. Juan, we both love football but I'm sorry I'll never support Real (unless they play Liverpool, Manchester City or Barca).

Daylan and Anela, I'll never forget the day we went up Mount Tam with that incredible view of the bay. In a parallel universe, you know I'd get barreled at Pipeline with probability $0.\overline{9}$. Kurt, I know I made a fool of myself with the chicken in the boiled water thing, but I promise you my cooking skills are way better than that. Ross, I know that we'll always be Manchester United till we die.

My PhD would not have been possible without the Malaysian community at Stanford, as well as my relatives in the Bay Area, Ching Yin, Chim Poh, Sin Ying, Sin Khoon, and Sian Yong. In addition, I lived with two fantastic roommates Anex Jose and Anita Kulkarni during my time here. At the end of a long day, I always felt like I would return to a place called home.

Finally, I am eternally grateful to my parents, for their unwavering love and kindness, both in good times and bad times.

To Mum and Pua, for their ∞-love that is concentrated in $D(\mathrm{Ben})^{\geq 0} \cap D(\mathrm{Ben})^{\leq 0}$.

Contents

List of Figures

Chapter 1

Introduction

Modern algebraic geometry is an extremely technical subject. It is very easy to lose sight of the forest for the trees. As a result, the primary focus of this introduction will be on a particular philosophy of doing things in algebraic geometry, at the cost of mathematical rigor. Consequently, many mathematical statements in this introduction are only true up to some $\epsilon > 0$. We hope that this enables a much broader audience to have a taste of the modern perspective on moduli theory.

1.1 The beginning of time

I always say that algebraic geometry - despite all the highfalutin stuff like derived categories and stacks - is fundamentally about the study of curves or surfaces defined by polynomial equations. Therefore, we will start with the example of elliptic curves, the theory of which is central to so many of the most exciting ideas in number theory and algebraic geometry.

Let $\mathbf{C}$ denote the complex numbers. An elliptic curve E over $\mathbf{C}$ is the zero locus of an equation in Legendre form

$$E_\lambda \colon y^2 = x(x-1)(x-\lambda),$$

for some $\lambda \in \mathbf{C} - \{0,1\}$. Now given the description of an elliptic curve in terms of a single parameter λ, it is only natural to ask the following question: How can we classify all elliptic curves? A starting guess would be that they are simply classified by the parameter λ. Unfortunately, this is not true because an elementary change of variables shows that E_λ is always isomorphic to E_μ, where μ is any of the six values

$$\left\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\right\}. \tag{1.1.1}$$

This suggests that a second, more refined approach to classifying all elliptic curves is to find a function of λ that is *invariant* under those six symmetries above. Thankfully, there is such a function, namely the j-invariant

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

The j-invariant has been studied since the 1800s and arose from the work of Felix Klein on tessellations of the upper half-plane $\mathbf{H}$.

Now we claim that the assignment

$$\begin{aligned} \{\text{Elliptic curves } E_\lambda\} &\to \mathbf{C} \\ E_\lambda &\mapsto j(\lambda) \end{aligned} \tag{1.1.2}$$

is bijective. We establish this as follows. Suppose two elliptic curves E_λ and E_μ have $j(\lambda) = j(\mu)$. Then μ must lie in the fiber over $j(\lambda)$. Since j is a rational function of degree 6 that satisfies the symmetry (1.1.1), it follows that μ must be equal to one of the six values (1.1.1). Consequently, $E_\lambda \simeq E_\mu$ which establishes that (1.1.2) is injective. To show it is surjective, we must show for any $z_0 \in \mathbf{C}$ that there exists an elliptic curve E_{λ_0} with $j(\lambda_0) = z_0$. This amounts to solving a degree 6 polynomial equation in $\mathbf{C}$, always possible by the deep and beautiful fundamental theorem of algebra.

Thus, it appears that elliptic curves should be classified by the complex numbers, or in algebro-geometric terms the affine line $\mathbf{A}^1$. Unfortunately, this is not true because the bijection (1.1.2) is only true at the level of $\mathbf{C}$-valued points, i.e. the j-invariant is only a pointwise criterion for checking isomorphy between elliptic curves. In other words, there exists a family of elliptic curves E over $\mathbf{C}^*$ whose fibers all have the same j-invariant, but which is not the trivial family. For a concrete example of such a family, consider

$$E_t \colon y^2 = x^3 + t$$

where $t \in \mathbf{C}^*$ is allowed to vary. The fibers of this family have the same j-invariant because they are all isomorphic to $y^2 = x^3 + 1$. Therefore, if E_t is trivial, we would have $E_t \simeq E'_t$, where E'_t is the family

$$E'_t \colon \{y'^2 = x'^3 + 1\} \times \mathbf{C}^*.$$

What must such an isomorphism look like? At the very least, being an isomorphism of families of elliptic curves, it must preserve the point at infinity. Therefore, the rational function $x' \in K(E'_t)$ with a double pole at infinity must pullback to a rational function in $K(E_t)$ with a double pole at infinity. By the Riemann-Roch theorem, the space of such functions is spanned by 1 and x, and therefore the isomorphism $E_t \simeq E'_t$ sends x' to $ax + b$ for some $a, b \in \mathbf{C}[t, t^{-1}]$. Similarly, $y' \in K(E'_t)$ has a triple pole at infinity and must pullback to a linear combination $cy + dx + e$ for some $c, d, e \in \mathbf{C}[t, t^{-1}]$.

We now deduce a contradiction as follows. First, the function y' is sent to its negative under the standard involution on E'_t. Since the isomorphism $E_t \simeq E'_t$ must commute with this involution, it follows that $d = e = 0$. Second, the equations $y^2 = x^3 + t$ and $(cy)^2 = (ax + b)^3 + 1$ must be equal up to a unit in $\mathbf{C}[t, t^{-1}]$. In other words, there exists $u \in \mathbf{C}[t, t^{-1}]^\times$ such that

$$y^2 - x^3 - t = u(cy)^2 - u(ax + b)^3 - u.$$

Now expand the right hand side above and compare coefficients. We get

$$uc^2 = 1, \tag{1.1.3}$$
$$ua^3 = 1, \tag{1.1.4}$$
$$3ua^2b = 0, \tag{1.1.5}$$
$$3uab^2 = 0, \tag{1.1.6}$$
$$ub^3 + u = t. \tag{1.1.7}$$

Equation (1.1.4) implies that $a \neq 0$, and hence by (1.1.5) that $b = 0$. Then, by (1.1.7) we deduce that $u = t$. In turn, (1.1.3) and (1.1.4) read respectively as

$$\begin{aligned} tc^2 &= 1, \\ ta^3 &= 1. \end{aligned}$$

In other words, t must admit a 6-th root in $\mathbf{C}[t, t^{-1}]$ which is evidently impossible.

In fact, the family E_t above shows there is is no scheme $\mathscr{M}_{1,1}$ that parametrizes all elliptic curves. What does it mean for $\mathscr{M}_{1,1}$ to parametrize all elliptic curves? Well, by Grothendieck's relative point of view, this means for every scheme S that

$$\mathrm{Hom}(S, \mathscr{M}_{1,1}) = \{\text{Families of elliptic curves } E \to S\}.$$

Suppose for the moment that such a scheme $\mathscr{M}_{1,1}$ exists. Then by Yoneda's Lemma, the identity morphism $\mathscr{M}_{1,1} \to \mathscr{M}_{1,1}$ would give rise to a family of elliptic curves $\mathscr{E} \to \mathscr{M}_{1,1}$ which is universal in the following sense: For any family of elliptic curves $E \to S$, there exists a unique map $S \to \mathscr{M}_{1,1}$ such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathscr{M}_{1,1} \end{array}$$

is Cartesian. In particular, the family of elliptic curves $E_t \to \mathbf{C}^*$ above gives rise to a map $\mathbf{C}^* \to \mathscr{M}_{1,1}$ which is constant because all the fibers of $E_t \to \mathbf{C}^*$ are isomorphic. Thus E_t is isomorphic to the pullback of $\mathscr{E}$ along a constant map, contradicting the fact that it is not trivial.

Having established that there is no scheme representing $\mathscr{M}_{1,1}$, the moduli problem of elliptic curves, in what sense then is it an algebro-geometric object? There are two ways we can try to make sense of $\mathscr{M}_{1,1}$:

1. Find a scheme U with a smooth surjective map $U \to \mathscr{M}_{1,1}$.

2. Find a scheme $M_{1,1}$ which admits a map $\mathscr{M}_{1,1} \to M_{1,1}$ that is a bijection on $\mathbf{C}$-points.

A positive answer to (1) would enable us to ask questions of geometry about $\mathscr{M}_{1,1}$. For instance, we could say that $\mathscr{M}_{1,1}$ is smooth over $\operatorname{Spec}\mathbf{C}$, if the same is true of U. A positive answer to (2) would give us the "best approximation" to $\mathscr{M}_{1,1}$ by a scheme, in the sense that even though $\mathscr{M}_{1,1}(S) \to M_{1,1}(S)$ is *not* a bijection on arbitrary S-valued points, it will be when $S = \operatorname{Spec}\mathbf{C}$. We now sketch why (1) and (2) have a positive answer.

Theorem 1.1.1. *There exists a scheme U with a smooth surjective morphism $U \to \mathscr{M}_{1,1}$.*

Proof. Consider the Legendre family $y^2 = x(x-1)(x-\lambda)$ over the λ-line $\mathbf{A}^1_\lambda \coloneqq \mathbf{A}^1 - \{0,1\}$. By definition of $\mathscr{M}_{1,1}$, this gives a map

$$\mathbf{A}^1_\lambda \to \mathscr{M}_{1,1}$$

which we claim is smooth and surjective. The surjectivity of this map can be shown as follows. Let $\pi\colon E \to S$ be a family of elliptic curves, i.e. a proper flat morphism of schemes whose fibers are elliptic curves together with a section $e\colon S \to E$. By Riemann-Roch and the theorem on cohomology and base change, the sheaves $\pi_*\mathscr{O}(e)$, $\pi_*\mathscr{O}(2e)$ and $\pi_*\mathscr{O}(3e)$ are locally free of ranks 1, 2 and 3 respectively. Thus Zariski locally on S, we can find bases for these vector bundles

$$1 \in \pi_*\mathscr{O}(e), \qquad 1, x, \in \pi_*\mathscr{O}(2e), \qquad 1, x, y \in \pi_*\mathscr{O}(3e).$$

A standard argument [Har77, Theorem 4.6] now shows that using these sections, we can embed E as a closed subscheme of $\mathbf{P}^2_S$ cut out by a Legendre equation $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \Gamma(S, \mathscr{O}_S)$ satisfying the condition that $\lambda, \lambda - 1 \in \Gamma(S, \mathscr{O}_S)^\times$. This says that locally on S, the point $(E \xrightarrow{\pi} S) \in \mathscr{M}_{1,1}$ comes from a point of $\mathbf{A}^1_\lambda$, proving that $\mathbf{A}^1_\lambda \to \mathscr{M}_{1,1}$ is surjective.

Finally, the morphism $\mathbf{A}^1_\lambda \to \mathscr{M}_{1,1}$ is not just smooth, but moreover étale. To see this, it is sufficient to show that the map on tangent spaces (at any point $\lambda_0\colon \operatorname{Spec}\mathbf{C}[\epsilon] \to \mathbf{A}^1_\lambda$)

$$\mathbf{A}^1_\lambda(\mathbf{C}[\epsilon]) \to \mathscr{M}_{1,1}(\mathbf{C}[\epsilon])$$

is an isomorphism. By a deformation theory calculation, both of these $\mathbf{C}$-vector spaces are 1-dimensional, and thus it is enough to show that the map on tangent spaces is non-zero. This follows from the fact that the pullback of the Legendre family along $\lambda_0\colon \operatorname{Spec}\mathbf{C}[\epsilon] \to \mathbf{A}^1_\lambda$, i.e. the family $y^2 = x(x-1)(x - \lambda_0^*(\lambda))$ over $\operatorname{Spec}\mathbf{C}[\epsilon]$, is not the trivial family. $\square$

Theorem 1.1.2. *There exists a scheme $M_{1,1}$ with a map $\mathscr{M}_{1,1} \to M_{1,1}$ that is a bijection on* $\mathbf{C}$*-points.*

Proof. Let $E \to S$ be a family of elliptic curves. We know that Zariski locally on S, E admits a description as a Legendre equation $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \Gamma(S, \mathscr{O}_S)$ satisfying the condition that $\lambda, \lambda - 1 \in \Gamma(S, \mathscr{O}_S)^\times$. The j-invariant of such a Legendre equation is compatible with intersections between different open neighborhoods, and thus glues to give well-defined morphism $\mathscr{M}_{1,1} \to \mathbf{A}^1$. The statement that this is a bijection on $\mathbf{C}$-valued points is exactly the fact that over $\mathbf{C}$, $j(E) = j(E')$ if and only if $E \simeq E'$, and that for any complex number z_0, there is an elliptic curve E_0 with $j(E_0) = z_0$. □

In the terminology of modern algebraic geometry, (1) says that the moduli stack of elliptic curves $\mathscr{M}_{1,1}$ is *algebraic*, while (2) says that $\mathscr{M}_{1,1}$ admits a coarse moduli space, namely the j-line $\mathbf{A}^1$. The reader will note that in showing (1) and (2), we used very particular properties of elliptic curves, such as the fact that an elliptic curve is described by an explicit Legendre equation. It is not at all clear how (1) and (2) should be answered in the case of an *arbitrary* moduli problem.

We now state two questions that will serve as our guide for the rest of this dissertation.

Question 1.1.3. Let $\mathscr{X}$ be a moduli stack. Does there exist a scheme U with a smooth surjective morphism $U \to \mathscr{X}$? In other words, is $\mathscr{X}$ *algebraic*?

Question 1.1.4. Suppose now that $\mathscr{X}$ is also *algebraic*. Does $\mathscr{X}$ admit a *coarse moduli space*? In other words, does there exist an algebraic space X with a morphism $\pi\colon \mathscr{X} \to X$ satisfying:

(i) π is universal for maps to algebraic spaces.

(ii) For every algebraically closed field k, π induces a bijection on k-points.

The outline for the rest of this introduction is as follows. We first review a general philosophy pioneered by Grothendieck for proving theorems in algebraic geometry. Then, we explain briefly at a high level how the "reverse" of this philosophy enables us to answer Questions 1.1.3 and 1.1.4. We then turn to a particular step in this philosophy (algebraization) that is the central theme of everything in this dissertation. Next, we show how algebraization is used to prove that stacks are algebraic (Question 1.1.3), or to construct coarse moduli spaces (Question 1.1.4). We then arrive at the beef of this dissertation, i.e. our results. Finally, we compare our work with that of Jack Hall [Hal18] and explain why his methods do not apply to our situation.

1.2 Soaking the walnut

Suppose we view the act of proving a theorem as opening a walnut. À la Grothendieck, there are two ways to do this. We may keep hitting it with a hammer until it ultimately cracks. Alternatively, we could soak it in water and let the shell soften over time.[1] Eventually, we arrive at the point where the shell is so soft that our bare hands are enough to open it. This second approach is Grothendieck's philosophy: Keep making a series of small reductions, each step being relatively trivial, until the result to be proven "falls out."

In practice, this translates to the following. Let $f: X \to S$ be a morphism of Noetherian schemes, satisfying certain hypotheses such as being finite type or separated. Under these hypotheses, suppose we wish to prove something about f. For instance, we may suppose that f is proper and quasi-finite, and we wish to prove that it is actually finite (Zariski's main theorem). Alternatively, we may suppose that f is proper, and we wish to prove that f_* preserves coherence. Then the analogy of soaking a walnut translates to repeatedly "shrinking" S until we arrive at the situation where S is the spectrum of a field, in which case everything is clear.

More precisely, we perform the following series of reductions. First, the statement that we wish to prove is usually Zariski local on S. Then, because the local ring at any point on S is a direct limit of the Zariski open neighborhoods around it, by spreading out we may assume that S is the spectrum of a Noetherian local ring A. Now the scheme S consists of a single closed point. However, it is not "small" enough in the sense the point might be non-reduced, or that S might have other non-closed points. Thus the question is: How do we shrink S further?

To this end, consider the following types of points, each of which is "smaller" than an arbitrary Noetherian local ring.

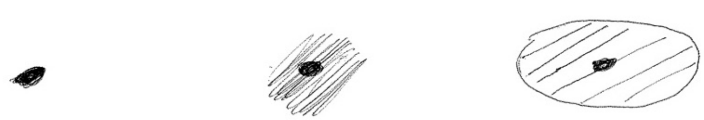

Figure 1.1: Three types of points in algebraic geometry

On the left, we have a bona fide point, namely the spectrum of a field. In the middle, we have a point with order n fuzz around it. This is the spectrum of an Artinian local ring. Finally on the right, we have a disk. In the land of schemes, these points correspond respectively to $\operatorname{Spec} \mathbf{C}$, $\operatorname{Spec} \mathbf{C}[\epsilon_n]$ and $\operatorname{Spec} \mathbf{C}[\![x]\!]$, where ϵ_n is nilpotent of order n. Note that there is a canonical relation

[1] A fellow grad student, Felipe Hernandez, tried this in his office. Unfortunately the walnut turned moldy in about a week and had to be thrown away.

between these points, expressed by the fact that we have maps

$$\operatorname{Spec}\mathbf{C} \to \operatorname{Spec}\mathbf{C}[\epsilon_n] \to \operatorname{Spec}\mathbf{C}[\![x]\!]$$

which correspond geometrically to the fact that $\operatorname{Spec}\mathbf{C}$ is the origin in the disk $\operatorname{Spec}\mathbf{C}[\![x]\!]$, while $\operatorname{Spec}\mathbf{C}[\epsilon_n]$ is the origin with fuzz of order n around it. We also make the very important observation that if we let $n \to \infty$, then eventually our fuzz will fill out the disk. This is true because $\mathbf{C}[\![x]\!]$ is the inverse limit of its thickenings $\mathbf{C}[\epsilon_n]$, and therefore we have $\operatorname{Spec}\mathbf{C}[\![x]\!] \simeq \varinjlim \operatorname{Spec}\mathbf{C}[\epsilon_n]$.

Now recall that our goal is to shrink S until it is an honest to God point. How can we leverage our discussion above on types of points? Suppose for the moment that we can shrink S to a disk $\operatorname{Spec}\mathbf{C}[\![x]\!]$. Then because $\operatorname{Spec}\mathbf{C}[\![x]\!] \simeq \varinjlim \operatorname{Spec}\mathbf{C}[\epsilon_n]$, we hope that by some kind of limit argument, reduce to the case of a single $\operatorname{Spec}\mathbf{C}[\epsilon_n]$, i.e. a fuzzy point. We will explore this idea of "taking a limit" more rigorously when we introduce the concept of *algebraization* later. Finally, to shrink from a fuzzy point to an actual point, we will use deformation theory. In summary, we perform the following series of reductions:

1. Any scheme $S = \operatorname{Spec} A$ with A Noetherian local admits a canonical map from a disk, namely the completion morphism $\operatorname{Spec}\widehat{A} \to \operatorname{Spec} A$.[2] Since this morphism is faithfully flat, by descent theory we may assume that A is a complete Noetherian local ring.

2. If $(A, \mathfrak{m})$ is a complete Noetherian local ring, then $A \simeq \varprojlim_n A/\mathfrak{m}^{n+1}$. Therefore, by an algebraization result such as formal GAGA (see Section 1.4), we may assume that A is an Artinian local ring.

3. Finally, use deformation theory to reduce from the case of an Artinian local ring to the case of a field.

We illustrate this "soaking the walnut" strategy with a brief proof of Zariski's main theorem.

Theorem 1.2.1. *A proper, quasi-finite morphism of Noetherian schemes $f\colon X \to S$ is finite.*

Proof. As usual, we easily reduce to the case that S is affine, in which case it is now enough to prove that X must be affine. Indeed, by [AM69, Exercise 3.35] we deduce that f is integral, a fortiori finite as desired. To this end, we use Serre's criterion for a scheme to be affine. Let $\mathscr{F}$ be a coherent sheaf on X. We must show that $R^1 f_* \mathscr{F}$ is zero. The condition of being zero can be checked on stalks, and therefore we may assume that S is the spectrum of a Noetherian local ring A.

Now we do Step 1. The formation of $R^1 f_* \mathscr{F}$ commutes with any flat base change $S' \to S$. Since $\operatorname{Spec}\widehat{A} \to \operatorname{Spec} A$ is faithfully flat, it follows that the vanishing of $R^1 f_* \mathscr{F}$ can be checked after base

[2] It is not true that any complete Noetherian local ring is the spectrum of a power series ring in one variable. Nonetheless, if the ring in question is the completion at a point of a smooth curve over $\mathbf{C}$, then this is true.

change to Spec $\widehat{A}$, and thus we may assume that A is complete local with maximal ideal $\mathfrak{m}$. For Step 2, we invoke the following algebraization result. If X_n denotes the base change $X \times_A A/\mathfrak{m}^{n+1}$, the theorem of formal functions [EGA, III_1, 4.1.5] states that

$$R^1 f_* \mathscr{F} = \varprojlim_n H^1(X_n, \mathscr{F}|_{X_n}).$$

If we can prove for every n that $H^1(X_n, \mathscr{F}|_{X_n})$ is zero, then we are done. However, X_n is a scheme over the Artinian local ring $A/\mathfrak{m}^{n+1}$, and therefore we may assume that A itself is Artinian local. For Step 3, let X_0 denote the special fiber $X \times_A A/\mathfrak{m}$. Observe that $|X| \simeq |X_0|$ and that a scheme whose underlying topological space is a finite discrete set is affine. Hence, it suffices to show that $|X_0|$ is a finite discrete set. This is trivial since X_0 is a quasi-finite scheme over a field. □

1.3 Unsoaking the walnut

In the 1970s, Michael Artin [Art69] [Art70] realized by analyzing the local behavior of moduli about a point that he could provide necessary and sufficient conditions for a stack to be algebraic (Question 1.1.3). At its heart, Artin's idea is very simple and is based on what we call the "unsoaking the walnut" strategy. Recall that in Grothendieck's "soaking the walnut" strategy, we solve a problem by repeatedly shrinking the size of the problem until we arrive at the situation of a point. However, in Artin's case, we start from a point instead and then repeatedly "expand outwards" to get the result that we want. Here is a sketch of Artin's idea:

1. Let k be a field, $\mathscr{X}/k$ a moduli stack and x a k-point of $\mathscr{X}$. Since $\mathscr{X}$ often parametrizes geometric objects, we should think of x as say an elliptic curve or surface over k. The goal is to prove that $\mathscr{X}$ is algebraic, which is equivalent to producing a smooth neighborhood $U \to \mathscr{X}$ about x.

2. "Expand x outwards" to produce a family of deformations $x_n \in \mathscr{X}(\mathscr{A}_n)$ that is *formally versal*. Here $\{A_n\}$ is an increasing sequence of Artinian local rings with transition maps nilpotent thickenings. The condition of formal versality roughly means "contains" all possible deformations of x.

3. "Take the limit" or *algebraize* to produce an object $\widetilde{x} := \varprojlim x_n$ of $\mathscr{X}$. This object $\widetilde{x}$ now lives over a complete Noetherian local ring $A := \varprojlim A_n$, which by Yoneda's Lemma is exactly a morphism $\widetilde{x}\colon \operatorname{Spec} A \to \mathscr{X}$.

The three step procedure outlined above is very close to producing a smooth neighborhood about x, in the sense that $\widetilde{x}\colon \operatorname{Spec} A \to \mathscr{X}$ is only formally smooth but not "legit" smooth.[3] However, we

[3] The morphism $\widetilde{x}\colon \operatorname{Spec} A \to \mathscr{X}$ is formally smooth because the family $\{x_n\}$ is formally versal. To get smoothness, one needs to apply Artin approximation and use the fact that versality is an open condition.

will not be concerned with this difference in this dissertation. This is because from our perspective, the core of Artin's idea is the three step procedure above to produce a "smooth" neighborhood of x: Start at a point x, "expand x outwards" (Step 2), and then take a "limit" (Step 3). For those familiar with the literature, we remark that Step 2 is exactly the Schlessinger-Rim condition, while Step 3 is the condition that formal deformations are effective.

More recently, the same idea has been applied by Jarod Alper, Jack Hall and David Rydh [AHR20] to address the existence of coarse moduli spaces (Question 1.1.4).[4] Their strategy is almost *mutatis mutandis* the same: Start at a closed point x of a stack $\mathscr{X}/k$. Does x (or technically the residual gerbe at x) admit a coarse space? Well yes, namely just $\operatorname{Spec} k$. Now let x_n be the n-th thickening of the closed point x. By deformation theory, argue that each x_n will admit a coarse space. Finally, algebraize or take a limit to produce a stack $\widetilde{x} \coloneqq \varprojlim x_n$ which admits a coarse space. This stack $\widetilde{x}$ also admits an "étale" morphism $\widetilde{x} \to \mathscr{X}$, and hence we have proven that $\mathscr{X}$ (étale-locally) admits a coarse space.

Unfortunately, Houston, we have two problems. The first problem is that old grasshopper Ben reading this in the year 2070 is confused. He cannot understand what Artin or Alper-Hall-Rydh have done, because young grasshopper Ben has given no details above! Our message to old grasshopper Ben is this: Fear not, for you will soon be out of the shroud of confusion and doubt. In the next section, we carry out the three step procedure in Artin's strategy in the case of the moduli space of elliptic curves. Then in the following section, we will provide most of the details of the Alper-Hall-Rydh strategy in the case of a smooth point of a stack.

The second problem is this. Old grasshopper Ben has heard us handwave about soaking walnuts, expanding points outwards and algebraization. None of these topics have been defined rigorously. Well ok, we have talked about soaking walnuts, so we only need to elaborate on the last two. Unfortunately, the margin in this dissertation is too small to write about both of them, so we can only choose one. We will choose algebraization, because it appears in the title of this dissertation. So what *is* algebraization?

[4] Technically for good moduli spaces, and even then only étale locally, but we will worry about this later.

1.4 Algebraization

Time to turn it up to eleven. Let $(A, \mathfrak{m})$ be a complete Noetherian local ring, and let $\{F_n\}$ be a system of "objects" over $A/\mathfrak{m}^{n+1}$. For instance, each F_n might be an elliptic curve or a coherent sheaf on $X_n \coloneqq X \times_A A/\mathfrak{m}^{n+1}$, for some fixed scheme X over A. Furthermore, suppose that this system of objects is compatible, i.e. an inverse system.

Definition 1.4.1. An inverse system $\{F_n\}$ is *algebraizable* if there exists an object F over A that pulls back to F_n over $A/\mathfrak{m}^{n+1}$. An *algebraization theorem* is a theorem that gives sufficient conditions for an inverse system $\{F_n\}$ to be algebraizable.

The first example of an algebraization theorem is the *deus ex machina* that is formal GAGA [EGA, III_1, 5.1.4]:

Theorem 1.4.2. *Let X be a proper scheme over a complete Noetherian local ring $(A, \mathfrak{m})$. Define $X_n \coloneqq X \times_A A/\mathfrak{m}^{n+1}$. Then there is an equivalence of categories*

$$\operatorname{Coh}(X) \xrightarrow{\sim} \varprojlim \operatorname{Coh}(X_n).$$

In other words, any inverse system of coherent sheaves $\{F_n\}$ is algebraizable

The abbreviation GAGA refers to "Géometrie Algébrique ét Géometrie Analytique", the title of a famous paper of Serre [Ser55] that compares the cohomology of coherent sheaves on a projective scheme $X/\mathbf{C}$ and its analytification X^{an}. Formal GAGA is thus a "formal" version of this theorem, where "formal" refers to the fact we compare coherent sheaves on a scheme and its formal completion $\widehat{X}$, rather than analytification X^{an}. For the moment, the reader unfamiliar with the theory of formal schemes may take $\widehat{X}$ to mean "a space whose category of coherent sheaves is isomorphic to $\varprojlim \operatorname{Coh}(X_n)$."

Now we alluded to algebraization previously as the idea of "taking a limit." Therefore, why is formal GAGA not just trivial? Given $F \in \operatorname{Coh}(X)$, by pullback we get $F_n \in \operatorname{Coh}(X_n)$. Conversely, given $\{F_n\} \in \varprojlim \operatorname{Coh}(X_n)$, the coherent sheaf to construct on X is surely $\varprojlim F_n$, the inverse limit in the category of ringed spaces. However, this is false because the inverse limit is *almost never* quasi-coherent! Consider the most basic situation where $X = \operatorname{Spec} \mathbf{Z}_p$ and each $F_n = (\mathbf{Z}_p/p^{n+1}\mathbf{Z}_p)^{\sim}$. Then $\varprojlim F_n$ is not quasi-coherent because its global sections are $\mathbf{Z}_p$ while it is zero at the generic point.

This points to two things. First, the equivalence in formal GAGA is more subtle than just taking an inverse limit. How do we prove that an abstract pullback functor is essentially surjective, without *explicitly* constructing the desired sheaf? The answer lies in the method of proof of formal GAGA. It uses a kind of Noetherian induction on the category of coherent sheaves - or in Grothendieck's terminology *dévissage* - to reduce to the case of projective space, where the equivalence is proven "by hand." We refer the reader to the excellent survey article of Illusie [FGI+05, Chapter 8] for more details.

Second, observe in the example above with $\operatorname{Spec} \mathbf{Z}_p$ that even though $\varprojlim F_n$ is not quasi-coherent, the tilde of its global sections trivially is. In fact, when X is equal to the base $\operatorname{Spec} A$, it is always true that taking the tilde of global sections *is* the inverse equivalence $\varprojlim \operatorname{Coh}(X_n) \to \operatorname{Coh}(X)$. The experienced reader will note that when X is affine, the functor sending an $\mathscr{O}_X$-module to the tilde of its global sections is exactly the mysterious right adjoint to the inclusion $\operatorname{QCoh}(X) \hookrightarrow \operatorname{Mod}(X)$, the *quasi-coherator* Q_X. This suggests that it is possible to construct the inverse equivalence by hand in the global situation, i.e. if $\mathfrak{F}$ is a formal coherent sheaf on $\widehat{X}$, then the sheaf it algebraizes to is $Q_X(\mathfrak{F})$. We will revisit this idea in Subsection 1.7.3 at the end of this introduction.

We now illustrate the usefulness of formal GAGA through two examples. The first example is elementary and gives a nice simple application of the theorem. The second, though proven in a more complicated way, is also a warm up to the Tannakian duality result [HR19] that is crucial to the étale slice theorem of [AHR20].

Example 1.4.3. Let A be a complete Noetherian local ring and X a family of curves over $\operatorname{Spec} A$, i.e. a proper flat morphism of schemes $X \to \operatorname{Spec} A$ with 1-dimensional fibers. Then $X \to \operatorname{Spec} A$ is projective.

Proof. The special fiber X_0 is projective since it is a proper curve over a field. In particular, there exists an ample line bundle $\mathscr{L}_0 \in \operatorname{Coh}(X_0)$. By deformation theory, the obstruction to lifting $\mathscr{L}_0$ to a *coherent sheaf* $\mathscr{L}_1 \in \operatorname{Coh}(X_1)$ lies in $H^2(X, \mathscr{O}_X)$ which is zero since X is a curve. Hence, there is a $\mathscr{L}_1 \in \operatorname{Coh}(X_1)$ such that $\mathscr{L} \otimes_{\mathscr{O}_{X_1}} \mathscr{O}_{X_0} = \mathscr{L}_0$. Continuing this procedure for every X_n, we arrive at an inverse system of coherent sheaves $\{\mathscr{L}_n\}$ which algebraizes to $\mathscr{L}$ on X by formal GAGA. The reader may check that $\mathscr{L}$ is in fact a line bundle, and that it is ample (by openness of the ample locus [EGA, III_1, 4.7.1]). □

Example 1.4.4. Let A be a complete Noetherian local ring and X a proper, flat scheme over $\operatorname{Spec} A$. Suppose the special fiber X_0 is isomorphic to $\mathbf{P}^d_k$, where k is the residue field of A. Then $X \simeq \mathbf{P}^d_A$.

Proof. Let X_n denote the base change $X \times_A A/\mathfrak{m}^{n+1}$. First, we will show that any inverse system of morphisms $X_n \to \mathbf{P}^d_{A/\mathfrak{m}^{n+1}}$ gives rise to a morphism $X \to \mathbf{P}^d_A$. To do this, we will use the Tannakian duality theorem of Hall and Rydh [HR19, Theorem 1.1], which implies that for Noetherian schemes X and Y,

$$\operatorname{Hom}(X, Y) = \operatorname{Hom}(\operatorname{Coh}(Y), \operatorname{Coh}(X)). \tag{1.4.1}$$

Hence:

$$\begin{aligned}
\mathrm{Hom}(X, \mathbf{P}^d_A) &= \mathrm{Hom}(\mathrm{Coh}(\mathbf{P}^d_A), \mathrm{Coh}(X)) && \text{(by 1.4.1)} \\
&= \mathrm{Hom}(\mathrm{Coh}(\mathbf{P}^d_A), \varprojlim_n \mathrm{Coh}(X_n)) && \text{(formal GAGA for } X\text{)} \\
&= \varprojlim_n \mathrm{Hom}(\mathrm{Coh}(\mathbf{P}^d_A), \mathrm{Coh}(X_n)) \\
&= \varprojlim_n \mathrm{Hom}(X_n, \mathbf{P}^d_A) && \text{(by 1.4.1)} \\
&= \varprojlim_n \mathrm{Hom}(X_n, \mathbf{P}^d_{A/\mathfrak{m}^{n+1}})
\end{aligned}$$

as desired. Now by assumption the special fiber X_0 is isomorphic to $\mathbf{P}^d_k$. Furthermore, projective space has no infinitesimal deformations. Therefore, we obtain that $X_n \simeq \mathbf{P}^d_{A/\mathfrak{m}^{n+1}}$ for all $n \geq 0$. By the algebraization result for morphisms above, this gives rise to a morphism $X \to \mathbf{P}^d_A$. Since the special fiber $X_0 \to \mathbf{P}^d_k$ is an isomorphism and X is flat over A, by [DR73, 7.4] the morphism $X \to \mathbf{P}^d_A$ is an isomorphism as desired. □

Remark 1.4.5. The experienced reader will note that the use of Tannakian duality here is overkill. Indeed, the example above follows from the following corollary of formal GAGA: Let X be a proper A-scheme and Y a finite type and separated A-scheme. Then the completion map

$$\mathrm{Hom}_{\mathrm{Spec}\,A}(X, Y) \to \mathrm{Hom}_{\mathrm{Spf}\,A}(\widehat{X}, \widehat{Y})$$

is bijective. The surjectivity follows from formal GAGA: For any morphism $f \colon \widehat{X} \to \widehat{Y}$, consider the graph of f as a closed formal subscheme of $\widehat{X \times_A Y}$. It is closed because Y is separated over A, and hence is cut out by a formal coherent ideal $\mathfrak{I}$ which we then algebraize using formal GAGA. Nonetheless, we use Tannakian duality as a warm up to the proof of the local quotient structure theorem of Alper-Hall-Rydh, where Tannakian duality is used to algebraize a morphism between formal algebraic stacks $\widehat{X} \to \widehat{Y}$. In that situation, Tannakian duality is absolutely necessary because the target stack Y is highly non-separated. In particular, the trick of algebraizing a morphism $\widehat{X} \to \widehat{Y}$ by considering it as a closed substack of $\widehat{X \times_A Y}$ will not work, because the graph will *not* be closed by non-separatedness.

We now explain exactly how algebraization plays an important role in answering Questions 1.1.3 (algebraicity) and 1.1.4 (existence of coarse moduli spaces), following the "unsoaking the walnut" strategy of Subsection 1.3.

1.5 Algebraization and the algebraicity of moduli stacks

Let us carry out three steps outlined in the "unsoaking the walnut" in the case of the moduli space of elliptic curves $\mathscr{M}_{1,1}$. So fix a point $\operatorname{Spec} \mathbf{C} \to \mathscr{M}_{1,1}$, i.e. an elliptic curve E_0 over $\mathbf{C}$. We want to produce a "smooth" disk about this point.

Now the obstruction to deforming $E_0/\mathbf{C}$ to an elliptic curve E_1 over $\operatorname{Spec} \mathbf{C}[\![x]\!]/(x^2)$ is given by $H^2(E_0, T_{E_0})$, which is zero since E_0 is a curve. Repeating this process for all higher thickenings, we get for all n an elliptic curve E_n over $\mathbf{C}[\![x]\!]/(x^{n+1})$ that reduces to E_0 on the special fiber. This is the "formally versal" family of objects $\{E_n\}$ in the "expand outwards" step of Artin's strategy.

We now algebraize the family $\{E_n\}$ as follows. Let e_0 be the point at infinity in E_0. By Riemann-Roch, the line bundle $\mathscr{L}_0 \coloneqq \mathscr{O}(3e_0)$ is very ample and embeds E_0 as a closed subscheme of $\mathbf{P}^2_{\mathbf{C}}$ defined by a cubic polynomial. Furthermore, by arguing *mutadis mutandis* as in Example 1.4.3, we arrive at an inverse system of very ample line bundles $\mathscr{L}_n \in \operatorname{Pic}(E_n)$, i.e. a compatible family of closed immersions

$$\begin{array}{ccccccc}
E_0 & \hookrightarrow & E_1 & \hookrightarrow & E_2 & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbf{P}^2_{\mathbf{C}} & \hookrightarrow & \mathbf{P}^2_{\mathbf{C}[\![x]\!]/(x^2)} & \hookrightarrow & \mathbf{P}^2_{\mathbf{C}[\![x]\!]/(x^3)} & \hookrightarrow & \cdots
\end{array}$$

Let $\mathscr{I}_n$ be the coherent ideal defining the closed immersion $E_n \hookrightarrow \mathbf{P}^2_{\mathbf{C}[\![x]\!]/(x^{n+1})}$. The family $\{\mathscr{I}_n\}$ is compatible since each E_n is flat over the base, and thus by the *deus ex machina* that is formal GAGA (Theorem 1.4.2), gives rise to a coherent ideal $\mathscr{I}$ on $\mathbf{P}^2_{\mathbf{C}[\![x]\!]}$. It is readily checked that $E \coloneqq V(\mathscr{I})$ is a genus 1 curve over $\mathbf{C}[\![x]\!]$ whose reduction mod x^{n+1} is exactly E_n. Finally, the section $e_0 \colon \operatorname{Spec} \mathbf{C} \to E_0$ lifts to a section $e \colon \operatorname{Spec} \mathbf{C}[\![x]\!] \to E$ by smoothness and the fact that $\mathbf{C}[\![x]\!]$ is Henselian. Hence E is a family of elliptic curves over the disk $\operatorname{Spec} \mathbf{C}[\![x]\!]$, i.e. a "smooth" neighborhood $\operatorname{Spec} \mathbf{C}[\![x]\!] \to \mathscr{M}_{1,1}$.

1.6 Algebraization and the existence of coarse moduli spaces

Let $\mathscr{X}$ be an algebraic stack over an algebraically closed field k. Then every point $x \in \mathscr{X}$ has a stabilizer group, which is either finite or infinite. A very important point about stacks with infinite stabilizers is that the coarse moduli space need not exist. For example, let $\mathscr{X} \coloneqq [\mathbf{G}_m \backslash \mathbf{A}^1]$, where $\mathbf{G}_m$ acts on $\mathbf{A}^1$ via $t \cdot x = tx$. Then one can show a coarse space for $\mathscr{X}$ - if it exists - would have to be the GIT quotient $\mathbf{G}_m \mathbin{\backslash\!\backslash} \mathbf{A}^1 = \operatorname{Spec} k$. However, $\mathscr{X}$ consists of two points, namely 0 and the orbit of 1. This contradicts the fact that the coarse moduli space must have the same points as $\mathscr{X}$, and thus no such space can exist.

We will answer Question 1.1.4 by replacing the notion of a coarse moduli space with that of a *good moduli space*. The notion of a good moduli space was introduced by Jarod Alper [Alp13] in order to deal with stacks with infinite - albeit linearly reductive - stabilizers. For the moment, the reader may take the canonical example of a good moduli space to be the following. Let A/k be an affine variety with the action of a linearly reductive group G/k. Then $\pi\colon [G\backslash \operatorname{Spec} A] \to \operatorname{Spec} A^G$ is a good moduli space. We now reformulate Question 1.1.4 as:

Question 1.6.1. Let $\mathscr{X}$ be an algebraic stack over k with linearly reductive stabilizers. Does there exist an algebraic space X with a good moduli space $\pi\colon \mathscr{X} \to X$?

It turns out that the question of the existence of a good moduli space is intimately related to the question of whether an algebraic stack $\mathscr{X}$ is étale locally a quotient stack. For example, if we know that $\mathscr{X}$ admits an étale local description as a quotient $[G\backslash \operatorname{Spec} A]$, then we can make the good moduli space locally as $[G\backslash \operatorname{Spec} A] \to \operatorname{Spec} A^G$. We then hope to be able to glue these to a global good moduli space for $\mathscr{X}$. The miracle from God is that *every* "reasonable" algebraic stack with linearly reductive stabilizers is étale-locally a quotient stack. This is the étale slice theorem of Alper-Hall-Rydh [AHR20].

Theorem 1.6.2. *Let $\mathscr{X}$ be a quasi-separated algebraic stack, locally of finite type over k with linearly reductive stabilizers. For every point $x \in \mathscr{X}(k)$, there exists an affine scheme* $\operatorname{Spec} A$ *with the action of the stabilizer G_x, a k-point $w \in \operatorname{Spec} A$ fixed by G_x, and an étale morphism*

$$([G_x\backslash \operatorname{Spec} A], w) \to (\mathscr{X}, x).$$

In our opinion, the hardest part about Theorem 1.6.2 is the existence of just *some* morphism from a quotient $[G_x\backslash \operatorname{Spec} A]$ to $\mathscr{X}$. Therefore, we now show how the "unsoaking the walnut" strategy of Section 1.3 produces such a morphism. We will assume (for simplicity) that $x \in \mathscr{X}(k)$ is a smooth point. Let N_x be the normal space to the point x and G_x the stabilizer of x. The stabilizer G_x acts on N_x, and we can view N_x as an affine scheme via $\operatorname{Spec} \operatorname{Sym} N_x^\vee$. In particular, there is a distinguished point $0 \in N_x$. Define:

$$\begin{aligned}
\mathscr{X}^{[n]} &= n\text{-th thickening of the residual gerbe } BG_x,\\
\mathscr{N}^{[n]} &= n\text{-th thickening of the origin } 0 \in [G_x\backslash N_x].
\end{aligned}$$

By smoothness of $x \in \mathscr{X}$ and a simple deformation theory argument (crucially using the fact that G_x is linearly reductive!), we obtain compatible isomorphisms $\mathscr{X}^{[n]} \simeq \mathscr{N}^{[n]}$, i.e. a diagram

$$\begin{array}{ccccccc}
\mathscr{X}^{[0]} & \hookrightarrow & \mathscr{X}^{[1]} & \hookrightarrow & \mathscr{X}^{[2]} & \hookrightarrow & \dots\\
\uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq & & \\
\mathscr{N}^{[0]} & \hookrightarrow & \mathscr{N}^{[1]} & \hookrightarrow & \mathscr{N}^{[2]} & \hookrightarrow & \dots
\end{array}$$

or more succinctly, an element of

$$\varprojlim_n \operatorname{Hom}(\mathscr{N}^{[n]}, \mathscr{X}).$$

This is the "expand outwards" step in the soaking the walnut strategy: We have shown that every n-th thickening of the residual gerbe at x admits a map from a quotient (in fact is a quotient!). We now use the datum of a morphism from an inverse system of quotients $\{\mathscr{N}^{[n]}\}$ to $\mathscr{X}$, to produce a morphism from a legitimate quotient $\mathscr{Y}$ to $\mathscr{X}$.

To this end, let $G_x \mathbin{/\!\!/} N_x$ denote the GIT quotient for the action of G_x on N_x, and let $\widehat{\mathscr{N}}$ be the fiber product of the diagram below:

$$\begin{array}{ccc} \widehat{\mathscr{N}} & \longrightarrow & [G_x \backslash N_x] \\ \downarrow & & \downarrow \\ \operatorname{Spec} \widehat{\mathscr{O}}_{G_x \mathbin{/\!\!/} N_x, 0} & \longrightarrow & G_x \mathbin{/\!\!/} N_x. \end{array}$$

Then we claim that $\mathscr{Y} \coloneqq \widehat{\mathscr{N}}$ has a morphism to $\mathscr{X}$. To see why this is the case, we first invoke the algebraization or *coherent completeness* result [AHR20, Theorem 1.3]: There is an equivalence of categories

$$\operatorname{Coh}(\widehat{\mathscr{N}}) \xrightarrow{\sim} \varprojlim_n \operatorname{Coh}(\mathscr{N}^{[n]}). \tag{1.6.1}$$

Second, we will need to invoke the Tannakian duality theorem of Hall and Rydh [HR19, Theorem 1.1]: For any excellent, Noetherian algebraic stacks $\mathscr{X}$ and $\mathscr{Y}$ with affine stabilizers,

$$\operatorname{Hom}(\mathscr{X}, \mathscr{Y}) = \operatorname{Hom}(\operatorname{Coh}(\mathscr{Y}), \operatorname{Coh}(\mathscr{X})). \tag{1.6.2}$$

Hence,

$$\begin{aligned} \varprojlim_n \operatorname{Hom}(\mathscr{N}^{[n]}, \mathscr{X}) &= \varprojlim_n \operatorname{Hom}(\operatorname{Coh}(\mathscr{X}), \operatorname{Coh}(\mathscr{N}^{[n]})) && \text{(from 1.6.2)} \\ &= \operatorname{Hom}(\operatorname{Coh}(\mathscr{X}), \varprojlim_n \operatorname{Coh}(\mathscr{N}^{[n]})) \\ &= \operatorname{Hom}(\operatorname{Coh}(\mathscr{X}), \operatorname{Coh}(\widehat{\mathscr{N}})) && \text{(from 1.6.1)} \\ &= \operatorname{Hom}(\widehat{\mathscr{N}}, \mathscr{X}) && \text{(from 1.6.2)} \end{aligned}$$

and we win! The data of a compatible family of morphisms $\mathscr{N}^{[n]}$ to $\mathscr{X}$ is exactly the data of a morphism from the legitimate quotient $\widehat{\mathscr{N}}$ to $\mathscr{X}$ as desired.

1.7 Results

People often ask me: "Ben, what did you do during your 5 years in math grad school?" My *de facto* answer is that I spent 5 years (and 10 if you count undergrad) in front of a blackboard staring at abstract symbols. Seriously, I proved two results:

1.7.1 Algebraization for complexes

In my second year of graduate school, I became interested in the story of the Tate conjecture for K3 surfaces in characteristic p. In particular, I was fascinated by the result of Lieblich-Maulik-Snowden [LMS14] that reduced this to a finiteness statement about the number of Fourier-Mukai partners a K3 surface has. Now by classical work of Mukai [Muk87], the study of Fourier-Mukai partners of a K3 surface $X/\mathbf{C}$ is intimately related to the study of the moduli space of sheaves on X. However, in the situation of [LMS14], it is necessary to consider a more general version of this, i.e. a moduli space of sheaves on a gerbe $\mathscr{X}$ over X. Typically, $\mathscr{X}$ corresponds to an ℓ-torsion element (for $\ell \neq p$) in the Brauer group $H^2(X, \mathbf{G}_m)$, and therefore is Deligne-Mumford.

In my third year of graduate school, my advisor Jarod Alper posed to me the broad question of studying the moduli of sheaves on an *arbitrary* gerbe $\mathscr{X}$ over a surface X. Naturally, the question of the existence of a moduli of objects in the derived category of $\mathscr{X}$ came up. As an arbitrary gerbe is typically not Deligne-Mumford or tame, I arrived at my first problem in graduate school: Let $\mathscr{X}$ be an algebraic stack that is proper and flat over an algebraic space S. Is the moduli of complexes $\mathscr{X}/S$ algebraic? I was not able to verify all of Artin's axioms, the chief problem being I did not know how to construct a tangent-obstruction theory for complexes on a stack. The reader may refer to the introduction of Chapter 2 for why this is the case.

However, I was able to prove algebraization theorems for pseudo-coherent, perfect, and relatively perfect complexes on an algebraic stack. Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A, and let $\widehat{\mathscr{X}}$ denote the formal completion of $\mathscr{X}$ with respect to this ideal. I proved:

Theorem 1.7.1 (Lim, 2018)**.** *There is a natural equivalence of (triangulated) categories of pseudo-coherent complexes*

$$D^-_{\mathrm{coh}}(\mathscr{X}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\widehat{\mathscr{X}}) \tag{1.7.1}$$

which restricts to an equivalence of categories of perfect complexes

$$\mathrm{Perf}(\mathscr{X}) \xrightarrow{\sim} \mathrm{Perf}(\widehat{\mathscr{X}}).$$

In addition, if $\mathscr{X}$ is also flat over A, then (1.7.1) also restricts to an equivalence of categories of *relatively perfect complexes*

$$\mathrm{RPerf}(\mathscr{X}) \xrightarrow{\sim} \mathrm{RPerf}(\widehat{\mathscr{X}}).$$

We refer the reader to Chapter 2 for the proof of this result. In the case of relatively perfect complexes on an algebraic space, this result was proven by Lieblich [Lie06]. In fact, he checked the rest of Artin's axioms and proved the algebraicity of the moduli space of complexes on a proper flat morphism of algebraic spaces. I am optimistic that in the near future, the algebraicity of the stack of complexes on a proper flat morphism of algebraic stacks will be within reach of the broader mathematical community on earth. Finally, we remark in the derived setting that Halpern-Leistner and Preygel [HP14] have proven the algebraicity of the moduli of complexes on a morphism of derived algebraic stacks.

1.7.2 Algebraization in characteristic p

During the summer of 2018, Jarod and I attended a conference in Salerno, Italy. One night at dinner, he brought up the idea of extending his étale slice theorem [AHR20] to the more general situation of stacks with *reductive* stabilizers. The reason this is more general than the setup in [AHR20] is the following.

- By the Lie-Kolchin theorem, any linearly reductive group is reductive. Recall that a k-group G is linearly reductive if every linear representation of G on a finite-dimensional k-vector space is completely reducible, and reductive if the unipotent radical of $G_{\overline{k}}$ is trivial.
- In characteristic zero, the converse is also true: Any reductive group is linearly reductive.
- In characteristic p, a group G is linearly reductive if and only if G^0 is a torus and $|G/G^0|$ has order prime to p [Nag62]. On the other hand, classical groups such as GL_d are still reductive.

The third point above is the main reason to want to generalize [AHR20] to stacks with reductive stabilizers: It says that in characteristic p, [AHR20] applies to an *extremely* narrow class of stacks! Damn, so we need to generalize the two ingredients that go into [AHR20]: (1) A deformation theory argument to show that every nilpotent thickening of the residual gerbe of a stack with linearly reductive stabilizers is also a quotient. (2) An algebraization argument to turn a morphism from an adic system of quotients to a morphism from a legitimate quotient.

Unfortunately, for stacks with reductive stabilizers in characteristic p, there are serious obstructions to extending both of these arguments. The deformation theory argument crucially relies on the fact that if G is linearly reductive, $\mathsf{R}\Gamma(BG, \mathsf{L}_{BG/k})$ is concentrated in $[0,1]$, where $\mathsf{L}_{BG/k}$ is the cotangent complex of BG. This is false if G is only assumed to be reductive. The algebraization argument uses two ingredients, namely the Tannakian duality result (1.6.2) and the coherent completeness result (1.6.1). The good news is that Tannakian duality is true independent of the characteristic of the ground field. The bad news is that the proof of the coherent completeness result crucially relies on the fact that the higher (coherent) cohomology of a quotient by a linearly reductive group scheme is zero - again false for G reductive.

Given these issues, Jarod suggested that I try to prove the coherent completeness of quotients by reductive groups. Indeed, in joint work with Jack Hall, I was able to prove the following results:

Theorem 1.7.2 (Hall-Lim, 2018)**.** *Let k be a field, G a split reductive group over k, and R a Noetherian k-algebra that is complete with respect to an ideal I. Let Z denote the fiber over* $\operatorname{Spec} R/I$ *in BG_R.*

- *The stack BG_R is cohomologically proper over* $\operatorname{Spec} R$.
- *The pair (BG_R, Z) satisfies formal functions.*
- *The pair (BG_R, Z) is coherently complete.*

Theorem 1.7.3 (Hall-Lim, 2019)**.** *Let k be a field and consider the standard action of* SL_d *on* $\mathbf{A}^d$. *Let Z denote the closed substack defined by the origin in* $[\mathrm{SL}_d \backslash \mathbf{A}^d]$.

- *The stack* $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ *is cohomologically proper over* $\operatorname{Spec} k$.
- *The pair* $([\mathrm{SL}_d \backslash \mathbf{A}^d], Z)$ *satisfies formal functions.*
- *The pair* $([\mathrm{SL}_d \backslash \mathbf{A}^d], Z)$ *is coherently complete.*

The reader may refer to Chapter 3 for a proof of these theorems and the relevant definitions involved in their statements. We remark that the first result above for BG was already known in the derived setting by Halpern-Leistner and Preygel [HP14]. However, the second result for the standard action of SL_d on $\mathbf{A}^d$ is new. I believe that this is the first result on the formal geometry of a non-trivial quotient by a reductive group that is *independent* of the characteristic of the ground field. In any case, the algebraization results for BG and $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ above are a consequence of Theorems 3.3.1, 3.4.1, and 3.5.1 on descending properties of formal geometry along a universally submersive morphism of Noetherian algebraic stacks.

Now given the generality of Theorems 3.3.1, 3.4.1, and 3.5.1, the reader may wonder why Jack and I only proved algebraization for a very specific quotient stack. The reason for this is that we only knew how to construct hypercovers in the $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ case and not in general. However, thanks to the blessing of the mathematical gods, moduli man Jarod Alper came along and provided the missing piece to the puzzle. Before stating our result, we need to recall an important fact. Let G/k be a reductive group acting on an affine k-scheme of finite type $\operatorname{Spec} A$. Then the structure morphism $[G\backslash \operatorname{Spec} A] \to \operatorname{Spec} A^G$ is an *adequate moduli space* [Alp14, Theorem 9.1.4]. Therefore if $A^G = k$, by [Alp14, Theorem 5.3.1] the stack $[G\backslash \operatorname{Spec} A]$ will contain a unique closed point. Our result is the following:

Theorem 1.7.4 (Alper-Hall-Lim, 2020)**.** *Let k be a field and A a finite type k-algebra. Let G/k be a reductive group acting on $\operatorname{Spec} A$ with the property that $A^G = k$. Define $X \coloneqq [G \backslash \operatorname{Spec} A]$ and let Z be the unique closed point in X.*

- *The stack X is cohomologically proper over $\operatorname{Spec} k$.*
- *The pair (X, Z) satisfies formal functions.*
- *The pair (X, Z) is coherently complete.*

We make two remarks about Theorem 1.7.4. First, note that Theorem 1.7.4 is not a full generalization of [AHR20, Theorem 1.3] to characteristic p because our result contains the additional assumption that $A^G = k$. At present, we do not know how to handle the case when the ring of invariants is a general complete Noetherian local ring. The reason for this is that when A^G is not a field, our approach constructs hypercovers of $[G \backslash \operatorname{Spec} A]$ that are not necessarily universally submersive, and therefore the descent theorems of Chapter 3 cannot be applied. However, we have good reason to believe that the general complete Noetherian local case is true (and will be proven soon). For instance, in the particular example of $[\mathrm{SL}_d \backslash \mathbf{A}^d]$, the proof over a field almost surely generalizes to any complete Noetherian local ring, though I have not checked all the details.

Second, Theorem 1.7.4 in the case that A^G is complete Noetherian local has applications beyond moduli theory. One such application is the work of Mitya Kubrak and Artem Prikhodko [KP20] which we now explain. Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$. Recall a result of Burt Totaro [Tot18, Theorem 12.1] that says for $G = \mathrm{Spin}(11)$ and $p = 2$,

$$\dim_{\mathbf{F}_2} H^{32}_{\mathrm{dR}}(B\,\mathrm{Spin}(11)_{\mathbf{F}_2}/\mathbf{F}_2) > \dim_{\mathbf{F}_2} H^{32}_{\text{ét}}(B\,\mathrm{Spin}(11)_{\mathbf{C}_2}, \mathbf{F}_2).$$

Given this, it is only natural to ask if the inequality above holds for any reductive group G. The answer is indeed yes and is the following result of Kubrak and Prikhodko.

Theorem 1.7.5. *Let G be a split reductive group over $\mathbf{Z}_p$. Then*

$$\dim_{\mathbf{F}_p} H^i_{\mathrm{dR}}(BG_{\mathbf{F}_p}/\mathbf{F}_p) \geq \dim_{\mathbf{F}_p} H^i_{\text{ét}}(BG_{\mathbf{C}_p}, \mathbf{F}_p).$$

We now explain briefly why this inequality holds in the case of a smooth proper scheme. Then, we say how Kubrak and Prikhodko extend this to the case of BG, proving Theorem 1.7.5. Consider the diagram below:

$$
\begin{array}{ccccc}
 & \mathsf{R}\Gamma_{\mathrm{dR}}(X_{\mathbf{F}_p}/\mathbf{F}_p) & \overset{(a)}{\dashleftarrow\dashrightarrow} & & \mathsf{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{F}_p) \\
\text{Bhatt-Morrow-Scholze} \nearrow\swarrow & & \overset{(b)}{\nwarrow\searrow} & \nearrow\swarrow \text{(c) Huber} & \\
\mathsf{R}\Gamma_{\mathrm{A_{inf}}}(\widehat{X}) & \xrightarrow{\text{Bhatt-Morrow-Scholze}} & \mathsf{R}\Gamma_{\text{ét}}(\widehat{X}_{\mathbf{C}_p}, \mathbf{F}_p). & &
\end{array}
\tag{1.7.2}
$$

Dashed arrow (a) points to the two cohomology complexes that are of interest. As the diagram indicates, the way (a) is established is by first introducing an intermediate object $\widehat{X}_{\mathbf{C}_p}$, namely the Raynaud generic fiber of the formal scheme $\widehat{X}_{\mathscr{O}_{\mathbf{C}_p}} \to \operatorname{Spf} \mathscr{O}_{\mathbf{C}_p}$. Then, arrow (b) is established using the groundbreaking work of Bhatt-Morrow-Scholze [BMS18] that relates $\mathsf{R}\Gamma_{\mathrm{dR}}(X_{\mathbf{F}_p}/\mathbf{F}_p)$ and $\mathsf{R}\Gamma_{\text{ét}}(\widehat{X}_{\mathbf{C}_p}, \mathbf{F}_p)$ via a cohomology theory with values in Fontaine's period ring $\mathrm{A_{inf}}$. Finally, arrow (c) follows from a base change theorem of Huber [Hub96, Theorem 3.8.1], noting crucially that since X is proper, the canonical morphism of rigid spaces from $\widehat{X}_{\mathbf{C}_p}$ to the rigid-analytification $(X_{\mathbf{C}_p})^{\mathrm{rig}}$ is an isomorphism. [Con99, Theorem A.3.1].

Now Kubrak and Prikhodko (building on [BMS18]) prove Theorem 1.7.5 by first establishing arrow (b) in (1.7.2) for a class of stacks that they call "Hodge-proper". Since BG for G reductive is Hodge-proper, this establishes arrow (b) for BG. However, in order to establish arrow (c) for BG, they cannot appeal to a generalization of [Hub96, Theorem 3.8.1] for stacks because BG is not proper, and therefore the Raynaud generic fiber $\widehat{BG}_{\mathbf{C}_p}$ is very different from the rigid-analytification $(BG_{\mathbf{C}_p})^{\mathrm{rig}}$. Instead, they prove arrow (c) by a careful reduction to the case of $B\mathbf{G}_m$, in which case the result can be deduced from the fact that $B\mathbf{G}_m$ and ind-projective space $\mathbf{P}^\infty$ have the same étale cohomology.

Thus, given that arrow (c) is true for BG, Kubrak and Prikhodko ask if it is also true for other cohomologically proper stacks satisfying formal GAGA.

Conjecture 1.7.6. Let K be a local field, $\mathscr{O}_K$ its ring of integers and C the completion of the algebraic closure of K. Let $\mathscr{X}$ be an algebraic stack that is smooth and cohomologically proper over $\operatorname{Spec} \mathscr{O}_K$. Suppose that $\mathscr{X}$ satisfies formal GAGA. Then for any torsion ring Λ,

$$\mathsf{R}\Gamma_{\text{ét}}(\mathscr{X}_C, \Lambda) \xrightarrow{\sim} \mathsf{R}\Gamma_{\text{ét}}(\widehat{\mathscr{X}}_C, \Lambda),$$

where $\widehat{\mathscr{X}}_C$ is the Raynaud generic fiber of $\widehat{\mathscr{X}}_{\mathscr{O}_C} \to \operatorname{Spf} \mathscr{O}_C$.

Now we see how my work is intertwined with that of Kubrak and Prikhodko: In order for Conjecture 1.7.6 to not be meaningless, they need to know that there are *interesting* examples of cohomologically proper stacks over $\operatorname{Spec} \mathscr{O}_K$ that satisfy formal GAGA! This is how Mitya became interested in my algebraization theorems for quotients by reductive groups (albeit only over a field currently). Conversely, I too have benefited extremely from mathematical conversations with Mitya: The results of Chapter 3 would never have been possible without a 3 hour conversation we had at San Francisco State University in November 2018.

Ok everybody, that's a wrap. One last thing though. Jack Hall [Hal18] has a unified approach to proving *all* GAGA theorems for schemes and algebraic spaces. So couldn't I just have generalized his methods to the case of stacks? No! In the next subsection, we will see exactly where this fails. The basic reason is that stacks can have unbounded cohomological dimension, which cannot happen for schemes or algebraic spaces. Weird, huh?

1.7.3 Young grasshopper Ben vs GAGA genius Jack Hall

This subsection will be the final and most technical part of the introduction. We first describe Jack's approach to proving formal GAGA in the special case of an algebraic space that is proper and flat over a DVR. Then, we say exactly where his proof breaks down for stacks.

Let X be an algebraic space that is separated and of finite type over an I-adically complete Noetherian ring A. Let Z denote the fiber over $V(I)$, and let $\widehat{X}$ be the formal completion of X with respect to the closed subspace Z. There is a morphism of ringed sites

$$\iota\colon \widehat{X} \to X$$

which induces a functor on the unbounded derived categories of modules

$$\mathsf{L}\iota^*\colon D(X) \to D(\widehat{X}).$$

Furthermore, since ι is flat, we note that $\mathsf{L}\iota^* = \iota^*$, and that this restricts to a functor on the full triangulated subcategories of bounded complexes with coherent cohomology

$$\iota^*\colon D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(\widehat{X}). \tag{1.7.3}$$

Now we are interested in the question of when (1.7.3) is an equivalence of categories. As we will see later, this is true if X is proper over $\operatorname{Spec} A$. However, for now, let us not assume properness and instead try to guess what a candidate inverse to ι^* might be. Recall that we always have an adjoint pair

$$\iota^*\colon D(X) \rightleftarrows D(\widehat{X})\colon \mathsf{R}\iota_*.$$

Therefore, we might guess that $\mathsf{R}\iota_*$ is the inverse to ι^*. Unfortunately, this is wrong: The functor $\mathsf{R}\iota_*$ does not even restrict to a functor

$$\mathsf{R}\iota_*\colon D^b_{\mathrm{Coh}}(\widehat{X}) \to D^b_{\mathrm{Coh}}(X),$$

because $\mathsf{R}\iota_*$ does not preserve quasi-coherence! To see why this is the case, recall again the simple example of $X = \operatorname{Spec} \mathbf{Z}_p$ and $F := \varprojlim (\mathbf{Z}/p^{n+1}\mathbf{Z})^{\sim}$. Then F is a coherent sheaf on $\operatorname{Spf} \mathbf{Z}_p$ that is not quasi-coherent when pushed forward to $\operatorname{Spec} \mathbf{Z}_p$.

Sigh, so $\mathsf{R}\iota_*$ cannot be the inverse to ι^*. Jack's idea to fix the problem (which goes back to Johan de Jong/Bhargav Bhatt) is this: Just force it to be quasi-coherent! How? Consider the quasi-coherator

$$\mathsf{R}Q_X \colon D(X) \to D_{\mathrm{Qcoh}}(X),$$

i.e. the mysterious right adjoint to the inclusion $D_{\mathrm{Qcoh}}(X) \hookrightarrow D(X)$ [Stacks, Tag 0CQZ]. Then, he defines a functor

$$\mathsf{R}\iota_{\mathrm{qc},*} \colon D(\widehat{X}) \to D_{\mathrm{Qcoh}}(X)$$

as the composition $\mathsf{R}\iota_{\mathrm{qc},*} := \mathsf{R}Q_X \circ \mathsf{R}\iota_*$. The functor $\mathsf{R}\iota_{\mathrm{qc},*}$ is right adjoint to ι^* and evidently solves the problem of quasi-coherence. However, we want more: we want *coherence*.

Aha! We claim that if X is proper, then $\mathsf{R}\iota_{\mathrm{qc},*}$ *does* indeed restrict to a functor

$$\mathsf{R}\iota_{\mathrm{qc},*} \colon D^b_{\mathrm{Coh}}(\widehat{X}) \to D^b_{\mathrm{Coh}}(X).$$

Jack's insight to proving this fact is the following realization.

Mental Ninja Leap 1.7.1. *Suppose that the functor* $\mathsf{R}\Gamma \colon D(X) \to D(A)$ *restricts to a functor* $\mathsf{R}\Gamma \colon D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(A)$. *Then* $\mathsf{R}\iota_{\mathrm{qc},*}$ *restricts to a functor*

$$\mathsf{R}\iota_{\mathrm{qc},*} \colon D^b_{\mathrm{Coh}}(\widehat{X}) \to D^b_{\mathrm{Coh}}(X).$$

Moreover, the adjunction

$$\iota^* \colon D_{\mathrm{Qcoh}}(X) \rightleftarrows D(\widehat{X}) \colon \mathsf{R}\iota_{\mathrm{qc},*}.$$

restricts to an adjunction

$$\iota^* \colon D^b_{\mathrm{Coh}}(X) \rightleftarrows D^b_{\mathrm{Coh}}(\widehat{X}) \colon \mathsf{R}\iota_{\mathrm{qc},*}.$$

Mental Ninja Leap 1.7.1 above is really a lot more powerful than just properness. Indeed, the only thing we need to know is that $\mathsf{R}\Gamma$ preserves coherence and boundedness. Note that if X is proper, then the former is a consequence of the proper mapping theorem, while the latter that algebraic spaces have bounded cohomological dimension. We refer the reader to [Hal18, Proposition 4.1] for the proof of Mental Ninja Leap 1.7.1. A very important fact that it relies on though is:

Yoda's Coherence Lemma 1.7.1. *Let* M *be a complex in* $D_{\mathrm{Qcoh}}(X)$. *If* $\mathsf{R}\mathrm{Hom}(P, M) \in D^b_{\mathrm{Coh}}(A)$ *for every perfect complex* P, *then* $M \in D^b_{\mathrm{Coh}}(X)$.

We call it Yoda's Coherence Lemma because it is kinda like Yoneda's lemma, in the sense that if the Hom from every doodad is coherent, then the original doodad is also coherent.[5] Now granting Mental Ninja Leap 1.7.1, let us prove that when X is proper, (1.7.3) is fully faithful. In order to simplify ideas, we will assume that A is a DVR with maximal ideal $I = (\varpi)$ and that X is flat over A. As we will see, the proof is essentially just "tensoring with a perfect complex" (at least that's what GAGA genius Jack Hall told me in a Google Hangouts call).

Theorem 1.7.7. *Let X be an algebraic space that is proper and flat over a discrete valuation ring A with maximal ideal (ϖ). Then*

$$\iota^*\colon D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(\widehat{X})$$

is fully-faithful, where the completion is taken with respect to $Z = V(\varpi)$.

Proof. By Mental Ninja Leap 1.7.1, the adjoint pair

$$\iota^*\colon D_{\mathrm{Qcoh}}(X) \leftrightarrows D(\widehat{X})\colon \mathsf{R}\iota_*$$

restricts to an adjoint pair

$$\iota^*\colon D^b_{\mathrm{Coh}}(X) \leftrightarrows D^b_{\mathrm{Coh}}(\widehat{X})\colon \mathsf{R}\iota_{\mathrm{qc},*}.$$

We must show for any $N \in D^b_{\mathrm{Coh}}(X)$ that the adjunction map

$$\eta_N\colon N \to \mathsf{R}\iota_{\mathrm{qc},*}\iota^* N$$

is a quasi-isomorphism. In fact, since N is bounded, it is enough to show this for N concentrated in degree 0, i.e. a coherent sheaf. Let K_N be the cone of this morphism. It is sufficient to show that $K_N \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \simeq 0$. The reason for this is the following. Nakayama's Lemma implies that K_N is zero in a Zariski open neighborhood U of Z. If $U \neq X$, the complement $X \setminus U$ would be a non-empty closed subspace of X whose image in $\operatorname{Spec} A$ is closed (by properness) and disjoint from $V(\varpi)$, contradicting the fact A is complete with respect to (ϖ).

Now we make a very careful observation: The sheaf $j_*\mathscr{O}_Z$ is a perfect complex. Indeed, since X is flat over A, $j_*\mathscr{O}_Z$ is quasi-isomorphic to the complex $[\mathscr{O}_X \xrightarrow{\varpi\cdot} \mathscr{O}_X]$ and thus is perfect as claimed. Hence, by abstract highfalutin nonsense [Hal18, Lemma 6.4], the projection map

$$(\mathsf{R}\iota_{\mathrm{qc},*}\iota^* N) \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \to \mathsf{R}\iota_{\mathrm{qc},*}\iota^*(N \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z)$$

is a quasi-isomorphism. But this means that $K_N \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \simeq K_{N\otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z}$ and hence we reduce to

[5] Amazingly, if for all $P \in \mathrm{Perf}(X)$ we have $\mathsf{RHom}(P, M) \in D^b_{\mathrm{Coh}}(A)$, then in fact for all $K \in D^b_{\mathrm{Coh}}(X)$, $\mathsf{RHom}(K, M) \in D^b_{\mathrm{Coh}}(A)$. This follows from the fact that any $K \in D_{\mathrm{Qcoh}}(X)$ can be "approximated" by perfect complexes.

showing that

$$\eta_{N\otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z} : N \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \to \mathsf{R}\iota_{\mathrm{qc},*}\iota^*(N \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z)$$

is a quasi-isomorphism.

To this end, consider the exact sequence

$$0 \to N_{\mathrm{tors}} \to N \to N/N_{\mathrm{tors}} \to 0$$

where N_{tors} is the ϖ-torsion in N. We reduce to showing that

$$\eta_{N_{\mathrm{tors}}\otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z}, \tag{1.7.4}$$

$$\eta_{N/N_{\mathrm{tors}}\otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z}, \tag{1.7.5}$$

are quasi-isomorphisms. For (1.7.4), observe that N_{tors} admits a (finite) filtration by subsheaves of the form j_*N' for some $N' \in \mathrm{Coh}(Z)$. Therefore, without loss of generality we may assume that $N_{\mathrm{tors}} \simeq j_*N'$ for some $N' \in \mathrm{Coh}(Z)$, in which case by abstract highfalutin nonsense again,

$$N_{\mathrm{tors}} \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \simeq j_*N' \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \simeq j_*(N' \otimes^{\mathsf{L}}_{\mathscr{O}_Z} \mathsf{L}j^*j_*\mathscr{O}_Z).$$

But the complex $N' \otimes^{\mathsf{L}}_{\mathscr{O}_Z} \mathsf{L}j^*j_*\mathscr{O}_Z$ is bounded since $\mathsf{L}j^*j_*\mathscr{O}_Z$ is perfect, and therefore we reduce to proving that η_{j_*M} is a quasi-isomorphism for $M \in \mathrm{Coh}(Z)$.

Similarly, for (1.7.5), observe that N/N_{tors} is a torsion-free ϖ-module, a fortiori A-flat. Hence

$$N/N_{\mathrm{tors}} \otimes^{\mathsf{L}}_{\mathscr{O}_X} j_*\mathscr{O}_Z \simeq N/N_{\mathrm{tors}} \otimes_{\mathscr{O}_X} j_*\mathscr{O}_Z \simeq j_*j^*(N/N_{\mathrm{tors}}).$$

Therefore, in any case, we reduce to showing that η_{j_*M} is a quasi-isomorphism for $M \in \mathrm{Coh}(Z)$. But now this is trivial:

$$\begin{aligned}
\mathsf{R}\iota_{\mathrm{qc},*}\iota^*(j_*M) &\simeq (\mathsf{R}Q_X \circ \mathsf{R}\iota_*)(\iota^*j_*M) \\
&\simeq (\mathsf{R}Q_X \circ \iota_*)(\iota^*j_*M) \\
&\simeq \mathsf{R}Q_X(\iota_*\iota^*j_*M) \\
&\simeq \mathsf{R}Q_X(j_*M) \\
&\simeq j_*M.
\end{aligned}$$

The first line is just the definition of $\mathsf{R}\iota_{\mathrm{qc},*}$, while the second line follows from the fact that ι_* is exact on formal coherent sheaves. The fourth line follows from the fact that for coherent sheaves pushed forward from Z, the adjunction map $\mathrm{id}_X \to \iota_*\iota^*$ is an isomorphism. The final line follows from the fact that j_*M is already quasi-coherent. □

Ok young grasshopper Ben, so where does the approach of GAGA genius Jack Hall fail in your situation? Recall that the two types of stacks we care about are proper algebraic stacks and stacks with reductive stabilizers. Then Jack's approach does not work for us because there are examples of such stacks with unbounded cohomological dimension. In particular, the hypotheses of Mental Ninja Leap 1.7.1 are not satisfied. In addition, in the case of stacks with reductive stabilizers, the method of proof of Mental Ninja Leap 1.7.1 fails because Yoda's Coherence Lemma 1.7.1 is false.

Counterexample 1.7.8. *There exists a proper algebraic stack $\mathscr{X}/\mathbf{F}_p$ such that $\mathsf{R}\Gamma\colon D(\mathscr{X}) \to D(\mathbf{F}_p)$ does not restrict to a functor $\mathsf{R}\Gamma\colon D^b_{\mathrm{Coh}}(\mathscr{X}) \to D^b_{\mathrm{Coh}}(\mathbf{F}_p)$.*

Proof. Let $\mathscr{X} = B\mathbf{Z}/p\mathbf{Z}$. Then for every $i \geq 0$, $H^i(\mathsf{R}\Gamma(B\mathbf{Z}/p\mathbf{Z}, \mathscr{O}_{B\mathbf{Z}/p\mathbf{Z}})) = H^i(\mathbf{Z}/p\mathbf{Z}, \mathbf{F}_p) = \mathbf{F}_p$. Hence $\mathsf{R}\Gamma$ does not preserve bounded complexes. □

Counterexample 1.7.9. *Let $\mathscr{X}$ be an algebraic stack of finite type over $\overline{\mathbf{F}}_p$, and $x \in |\mathscr{X}|$ a closed point whose stabilizer G_x is reductive but not linearly reductive. Then $\mathsf{R}\Gamma\colon D(\mathscr{X}) \to D(\overline{\mathbf{F}}_p)$ does not restrict to a functor $\mathsf{R}\Gamma\colon D^b_{\mathrm{Coh}}(\mathscr{X}) \to D^b_{\mathrm{Coh}}(\overline{\mathbf{F}}_p)$.*

Proof. Since G_x is reductive but not linearly reductive, there exists a coherent sheaf $F \in \mathrm{Coh}(BG_x)$ such that $\mathsf{R}\Gamma(BG_x, F)$ has cohomology in arbitrarily high degree. Let $i_*\colon BG_x \hookrightarrow \mathscr{X}$ denote the canonical closed immersion. Then $\mathsf{R}\Gamma(\mathscr{X}, i_*F) \simeq \mathsf{R}\Gamma(BG_x, F)$ and therefore $\mathsf{R}\Gamma\colon D(\mathscr{X}) \to D(\overline{\mathbf{F}}_p)$ does not preserve boundedness. □

Counterexample 1.7.10. *There exists an algebraic stack $\mathscr{X}/\overline{\mathbf{F}}_p$ with reductive stabilizers such that Yoda's Coherence Lemma 1.7.1 is false.*

Proof. Let $\mathscr{X} = B\mathbf{G}_m$. For every positive integer n, let M_n be the quasi-coherent sheaf on $B\mathbf{G}_m$ corresponding to the irreducible representation $\mathbf{G}_m \to \overline{\mathbf{F}}_p^\times$ given by $x \mapsto x^n$. Consider the complex $M := \bigoplus_{n\geq 0} M_n[-n]$. We claim for every perfect complex P that $\mathsf{RHom}(P, M)$ is bounded with coherent cohomology. Indeed, it is enough to check this with $P = M_k$ for some $k \in \mathbf{Z}$. Then

$$\begin{aligned}
\mathsf{RHom}(P, M) &\simeq \mathsf{RHom}(M_k, M) \\
&\simeq \mathsf{RHom}\left(M_k, \bigoplus_{n\geq 0} M_n[-n]\right) \\
&\simeq \prod_{n\geq 0} \mathsf{RHom}\left(M_k, M_n[-n]\right) \\
&\simeq \prod_{n\geq 0} \mathrm{Hom}\left(M_k, M_n\right)[-n] \\
&\simeq \overline{\mathbf{F}}_p[-k]
\end{aligned}$$

by Schur's Lemma. However, M is not bounded because it has cohomology in every positive degree. □

Chapter 2

Grothendieck's Existence Theorem for Relatively Perfect Complexes on Algebraic Stacks

The results in this chapter appear in [Lim19].

2.1 Introduction

Recently, pioneering work of Bridgeland [Bri07] has shown that it is possible to define a notion of stability for objects in any triangulated category, vastly generalizing the notion of stability of vector bundles as considered in GIT. However, unlike moduli of vector bundles in GIT, it is a subtler question to construct a moduli stack of objects in $D^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves on a scheme X. At the very least, such a stack should be *algebraic* in order to ask geometric questions.

In [Art74], building on the seminal papers [Art69] and [Art70], Artin gave a list of sufficient axioms for a stack to be algebraic. One of these axioms is that formal deformations of objects can be algebraized, i.e., that Grothendieck's existence theorem is satisfied. Therefore, in order to construct an algebraic stack parametrizing objects in $D^b(\mathrm{Coh}(X))$, we must at least be able to prove Grothendieck's existence theorem for such objects. The first result of this nature is that of Lieblich [Lie06, Proposition 3.6.1]. He proves that if one restricts to relatively perfect complexes (see Definition 2.7.1) on an algebraic space X, proper and flat over a complete local Noetherian ring, then Grothendieck's existence theorem holds. In fact, he checks the rest of Artin's axioms and proves that if $X \to S$ is a proper flat morphism of algebraic spaces, the moduli space of relatively perfect and universally gluable complexes $\mathscr{D}_{X/S}$ is represented by an algebraic stack, locally of finite type over S [Lie06, Theorem 4.2.1].

The goal of this article is to establish an important first step towards constructing a moduli of relatively perfect objects on an *algebraic stack*. More precisely, we extend Lieblich's result on Grothendieck's existence theorem for relatively perfect complexes to the setting of algebraic stacks. This will be done by way of two more general results, namely formal GAGA and Grothendieck's existence theorem for pseudo-coherent complexes (c.f. Definition 2.4.1 and Remark 2.4.6).

Theorem 2.1.1 (Formal GAGA for pseudo-coherent complexes)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Then there is a natural equivalence of triangulated categories*

$$D^-_{\mathrm{coh}}(\mathscr{X}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\widehat{\mathscr{X}})$$

of pseudo-coherent complexes.

Theorem 2.1.2 (Grothendieck existence for pseudo-coherent complexes)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Then any adic system of pseudo-coherent complexes on $\mathscr{X}_n$ algebraizes to a pseudo-coherent complex on $\mathscr{X}$.*

Using these results, we can prove formal GAGA and Grothendieck's existence theorem for perfect complexes on algebraic stacks:

Theorem 2.1.3 (Formal GAGA for perfect complexes)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Then there is a natural equivalence of triangulated categories*

$$\mathrm{Perf}(\mathscr{X}) \xrightarrow{\sim} \mathrm{Perf}(\widehat{\mathscr{X}})$$

of perfect complexes.

Theorem 2.1.4 (Grothendieck existence for perfect complexes)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Then any adic system of perfect complexes on $\mathscr{X}_n$ algebraizes to a perfect complex on $\mathscr{X}$.*

Finally, using Grothendieck's existence theorem for pseudo-coherent complexes, we deduce a generalization of [Lie06, Proposition 3.6.1] to the setting of algebraic stacks. Note that unlike the case of pseudo-coherent or perfect complexes, we need a flatness hypothesis here.

Theorem 2.1.5 (Grothendieck existence for relatively perfect complexes)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper and flat over an I-adically complete Noetherian ring A. Define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Then any adic system of relatively perfect complexes on $\mathscr{X}_n$ algebraizes to a relatively perfect complex on $\mathscr{X}$.*

Remark 2.1.6. The reader may wonder if the triangulated category $D^-_{\text{coh}}(\widehat{\mathscr{X}})$ can be described as some kind of inverse limit of triangulated categories $\lim D^-_{\text{coh}}(\mathscr{X}_n)$. At present, we do not know if there is an appropriate definition for such an inverse limit that does not use the language of ∞-categories. One issue is the definition of what morphisms should be in this category. For example, given two systems $\{P_n\}$ and $\{Q_n\}$, defining $\text{Hom}(\{P_n\}, \{Q_n\}) \coloneqq \lim \text{Hom}(P_n, Q_n)$ is *not* the right definition. This is because if we suppose that P_n, Q_n are the reductions of P, Q to $\mathscr{X}_n$, the natural surjective map

$$\text{Hom}(P, Q) \to \lim \text{Hom}(P_n, Q_n)$$

has kernel $R^1 \lim \text{Hom}(P_n, Q_n[-1])$ which may not vanish. Since it is not necessary to deal with inverse limits of triangulated categories to state our theorems, we will avoid the issue completely.

2.1.1 Comparison of Lieblich's result with ours

The classical form of Grothendieck's existence theorem asserts that given a scheme X proper over an I-adically complete Noetherian ring A, any adic system of coherent sheaves F_n on $X_n \coloneqq X \times_A A/I^{n+1}$ can be algebraized to a coherent sheaf on X. The theorem has been extended to the case of algebraic spaces by [Knu71, Theorem 6.3] and most recently by [Ols05, Theorem 1.4] and [Con05, Theorem 4.1] for algebraic stacks. In all these cases, the proofs rely on Chow's Lemma and standard dévissage techniques.

Lieblich's proof of [Lie06, Proposition 3.6.1] ultimately reduces to the case of coherent sheaves on algebraic spaces [Knu71, Theorem 6.3]. Therefore, given the results of [Ols05, Theorem 1.4] and [Con05, Theorem 4.1], it is reasonable to ask if the proof of [Lie06, Proposition 3.6.1] generalizes verbatim to the case of algebraic stacks. We now give a rough idea for why this is not the case. Suppose that X_A is an algebraic space, proper and flat over an Artinian local ring A. Let $A \to A_0$ be a square-zero thickening, E_0 an object of $D^b(\text{Coh}(X_{A_0}))$ that is relatively perfect over A_0, and $E \in D^b(\text{Coh}(X_A))$ such that $E \otimes^{\mathbf{L}}_A A_0 \xrightarrow{\sim} E_0$. Lieblich constructs a tangent-obstruction theory for deformations of objects in the derived category [Lie06, Theorem 3.1.1], crucially relying on the fact that the (small) étale site is invariant under infinitesimal thickenings. Lieblich uses this to show that E_0 is quasi-isomorphic to a bounded above complex $J_0^\bullet$ that lifts to a complex $J^\bullet$ satisfying $J^\bullet \otimes_A A_0 = J_0^\bullet$.

The upshot of this procedure is that if X is an algebraic space over a now complete Noetherian local ring A, any adic system of relatively perfect objects $\{E_n\} \in D^b(\text{Coh}(X_n))$ can be replaced by an adic system of *complexes* $\{J_n^\bullet\}$. In other words, the transition maps $J_n^\bullet \to J_{n-1}^\bullet$ are maps of complexes such that $J_n^\bullet \otimes_{A_n} A_{n-1} = J_{n-1}^\bullet$. Lieblich then completes the proof by algebraizing the *inverse limit* $\lim J_i^\bullet$ to a relatively perfect complex on X using [Knu71, Theorem 6.3]. However, if we now wish to work on an *algebraic stack* $\mathscr{X}$, Lieblich's proof does not generalize for the following reason: The lisse-étale site is not invariant under infinitesimal thickenings and therefore Lieblich's tangent-obstruction theory cannot be applied to replace an adic system of objects in the derived category with an adic system of complexes. It follows that any attempt to generalize Lieblich's result to algebraic stacks must either develop a deformation theory of complexes on the lisse-étale site, or avoid any recourse to deformation theory completely.

We will take the latter route in this article and show that it is not necessary to represent an adic system of relatively perfect objects in $D^b(\mathrm{Coh}(\mathscr{X}_n))$ by an adic system of relatively perfect complexes. Instead, given an adic system of relatively perfect objects P_n, we will directly construct a pseudo-coherent object on $\widehat{\mathscr{X}}$ (the *derived limit* $R\lim P_n$) that recovers each P_n at the finite level. This object will then be algebraized to a pseudo-coherent object P on $\mathscr{X}$ by formal GAGA. Finally, we use the fact that $\mathscr{X} \to \operatorname{Spec} A$ is flat, and the fact that P_0 is relatively perfect over $\operatorname{Spec} A/I$ to prove that the algebraized object P is relatively perfect over $\operatorname{Spec} A$.

The outline of this article is as follows. First, in Section 2.3 we provide a very general criterion for when a morphism of ringed sites $f\colon (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ gives rise to an equivalence of triangulated categories $f^*\colon D^-_{\mathscr{A}_X}(X) \to D^-_{\mathscr{A}_Y}(Y)$ (see Section 2.3 for notation). This will be done using a slight variant of [LO08, Lemma 2.1.10]. We apply this criterion to prove formal GAGA for pseudo-coherent complexes in Section 2.4. As a corollary of formal GAGA, we prove Grothendieck's existence theorem for pseudo-coherent complexes in Section 2.5. In Section 2.6, we apply the preceding results to prove formal GAGA and Grothendieck's existence theorem for perfect complexes. Finally, in Section 2.7 we prove Grothendieck's existence theorem for relatively perfect complexes.

2.1.2 Recent results in the literature

Hall [Hal18] has proven an existence theorem for pseudo-coherent complexes in a non-Noetherian setting. Unlike previous proofs of the existence theorem, Hall's proof does not rely on dévissage to the projective situation. Instead, he proves a very general result [Hal18, Theorem 6.1] that is sufficient to imply all existing GAGA results in the setting of $\mathbf{C}$-analytic spaces, rigid-analytic spaces, Berkovich spaces or formal algebraic spaces [Hal18, Examples 7.5, 7.7 and 7.8]. At this point, we make the important remark that our work is *not* a generalization of [Hal18, Example 7.8] to the setting of algebraic stacks because we assume Noetherian hypotheses. Conversely, Hall's method does not generalize to algebraic stacks because a key hypothesis of [Hal18, Theorem 6.1] fails in this setting, namely that $R\Gamma(\mathscr{X}, -)$ preserves pseudo-coherence. Indeed, the algebraic stack $B\mathbf{Z}/p\mathbf{Z}$ is proper over $\operatorname{Spec} \mathbf{F}_p$, but $R\Gamma(B\mathbf{Z}/p\mathbf{Z}, \mathscr{O}_{B\mathbf{Z}/p\mathbf{Z}})$ has non-zero cohomology in all positive degrees. Therefore, any generalization of Hall's method to prove non-Noetherian GAGA for stacks should at least include the hypothesis that $\mathscr{X}$ is tame, or admits a good moduli space in the sense of Alper [Alp13, Definition 4.1]. In summary, Hall's results are essentially *orthogonal* to ours.

Finally we mention results of Halpern-Leistner–Preygel [HP14]. They have established various

GAGA results for algebraic stacks in the context of derived algebraic geometry, with a focus on those with affine diagonal. Since the diagonals of the algebraic stacks considered in the present paper are proper, our results only overlap in the case of finite diagonals.

2.2 Acknowledgments

I am grateful to Max Lieblich for suggesting the problem to me, in addition to his generosity with his time. I am very grateful to my advisors Jarod Alper and Ravi Vakil for their guidance, time and many useful comments regarding the project. I am very indebted to Jack Hall for his technical expertise in formal algebraic geometry. This project benefited from numerous helpful discussions with Daniel Bragg, Brian Conrad, Daniel Dore, Alessandro M. Masullo, Siddharth Mathur, Zev Rosengarten, Jesse Silliman and Bogdan Zavyalov. Part of this work was carried out at the mathematics department of the University of Washington. We thank them for their hospitality.

2.3 A preliminary result

The goal of this section is to establish a very general result (Theorem 2.3.7) that we will use to deduce formal GAGA for pseudo-coherent complexes. Roughly, given a flat morphism of ringed sites $f\colon (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$, we are interested in sufficient conditions for there to exist an equivalence of categories

$$f^*\colon D^-_{\mathscr{A}_X}(X) \xrightarrow{\sim} D^-_{\mathscr{A}_Y}(Y).$$

The notation $D^-_{\mathscr{A}_X}(X)$ refers to the following. Write $D(X)$ for the unbounded derived category of $\mathscr{O}_X$-modules, $D(\mathrm{Mod}(\mathscr{O}_X))$. Fix a weak Serre subcategory $\mathscr{A}_X \subseteq \mathrm{Mod}(\mathscr{O}_X)$. Then $D^-_{\mathscr{A}_X}(X)$ is the full triangulated subcategory of $D(X)$ consisting of bounded above objects whose cohomology sheaves lie in $\mathscr{A}_X \subseteq \mathrm{Mod}(\mathscr{O}_X)$ (here we use the fact that $\mathscr{A}_X$ is a weak Serre subcategory in order that $D^-_{\mathscr{A}_X}(X)$ be a full triangulated subcategory of $D(X)$). Analogously, we can define $D^-_{\mathscr{A}_Y}(Y)$. Before we can state our result, we will need a rather technical lemma which is basically [LO08, Lemma 2.1.10]. We will include a proof of it here because the proof in [LO08] relies on [Spa88, Lemma 0.11] whose proof is light on details. In addition, our hypotheses are a little different than [LO08, Lemma 2.1.10].

Definition 2.3.1. Let X be a (Grothendieck) site. A *prebasis* for the topology on X is a subclass $\mathscr{B}_X \subseteq \mathrm{Ob}(X)$ such that every covering $\{V_i \to V\}$ can be refined to a covering by elements of $\mathscr{B}_X$.

Example 2.3.2. If X is an algebraic stack, we can take affines with a smooth morphism to X as a prebasis for the lisse-étale topology on X.

Lemma 2.3.3. *Let F be a presheaf of abelian groups on a site X with a prebasis $\mathscr{B}_X$. Let $F^{\#}$ denote the sheafification of F. Suppose for every $U \in \mathscr{B}_X$ that $F(U) = 0$. Then $F^{\#} = 0$.*

Proof. By the universal property of sheafification, it is enough to prove for any sheaf G and morphism of presheaves $\varphi\colon F \to G$ that $\varphi = 0$. Choose any $V \in \mathrm{Ob}(X)$ and section $s \in F(V)$. By assumption, there exists a cover $\{U_i \to V\}$ with $U_i \in \mathscr{B}_X$ such that $s|_{U_i} = 0$. Then $\varphi(s)|_{U_i} = \varphi(s|_{U_i}) = 0$. Since G is a sheaf, $\varphi(s) = 0$. We conclude that $\varphi = 0$ as desired. □

Let $f : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ be a morphism of ringed sites. Recall that f is said to be *flat* if the ring map $f^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ is flat. In particular, this implies that the pullback $f^*\colon \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_Y)$ is an exact functor. The technical lemma that we need is this:

Lemma 2.3.4. *Let $f\colon (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ be a flat morphism of ringed sites which admit prebases $\mathscr{B}_X$ and $\mathscr{B}_Y$ respectively. Let $\mathscr{A}_Y \subseteq \mathrm{Mod}(\mathscr{O}_Y)$ be a weak Serre subcategory and fix $\mathscr{G} \in D(Y)$. Suppose the following conditions hold:*

1. *For any object U in $\mathscr{B}_X$, $u_f(U)$ is in $\mathscr{B}_Y$ where $u_f\colon \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ is the functor on objects associated to the morphism of sites f.*
2. *For any $V \in \mathscr{B}_Y$ and $G \in \mathscr{A}_Y$, $H^p(V, G) = 0$ for all $p > 0$.*
3. *For any $q \in \mathbf{Z}$, $H^q(\mathscr{G}) \in \mathscr{A}_Y$.*

Choose any integer $n \in \mathbf{Z}$. Then for any non-negative integer j_0 such that $n + j_0 \geq 0$, the natural map

$$H^n(Rf_*\mathscr{G}) \to H^n(Rf_*\tau_{\geq -j_0}\mathscr{G})$$

is an isomorphism.

Proof. The exact triangle

$$Rf_*\tau_{< -j_0}\mathscr{G} \to Rf_*\mathscr{G} \to Rf_*\tau_{\geq -j_0}\mathscr{G}$$

gives rise to an exact sequence in cohomology

$$H^n(Rf_*\tau_{< -j_0}\mathscr{G}) \to H^n(Rf_*\mathscr{G}) \to H^n(Rf_*\tau_{\geq -j_0}\mathscr{G}) \to H^{n+1}(Rf_*\tau_{< -j_0}\mathscr{G}).$$

We need to show that the left and right-most terms are zero. To do this, first recall that If $(C, \mathscr{O}_C)$ is a ringed site and K an object of $D(\mathscr{O}_C)$, the n-th cohomology sheaf of K, $H^n(K)$, is the sheafification of the presheaf $U \mapsto H^n(U, K)$, where $H^n(U, K)$ is the n-th cohomology group $H^n(R\Gamma(U, K))$. Therefore, by this and Lemma 2.3.3, it is enough to show for any object $U \in \mathscr{B}_X$ that the following is true.

Claim 2.3.5. *The map*

$$H^i(U, Rf_*\mathscr{G}) \to H^i(U, Rf_*\tau_{\geq -j_0}\mathscr{G})$$

has the following properties. It is:

- *Surjective for* $i = n - 1$*.*
- *An isomorphism for* $i = n$*.*
- *Injective for* $i = n + 1$*.*

We show the $i = n$ case is an isomorphism as follows. First, using assumptions (2) (3) we can apply [Stacks, Tag 0D6P] (which relies on the techniques of [Spa88] on K-injective resolutions) to conclude that the natural map

$$\mathscr{G} \to R\lim \tau_{\geq -j}\mathscr{G}$$

is an isomorphism, where j runs through all positive integers. Then, since derived pushforward commutes with derived limits [Stacks, Tag 0A07],

$$H^n(U, Rf_*\mathscr{G}) \xrightarrow{\sim} H^n(U, R\lim Rf_*\tau_{\geq -j}\mathscr{G}).$$

It is now sufficient to prove that

$$H^n(U, R\lim Rf_*\tau_{\geq -j}\mathscr{G}) \xrightarrow{\sim} H^n(U, Rf_*\tau_{\geq -j_0}\mathscr{G}).$$

To get a hold on the derived limit, we use the Milnor exact sequence

$$0 \to R^1 \lim H^{n-1}(U, Rf_*\tau_{\geq -j}\mathscr{G}) \to H^n(U, R\lim Rf_*\tau_{\geq -j}\mathscr{G})) \to \lim H^n(U, Rf_*\tau_{\geq -j}\mathscr{G}) \to 0.$$

We will show that

$$\begin{aligned} R^1 \lim H^{n-1}(U, Rf_*\tau_{\geq -j}\mathscr{G}) &= 0, \\ \lim H^n(U, Rf_*\tau_{\geq -j}\mathscr{G}) &= H^n(U, Rf_*\tau_{\geq -j_0}\mathscr{G}). \end{aligned}$$

To this end, consider the exact triangle

$$H^{-(j_0+k)}(\mathscr{G})[j_0 + k] \to \tau_{\geq -(j_0+k)}\mathscr{G} \to \tau_{\geq -(j_0+k)+1}\mathscr{G},$$

where k is a positive integer that we will allow to vary. Applying $R\Gamma(U, -)$ and then taking cohomology, we get an exact sequence

$$H^{n+j_0+k}(U, Rf_*H^{-(j_0+k)}(\mathscr{G})) \longrightarrow H^n(U, Rf_*\tau_{\geq -(j_0+k)}\mathscr{G}) \longrightarrow H^n(U, Rf_*\tau_{\geq -(j_0+k)+1}\mathscr{G}) \longrightarrow H^{n+j_0+k+1}(U, Rf_*H^{-(j_0+k)}(\mathscr{G})).$$

The key observation now is that for $k \geq 1$,

$$\begin{aligned} H^{n+j_0+k}(U, Rf_*H^{-(j_0+k)}(\mathscr{G})) &\stackrel{\text{def}}{\equiv} H^{n+j_0+k}(R\Gamma(U, Rf_*H^{-(j_0+k)}(\mathscr{G}))) \\ &= H^{n+j_0+k}(u_f(U), H^{-(j_0+k)}(\mathscr{G})) \\ &= 0. \end{aligned}$$

Indeed by (1), $u_f(U) \in \mathscr{B}_Y$ and the fact that $k \geq 1$ implies $n + j_0 + k > 0$. The vanishing of the above cohomology group then follows from assumptions (2) and (3). Similarly,

$$H^{n+j_0+k+1}(U, Rf_*H^{-(j_0+k)}(\mathscr{G})) = 0$$

for $k \geq 1$ as well. The conclusion is that

$$G_{-j} \coloneqq H^n(U, Rf_*\tau_{\geq -j}\mathscr{G})$$

is an inverse system of abelian groups such that $G_{-(j_0+k)} \xrightarrow{\sim} G_{-(j_0+k)+1}$ for all $k \geq 1$. By Lemma 2.3.6 (c) below, we conclude that

$$\lim H^n(U, Rf_*\tau_{\geq -j}\mathscr{G}) = H^n(U, Rf_*\tau_{\geq -j_0}\mathscr{G}).$$

Reasoning similarly using Lemma 2.3.6 (a),

$$R^1 \lim H^{n-1}(U, Rf_*\tau_{\geq -j}\mathscr{G}) = 0.$$

This finishes the $i = n$ case of Claim 2.3.5. The rest of the aforementioned claim is proven similarly. □

Lemma 2.3.6. *Let $\{G_{-j}\}_{j \in \mathbf{N}}$ be an inverse system of abelian groups*

$$\ldots \to G_{-3} \to G_{-2} \to G_{-1}.$$

Suppose there exists $j_0 \geq 1$ such that for all $k \geq 1$, $G_{-(j_0+k)} \to G_{-(j_0+k)+1}$ is surjective. Then the following hold:

(a) The system $\{G_{-j}\}_j$ is Mittag-Leffler.

(b) The natural map $\lim G_{-j} \to G_{-j_0}$ is surjective.

(c) In addition, if the transition maps $G_{-(j_0+k)} \to G_{-(j_0+k)+1)}$ are injective (a fortiori isomorphisms) for $k \geq 1$, then $\lim G_{-j} \to G_{-j_0}$ is an isomorphism.

Proof. Statement (a) follows from the fact that the transition maps are all eventually surjective. For (b), we must show that any $x \in G_{-j_0}$ can be lifted to G_{-j_0+k} for all $k \geq 1$. This follows from the assumption that all transition maps $G_{-(j_0+k)} \to G_{-(j_0+k)+1}$ are surjective. Finally, suppose that $G_{-(j_0+k)} \to G_{-j_0}$ is an isomorphism for all $k \geq 1$. If an element

$$\overrightarrow{x} := (\dots, x_{-(j_0+1)}, x_{-(j_0)}, x_{-(j_0-1)}, \dots, x_{-1}) \in \lim G_{-j}$$

maps to zero in G_{-j_0}, then necessarily $x_{-j_0} = 0$. By definition of the inverse limit,

$$x_{-(j_0-1)} = \dots = x_{-1} = 0.$$

By injectivity, $x_{-(j_0+k)} = 0$ for all $k \geq 1$ and therefore $\overrightarrow{x} = 0$. □

Here is the main result of this section:

Theorem 2.3.7. *Let $f\colon (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ be a flat morphism of ringed sites which admit prebases $\mathscr{B}_Y$ and $\mathscr{B}_X$ respectively. Fix weak Serre subcategories $\mathscr{A}_Y \subseteq \mathrm{Mod}(\mathscr{O}_Y)$ and $\mathscr{A}_X \subseteq \mathrm{Mod}(\mathscr{O}_X)$. Suppose that the following conditions hold:*

(i) The pullback map $f^*\colon \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_Y)$ restricts to an equivalence of categories

$$f^*\colon \mathscr{A}_X \xrightarrow{\sim} \mathscr{A}_Y.$$

(ii) For every $F, F' \in \mathscr{A}_X$ and $n \in \mathbf{Z}$, the natural map

$$\mathrm{Ext}^n_{\mathscr{O}_X}(F, F') \to \mathrm{Ext}^n_{\mathscr{O}_Y}(f^*F, f^*F')$$

is an isomorphism of abelian groups.

(iii) For any object U in $\mathscr{B}_X$, $u_f(U)$ is in $\mathscr{B}_Y$ where $u_f\colon \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ is the functor associated to the morphism of sites f.

(iv) For every $G \in \mathscr{A}_Y$ and $V \in \mathscr{B}_Y$, $H^p(V, G) = 0$ for all $p > 0$.

Then the pullback $f^*\colon D^-(X) \to D^-(Y)$ restricts to an equivalence of categories

$$f^*\colon D^-_{\mathscr{A}_X}(X) \xrightarrow{\sim} D^-_{\mathscr{A}_Y}(Y).$$

Proof. We will first prove that f^* restricts to a fully faithful functor

$$f^*\colon D^b_{\mathscr{A}_X}(X) \to D^b_{\mathscr{A}_Y}(Y).$$

The fact that f^* sends bounded objects to bounded objects is clear because f is flat, a fortiori commutes with cohomology. Fix $\mathscr{F} \in D^b_{\mathscr{A}_X}(X)$. It is enough to prove for any $\mathscr{F}' \in D^b_{\mathscr{A}_X}(X)$ that

$$R\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}') = R\operatorname{Hom}_{\mathscr{O}_Y}(f^*\mathscr{F}, f^*\mathscr{F}'). \tag{2.3.1}$$

Let us first assume that $\mathscr{F}'$ is concentrated in a single degree, so in particular is in $\mathscr{A}_X$. We will prove the equality above by induction on the length of $\mathscr{F}$. If $\mathscr{F}$ is concentrated in a single degree, then this is condition (ii) above. Now suppose that $\mathscr{F}$ is a bounded complex concentrated in the interval $[a, b]$. Consider the exact triangle

$$H^a(\mathscr{F})[-a] \to \mathscr{F} \to \tau_{\geq a+1}\mathscr{F}.$$

Applying f^* and then $R\operatorname{Hom}_{\mathscr{O}_Y}(-, f^*\mathscr{F}')$ gives a diagram

$$\begin{array}{ccccc} R\operatorname{Hom}_{\mathscr{O}_X}(\tau_{\geq a+1}\mathscr{F}, \mathscr{F}') & \longrightarrow & R\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}') & \longrightarrow & R\operatorname{Hom}_{\mathscr{O}_X}(H^a(\mathscr{F})[-a], \mathscr{F}') \\ \downarrow & & \downarrow & & \downarrow \\ R\operatorname{Hom}_{\mathscr{O}_Y}(f^*\tau_{\geq a+1}\mathscr{F}, f^*\mathscr{F}') & \longrightarrow & R\operatorname{Hom}_{\mathscr{O}_Y}(f^*\mathscr{F}, f^*\mathscr{F}') & \longrightarrow & R\operatorname{Hom}_{\mathscr{O}_Y}(f^*H^a(\mathscr{F})[-a], f^*\mathscr{F}'). \end{array}$$

By induction, the left vertical arrow is an isomorphism. Since all the cohomology sheaves of $\mathscr{F}$ are in $\mathscr{A}_X$, the right vertical arrow is an isomorphism. Therefore the middle vertical arrow is an isomorphism. This shows (2.3.1) in the case that $\mathscr{F}'$ is concentrated in a single degree, and repeating the same argument above shows (2.3.1) in general. Therefore $f^*\colon D^b_{\mathscr{A}_X}(X) \to D^b_Y(Y)$ is fully faithful.

We now prove that

$$f^*\colon D^b_{\mathscr{A}_X}(X) \to D^b_{\mathscr{A}_Y}(Y).$$

is essentially surjective, a fortiori an equivalence of categories. Fix $\mathscr{G} \in D^b_{\mathscr{A}_Y}(Y)$. Suppose that $\mathscr{G}$ is concentrated in the interval $[a, b]$. Consider the exact triangle

$$(\tau_{\geq a+1}\mathscr{G})[-1] \to H^a(\mathscr{G})[-a] \to \mathscr{G}.$$

By (i) and the inductive hypothesis, there exists $\mathscr{F}, \mathscr{F}'$ so that

$$\begin{aligned} f^*\mathscr{F} &= (\tau_{\geq a+1}\mathscr{G})[-1], \\ f^*\mathscr{F}' &= H^a(\mathscr{G})[-a]. \end{aligned}$$

By full faithfulness, the morphism $(\tau_{\geq a+1}\mathscr{G})[-1] \to H^a(\mathscr{G})[-a]$ is equal to $f^*\varphi$ for some $f\colon \mathscr{F} \to \mathscr{F}'$. It follows that $\mathscr{G} \cong f^*\operatorname{Cone}(\mathscr{F} \to \mathscr{F}')$ and so we have the equivalence on bounded derived categories as claimed.

Finally, we will use Lemma 2.3.4 to reduce to the bounded case as follows. We must show for any $\mathscr{G} \in D^-_{\mathscr{A}_Y}(Y)$ and $\mathscr{F} \in D^-_{\mathscr{A}_X}(X)$ that the counit $f^*Rf_*\mathscr{G} \to \mathscr{G}$ and unit $\mathscr{F} \to Rf_*f^*\mathscr{F}$ are isomorphisms. Equivalently, for any $n \in \mathbf{Z}$ that

$$H^n(f^*Rf_*\mathscr{G}) \to H^n(\mathscr{G}) \quad \text{and} \quad H^n(\mathscr{F}) \to H^n(Rf_*f^*\mathscr{F})$$

are isomorphisms. Consider first the case of the counit. Fix $n \in \mathbf{Z}$ and choose a non-negative integer $j_0 \geq 0$ such that $n + j_0 \geq 0$. We want to show that the arrow (1) in the diagram below is an isomorphism.

$$\begin{array}{ccc}
H^n(f^*Rf_*\mathscr{G}) & \xrightarrow{(1)} & H^n(\mathscr{G}) \\
\| & & \\
f^*H^n(Rf_*\mathscr{G}) & & \\
\downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (4)} \\
f^*H^n(Rf_*\tau_{\geq -j_0}\mathscr{G}) & & \\
\| & & \\
H^n(f^*Rf_*\tau_{\geq -j_0}\mathscr{G}) & \xrightarrow{(3)} & H^n(\tau_{\geq -j_0}\mathscr{G})
\end{array}$$

The displayed equalities follow from flatness and arrow (2) is an isomorphism from Lemma 2.3.4. Furthermore, the object $\tau_{\geq -j_0}\mathscr{G}$ is bounded and therefore (3) is an isomorphism. Arrow (4) is trivially an isomorphism and therefore (1) is an isomorphism as desired. Reasoning similarly with the diagram

$$\begin{array}{ccc}
H^n(\mathscr{F}) & \longrightarrow & H^n(Rf_*f^*\mathscr{F}) \\
\downarrow & & \downarrow \\
 & & H^n(Rf_*\tau_{\geq -j_0}f^*\mathscr{F}) \\
 & & \| \\
H^n(\tau_{\geq -j_0}\mathscr{F}) & \longrightarrow & H^n(Rf_*f^*\tau_{\geq -j_0}\mathscr{F})
\end{array}$$

allows us to conclude that the unit is an isomorphism. This completes the proof of Theorem 2.3.7. □

2.4 Formal GAGA for pseudo-coherent complexes

2.4.1 Preliminary definitions

Let R be a ring. Recall that an R-module M is said to be finitely presented if there exists an exact sequence $R^{n_{-1}} \to R^{n_0} \to M \to 0$. Alternatively, thinking of M as being the complex $M[0]$, there exists a morphism of complexes

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & R^{n_{-1}} & \longrightarrow & R^{n_0} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

with the following properties.

- The map on cohomology is an isomorphism in degree 0.
- The map on cohomology is surjective in degree -1.

This motivates the following definition:

Definition 2.4.1 ([Stacks, Tag 064Q])**.** Let R be a ring. A complex of R-modules $P^\bullet$ is said to be *m-pseudo coherent* (for some $m \in \mathbf{Z}$) if there exists a bounded complex of finite free modules $F^\bullet$ and a morphism of complexes $F^\bullet \to P^\bullet$ such that the following hold:

- $H^i(F^\bullet) \to H^i(P^\bullet)$ is an isomorphism for $i > m$.
- $H^i(F^\bullet) \to H^i(P^\bullet)$ and surjective for $i = m$.

A complex $P^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for every $m \in \mathbf{Z}$. An object $P \in D(R)$ is *pseudo-coherent* if it is quasi-isomorphic to a pseudo-coherent complex.

Example 2.4.2. An R-module M is finite if and only if $M[0] \in D(R)$ is 0-pseudo-coherent, and finitely presented if and only if $M[0]$ is -1-pseudo-coherent. This is clear from the definition above.

Remark 2.4.3. Let P be an object of $D(R)$. Then P is pseudo-coherent if and only if there exists a quasi-isomorphism $F^\bullet \to P$, where $F^\bullet$ is a bounded above complex of finite free R-modules. This is [Stacks, Tag 064T].

We will let $D^-_{\mathrm{pc}}(R)$ denote the full triangulated subcategory of $D(R)$ consisting of objects of $D(R)$ quasi-isomorphic to a pseudo-coherent complex. The category D^-_{pc} is triangulated by [Stacks, Tag 064V] and [Stacks, Tag 064X]. The definition for rings generalizes to the following situation:

Definition 2.4.4 ([Stacks, Tag 08FT])**.** Let $(C, \mathscr{O}_C)$ be a ringed site. An object $P \in D(C)$ is said to be m-pseudo-coherent, if every $U \in \mathrm{Ob}(C)$ admits a covering $\{U_j \to U\}$, and morphisms $\alpha_j \colon F_j^\bullet \to P|_{U_j}$ (with each $F_j^\bullet$ a bounded complex of finite projective modules) such that the following hold:

- $H^i(\alpha_j)$ is an isomorphism for all $i > m$.
- $H^i(\alpha_j)$ is surjective for $i = m$.

We say that an object $P \in D(C)$ is pseudo-coherent if it is m-pseudo-coherent for all $m \in \mathbf{Z}$. As in the case of rings, we will let $D^-_{\mathrm{pc}}(C)$ denote the full triangulated subcategory of $D(C)$ of objects quasi-isomorphic to a bounded above pseudo-coherent complex. We need an important permanence property of pseudo-coherent objects:

Lemma 2.4.5. *Let $f\colon (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ be a morphism of ringed sites. Then $Lf\colon D(X) \to D(Y)$ restricts to a functor $D^-_{\mathrm{pc}}(X) \to D^-_{\mathrm{pc}}(Y)$.*

Proof. See [Stacks, Tag 08H4]. □

Remark 2.4.6. If $(C, \mathscr{O}_C)$ is a ringed site with coherent structure sheaf, then $D^-_{\mathrm{pc}}(C) = D^-_{\mathrm{coh}}(C)$, where the latter is the full subcategory of $D(C)$ of objects quasi-isomorphic to a bounded above complex of $\mathscr{O}_C$-modules with coherent cohomology [SGA6, Exposé 1, Corollary 3.5]. As all the ringed sites we consider in this article have coherent structure sheaf, we will henceforth take pseudo-coherent to mean "bounded above with coherent cohomology".

2.4.2 The GAGA theorem

Let X be a locally Noetherian scheme and $I \subseteq \mathscr{O}_X$ a coherent ideal defining a closed subscheme $X_0 \subseteq X$. The *formal completion* of X along X_0 (denoted $\widehat{X}_{\mathrm{formal,Zar}}$) is the topologically ringed site $(X_0, \widehat{\mathscr{O}}_{X,\mathrm{formal}})$, where $\widehat{\mathscr{O}}_{X,\mathrm{formal}}$ is the sheaf of rings

$$\widehat{\mathscr{O}}_{X_{\mathrm{formal}}}(U_0) \coloneqq \lim H^0(U_n, \mathscr{O}_{U_n}).$$

Here U_n is the unique lifting of the Zariski open $U_0 \to X_0$ to a Zariski open $U_n \to X_n$. For example, if $X = \operatorname{Spec} A$, then $\widehat{X}_{\mathrm{formal,Zar}}$ is nothing more than $\operatorname{Spf} \widehat{A}$, where $\widehat{A}$ is the I-adic completion of A. To state our GAGA theorem, we will need an analogous notion of formal completion for stacks. However, the definition above for schemes does not generalize because the lisse-étale site is not invariant under infinitesimal deformations. Instead, we will work with a surrogate definition of formal completion.

Definition 2.4.7. [Con05, Definition 1.2] The ringed site $\widehat{\mathscr{X}}$ is the lisse-étale site of $\mathscr{X}$ equipped with the sheaf of rings

$$\widehat{\mathscr{O}}_{\mathscr{X}} \coloneqq \lim \mathscr{O}_{\mathscr{X}}/\mathscr{I}^{n+1},$$

where the limit is taken in the category of lisse-étale $\mathscr{O}_{\mathscr{X}}$-modules.

There is a canonical morphism of ringed sites

$$\iota\colon \widehat{\mathscr{X}} \to \mathscr{X}$$

which on objects is simply the identity functor. Furthermore, ι is flat by [GZB15, Lemma 3.3]. We now recall the main result of [Con05].

Theorem 2.4.8. *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Let $\mathscr{I}$ denote the pullback of I to $\mathscr{X}$ and $\widehat{\mathscr{X}}$ the ringed site as defined above with respect to $\mathscr{I}$. Define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Then there are natural equivalences of categories*

$$\mathrm{Coh}(\mathscr{X}) \xrightarrow{\sim} \mathrm{Coh}(\widehat{\mathscr{X}}) \xrightarrow{\sim} \lim \mathrm{Coh}(\mathscr{X}_n).$$

The first arrow is given by sending a coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ to $\iota^*\mathscr{F} \simeq \lim \mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}$, while the second arrow is given by sending a coherent sheaf $\mathscr{G}$ on $\widehat{\mathscr{X}}$ to the adic system $(\mathscr{G}/\mathscr{I}^{n+1}\mathscr{G})_n$.

Remark 2.4.9. In the case that $\mathscr{X} = \operatorname{Spec} A$, we have $\mathrm{Coh}(\widehat{\operatorname{Spec} A}) = \mathrm{Coh}(\operatorname{Spf} A)$ by [Con05, Remark 1.6]. Therefore, the proposition above says that

$$\mathrm{Coh}(\operatorname{Spec} A) \xrightarrow{\sim} \mathrm{Coh}(\operatorname{Spf} A) \xrightarrow{\sim} \lim \mathrm{Coh}(\operatorname{Spec} A/I^{n+1}).$$

It is tempting to think that given an adic system of finite A/I^{n+1}-modules M_n, the corresponding coherent sheaf on $\operatorname{Spec} A$ is $\lim \widetilde{M_n}$. This however is false. Let A be a complete DVR with uniformizer ϖ. The sheaf $\lim \widetilde{A/\varpi^n}$ on $\operatorname{Spec} A$ is not quasi-coherent because its global sections are A while its value on the basic open $D(\varpi)$ is zero. The issue is that the functor sending an A-module M to its associated quasi-coherent sheaf $\widetilde{M}$ does not commute with limits. However, notice that $\lim \widetilde{A/\varpi^n}$ *is* a coherent sheaf on $\operatorname{Spf} A$.

We can now prove the following theorem:

Theorem 2.4.10 (Formal GAGA for pseudo-coherent complexes)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. The flat morphism of sites $\iota\colon \widehat{\mathscr{X}} \to \mathscr{X}$ induces an equivalence of categories*

$$\iota^*\colon D^-_{\mathrm{coh}}(\mathscr{X}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\widehat{\mathscr{X}}).$$

Proof. We will apply Theorem 2.3.7 to the morphism of sites $\iota\colon \widehat{\mathscr{X}} \to \mathscr{X}$. We will take smooth morphisms $U \to \mathscr{X}$, where U is affine as a prebasis for the topology on $\mathscr{X}$, and similarly for $\widehat{\mathscr{X}}$ (the underlying sites of $\mathscr{X}$ and $\widehat{\mathscr{X}}$ are the same). The weak Serre subcategories in question are $\mathrm{Coh}(\widehat{\mathscr{X}})$ [Con05, Corollary 1.7] and $\mathrm{Coh}(\mathscr{X})$[1]. Conditions (i) and (ii) are respectively Theorem 2.4.8 above and [Con05, Lemma 4.3]. Condition (iv) is Lemma A.0.2 below. Condition (iii) is trivial because the functor on objects $u_\iota\colon \mathrm{Ob}(\mathscr{X}) \to \mathrm{Ob}(\widehat{\mathscr{X}})$ is the identity functor. All the conditions of Theorem 2.3.7 are satisfied and we are done. □

[1]The careful reader will note here that it is important to use the lisse-étale topology. In the big fppf site, the inclusion functor $\mathrm{Coh}(\mathscr{X}) \subseteq \mathrm{Mod}(\mathscr{O}_{\mathscr{X}})$ is not exact.

2.5 Grothendieck's existence theorem for pseudo-coherent complexes

2.5.1 Preparations

Let $j\colon \mathscr{Y} \to \mathscr{X}$ be a morphism of algebraic stacks. For any sheaf F on $\mathscr{Y}$ and object U of $\mathscr{X}_{\text{lis-ét}}$, recall that j_*F is the sheaf on $\mathscr{X}_{\text{lis-ét}}$ whose value on an object U of $\mathscr{X}_{\text{lis-ét}}$ is $F(\mathscr{Y} \times_{\mathscr{X}} U)$. In addition, also recall that for any sheaf G on $\mathscr{X}_{\text{lis-ét}}$, $j^{-1}G$ is the sheafification of the presheaf whose value on $V \in \mathscr{Y}_{\text{lis-ét}}$ is

$$\varinjlim_{V \to U} F(U),$$

where the limit is taken over all $U \in \mathscr{X}_{\text{lis-ét}}$ that fit into a commutative square

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathscr{Y} & \longrightarrow & \mathscr{X}. \end{array}$$

The functor j^{-1} is left adjoint to j_*. However, it was observed by Gabber and Behrend that the functor j^{-1} is *not* exact (even if $\mathscr{X}$ and $\mathscr{Y}$ are schemes!) and therefore j does not induce a morphism of ringed sites $\mathscr{Y}_{\text{lis-ét}} \to \mathscr{X}_{\text{lis-ét}}$. Hence, it is a subtle question to construct a derived pullback on the level of derived categories of *sheaves of modules*

$$Lj^*\colon D(\mathscr{X}) \to D(\mathscr{Y}).$$

On the other hand, the *derived pushforward*

$$Rj_*\colon D(\mathscr{Y}) \to D(\mathscr{X}) \tag{2.5.1}$$

always exists for formal reasons. Indeed,

$$j_*\colon \operatorname{Mod}(\mathscr{O}_{\mathscr{Y}}) \to \operatorname{Mod}(\mathscr{O}_{\mathscr{X}})$$

is a left exact functor between Grothendieck abelian categories, and therefore Rj_* exists by [Ser03, Corollary 3.14]. Of course, work must be done to show that Rj_* has good properties such as a Leray spectral sequence for cohomology, precisely because we do not know that j_* preserves injectives[2]. Indeed, this is the content of [Ols07].

To remedy the issue of the non-functoriality of the lisse-étale site, Hall and Rydh [HR17] (building

[2]If j^{-1} were exact, then j_* would be right adjoint to an exact functor and therefore preserve injectives.

on work of [LO08] on unbounded cohomological descent) have defined a surrogate functor

$$L(j_{\mathrm{qc}})^* \colon D_{\mathrm{qcoh}}(\mathscr{X}) \to D_{\mathrm{qcoh}}(\mathscr{Y}).$$

We briefly recall the construction of this functor in the case that j is representable, as this is the only situation we need. The general situation is no more difficult. Let $j \colon \mathscr{Y} \to \mathscr{X}$ be a representable morphism of algebraic stacks. Let $p : V \to \mathscr{X}$ be a smooth surjection from an algebraic space. Define $U \coloneqq V \times_{\mathscr{X}} \mathscr{Y}$ (which is an algebraic space by representability of j), and let $q \colon U \to \mathscr{Y}$ denote the corresponding smooth surjective morphism. Let $p_\bullet \colon V_\bullet \to \mathscr{X}$ and $q_\bullet \colon U_\bullet \to \mathscr{Y}$ denote the corresponding morphisms of simplicial algebraic spaces. Associated to each of these simplicial algebraic spaces are certain ringed topoi $U^+_{\bullet,\text{ét}}$ and $V^+_{\bullet,\text{ét}}$ [HR17, Section 1.1]. Similarly, in the lisse-étale topology we have $U^+_{\bullet,\text{lis-ét}}$ and $V^+_{\bullet,\text{lis-ét}}$. These ringed topoi, together with the original $\mathscr{X}$ and $\mathscr{Y}$ sit in a 2-commutative diagram

$$\begin{array}{ccccc} \mathscr{Y} & \xleftarrow{q^+_\bullet} & U^+_{\bullet,\text{lis-ét}} & \xrightarrow{\mathrm{res}_U} & U^+_{\bullet,\text{ét}} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j^+_{\bullet,\text{lis-ét}}} & & \downarrow{\scriptstyle j^+_{\bullet,\text{ét}}} \\ \mathscr{X} & \xleftarrow[p^+_\bullet]{} & V^+_{\bullet,\text{list-ét}} & \xrightarrow[\mathrm{res}_V]{} & V^+_{\bullet,\text{ét}}. \end{array} \tag{2.5.2}$$

The key idea to defining $L(j_{\mathrm{qc}})^*$ now is the following. By [LO08, Ex. 2.2.5], we have equivalences of categories

$$D_{\mathrm{qcoh}}(\mathscr{Y}) \xleftarrow{(q^+_\bullet)_*} D_{\mathrm{qcoh}}(U^+_{\bullet,\text{lis-ét}}) \xrightarrow{(\mathrm{res}_U)_*} D_{\mathrm{qcoh}}(U^+_{\bullet,\text{ét}}) \tag{2.5.3}$$

$$D_{\mathrm{qcoh}}(\mathscr{X}) \xleftarrow{(p^+_\bullet)_*} D_{\mathrm{qcoh}}(V^+_{\bullet,\text{lis-ét}}) \xrightarrow{(\mathrm{res}_V)_*} D_{\mathrm{qcoh}}(V^+_{\bullet,\text{ét}}). \tag{2.5.4}$$

On the other hand, functoriality of the étale topos gives a pullback $L(j^+_{\bullet,\text{ét}})^* \colon D(V^+_{\bullet,\text{ét}}) \to D(U^+_{\bullet,\text{ét}})$ which restricts to $L(j^+_{\bullet,\text{ét}})^* \colon D_{\mathrm{qcoh}}(V^+_{\bullet,\text{ét}}) \to D_{\mathrm{qcoh}}(U^+_{\bullet,\text{ét}})$. This allows us to define the functor $L(j_{\mathrm{qc}})^*$ as

$$L(j_{\mathrm{qc}})^* \coloneqq R(q^+_\bullet)_* \circ L(\mathrm{res}_U)^* \circ L(j^+_{\bullet,\text{ét}})^* \circ R(\mathrm{res}_V)_* \circ L(p^+_\bullet)^*.$$

It is readily checked that the definition of $L(j_{\mathrm{qc}})^*$ is independent of the choice of smooth cover of $\mathscr{X}$. We mention two important properties of this functor. The first property is that if j is flat, then given any object $P \in D_{\mathrm{qcoh}}(\mathscr{X})$ and integer $n \in \mathbf{Z}$, we have $H^n(L(j_{\mathrm{qc}})^* P) \cong j^* H^n(P)$. The second property is the following. Suppose that $\mathscr{Y}, \mathscr{X}$ are Deligne-Mumford stacks. The étale topos is functorial and therefore there is a pullback $Lj^* \colon D(\mathscr{X}_{\text{ét}}) \to D(\mathscr{Y}_{\text{ét}})$. Since the lisse-étale and étale topoi agree for Deligne-Mumford stacks, it follows we have a pullback $Lj^* \colon D(\mathscr{X}) \to D(\mathscr{Y})$. This pullback restricts to $D_{\mathrm{qcoh}}(\mathscr{X}) \to D_{\mathrm{qcoh}}(\mathscr{Y})$ and agrees with $L(j_{\mathrm{qc}})^*$.

The following results will be used to prove the existence theorem in the next subsection.

Proposition 2.5.1. *Let $j\colon \mathscr{Y} \to \mathscr{X}$ be a morphism of locally Noetherian algebraic stacks. Then the functor $L(j_{\mathrm{qc}})^*\colon D_{\mathrm{qcoh}}(\mathscr{X}) \to D_{\mathrm{qcoh}}(\mathscr{Y})$ restricts to a functor on pseudo-coherent complexes*

$$L(j_{\mathrm{qc}})^*\colon D^-_{\mathrm{coh}}(\mathscr{X}) \to D^-_{\mathrm{coh}}(\mathscr{Y}).$$

Proof. From the construction of $L(j_{\mathrm{qc}})^*$, we see that it is constructed "locally" at the level of the étale topology, where it is given as a pullback arising from a morphism of ringed topoi. Since pseudo-coherent complexes are preserved under arbitrary pullback, we are done. □

Proposition 2.5.2. *Let $j\colon \mathscr{Y} \to \mathscr{X}$ be a representable, quasi-compact and quasi-separated morphism of algebraic stacks. Then the restriction of Rj_* to $D_{\mathrm{qcoh}}(\mathscr{Y})$ factors through $D_{\mathrm{qcoh}}(\mathscr{X})$. Furthermore in this situation, $Rj_* = R(j_{\mathrm{qc}})_*$ where Rj_* is the derived pushforward (2.5.1), and therefore we have an adjoint pair*

$$L(j_{\mathrm{qc}})^*\colon D_{\mathrm{qcoh}}(\mathscr{Y}) \leftrightarrows D_{\mathrm{qcoh}}(\mathscr{X})\colon Rj_*.$$

Proof. A representable morphism is concentrated by [HR17, Lemma 2.5 (3)]. The result follows from [HR17, Theorem 2.6 (2)]. □

Proposition 2.5.3. *Let $j\colon \mathscr{Y} \to \mathscr{X}$ be a closed immersion of locally Noetherian algebraic stacks. Then there is a natural equivalence of categories*

$$\overline{j}^*\colon D^-_{\mathrm{coh}}(\mathscr{X}, j_*\mathscr{O}_{\mathscr{Y}}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\mathscr{Y})$$

Proof. Pick a a diagram as in (2.5.2). Then the morphism of ringed topoi $j^+_{\bullet,\text{ét}}\colon U^+_{\bullet,\text{ét}} \to V^+_{\bullet,\text{ét}}$ factors as

$$U^+_{\bullet,\text{ét}} \xrightarrow{\overline{j}^+_{\bullet,\text{ét}}} (V^+_{\bullet,\text{ét}}, (j^+_{\bullet,\text{ét}})_*\mathscr{O}_{U^+_{\bullet,\text{ét}}}) \xrightarrow{k^+_{\bullet,\text{ét}}} V^+_{\bullet,\text{ét}}.$$

Now in [HR17, Corollary 2.7], it is shown that

$$(\overline{j}^+_{\bullet,\text{ét}})^*\colon D_{\mathrm{qcoh}}(V^+_{\bullet,\text{ét}}, (j^+_{\bullet,\text{ét}})_*\mathscr{O}_{U^+_{\bullet,\text{ét}}}) \to D_{\mathrm{qcoh}}(U^+_{\bullet,\text{ét}})$$

is an equivalence of categories. Therefore by (2.5.3) and (2.5.4) this induces an equivalence $\overline{j}^*$ of triangulated categories

$$\overline{j}^*\colon D_{\mathrm{qcoh}}(\mathscr{X}, j_*\mathscr{O}_{\mathscr{Y}}) \xrightarrow{\sim} D_{\mathrm{qcoh}}(\mathscr{Y}).$$

To complete the proof, we must show that $\overline{j}^*$ descends to an equivalence at the level of pseudo-coherent complexes. From the construction of $\overline{j}$, it is clear that it restricts to a fully faithful functor

$$\overline{j}^*\colon D^-_{\mathrm{coh}}(\mathscr{X}, j_*\mathscr{O}_{\mathscr{Y}}) \to D^-_{\mathrm{coh}}(\mathscr{Y}).$$

The formation of $\overline{j}^*$ commutes with cohomology and therefore to conclude it is enough to show the following. Let F be any quasi-coherent sheaf on $(\mathscr{X}, j_*\mathscr{O}_{\mathscr{Y}})$. If $\overline{j}^* F$ is coherent, then the same is true of F. Now the question of coherence is local and therefore we may reduce to the case that $\mathscr{X}, \mathscr{Y}$ are affine schemes with the lisse-étale topology. By descent, we reduce to the setting of the Zariski topology from which the result is clear. □

2.5.2 Grothendieck's existence theorem for pseudo-coherent complexes

Let X be a scheme that is proper over an I-adically complete Noetherian ring A. Recall in this case that the existence theorem for coherent sheaves follows immediately from formal GAGA. Indeed, the theory of formal schemes tells us that the reduction map from $\mathrm{Coh}(\widehat{X})$ to the category of adic systems of coherent sheaves F_n on $X_n \coloneqq X \times_A A/I^{n+1}$ is an equivalence of categories. However, in the situation of complexes on an algebraic stack $\mathscr{X}$, the non-functoriality of the lisse-étale site means that extra care must be taken. There are three issues we must clarify:

1. The notion of an adic system of complexes on $\mathscr{X}_n$.
2. The notion of an adic system of complexes on $\mathscr{X}_n$ algebraizing to a complex on $\mathscr{X}$.
3. The definition of a reduction functor $D^-_{\mathrm{coh}}(\widehat{\mathscr{X}}) \to D^-_{\mathrm{coh}}(\mathscr{X}_n)$.

We address these as follows:

1. For $\mathscr{X}$ an algebraic stack over an I-adically complete Noetherian ring A, define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Let

$$\begin{aligned} i_{n,n+1}\colon \quad & \mathscr{X}_n \hookrightarrow \mathscr{X}_{n+1} \\ j_n\colon \quad & \mathscr{X}_n \hookrightarrow \mathscr{X} \end{aligned}$$

denote the canonical closed immersions of algebraic stacks. By Proposition 2.5.2 $L(i_{n,n+1,\mathrm{qc}})^*$ is left adjoint to $R(i_{n,n+1})_*$. A *system* of pseudo-coherent complexes P_n on $\mathscr{X}_n$ is the data for every $n \geq 0$ an object $P_n \in D^-_{\mathrm{coh}}(\mathscr{X}_n)$ with maps

$$P_{n+1} \to R(i_{n,n+1})_* P_n. \tag{2.5.5}$$

The system is said to be *adic* if (2.5.5) is adjoint to an isomorphism

$$L(i_{n,n+1,\mathrm{qc}})^* P_{n+1} \xrightarrow{\sim} P_n.$$

2. We say that an adic system of pseudo-coherent complexes P_n on $\mathscr{X}_n$ *algebraizes* to a pseudo-coherent complex P on $\mathscr{X}$ if there exists an object $P \in D^-_{\mathrm{coh}}(\mathscr{X})$ such that $L(j_{n,\mathrm{qc}})^* P \simeq P_n$.

3. By Proposition 2.5.3 that there is a natural equivalence of categories

$$\overline{j}_n^*\colon D^-_{\mathrm{coh}}(\mathscr{X}, j_{n,*}\mathscr{O}_{\mathscr{X}_n}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\mathscr{X}_n).$$

Therefore, it is enough to define a functor

$$D^-_{\mathrm{coh}}(\widehat{\mathscr{X}}) \to D^-_{\mathrm{coh}}(\mathscr{X}, j_{n,*}\mathscr{O}_{\mathscr{X}_n}).$$

However, because the underlying sites of $\widehat{\mathscr{X}}$ and $(\mathscr{X}, j_{n,*}\mathscr{O}_{\mathscr{X}_n})$ are the same, we can simply define this functor by extension of scalars. In summary, we define $D^-_{\mathrm{coh}}(\widehat{\mathscr{X}}) \to D^-_{\mathrm{coh}}(\mathscr{X}_n)$ as the composition

$$D^-_{\mathrm{coh}}(\widehat{\mathscr{X}}) \xrightarrow{Q \mapsto Q \otimes^{\mathrm{L}}_{\widehat{\mathscr{O}}_{\mathscr{X}}} j_{n,*}\mathscr{O}_{\mathscr{X}_n}} D^-_{\mathrm{coh}}(\mathscr{X}, j_{n,*}\mathscr{O}_{\mathscr{X}_n}) \xrightarrow{\overline{j}_n^*} D^-_{\mathrm{coh}}(\mathscr{X}_n). \tag{2.5.6}$$

The proposition below makes precise the relationship between (2.5.6) and the pullback

$$L(j_{n,\mathrm{qc}})^*\colon D^-_{\mathrm{coh}}(\mathscr{X}) \to D^-_{\mathrm{coh}}(\mathscr{X}_n).$$

obtained from [HR17].

Proposition 2.5.4. *Let $j\colon \mathscr{Y} \to \mathscr{X}$ be a closed immersion of locally Noetherian algebraic stacks. There is a 2-commutative diagram*

$$\begin{array}{ccccccc}
D_{\mathrm{qcoh}}(\mathscr{X}) & & \xrightarrow{\quad L(j_{\mathrm{qc}})^* \quad} & & & & D_{\mathrm{qcoh}}(\mathscr{Y}) \\
\uparrow & & & & & & \uparrow \\
D^-_{\mathrm{coh}}(\mathscr{X}) & \longrightarrow & D^-_{\mathrm{coh}}(\mathscr{X}, \widehat{\mathscr{O}}_{\mathscr{X}}) & \longrightarrow & D^-_{\mathrm{coh}}(\mathscr{X}, j_*\mathscr{O}_{\mathscr{Y}}) & \xrightarrow{\overline{j}^*} & D^-_{\mathrm{coh}}(\mathscr{Y})
\end{array}$$

where $\widehat{\mathscr{O}}_{\mathscr{X}}$ is the completion of $\mathscr{O}_{\mathscr{X}}$ with respect to the coherent ideal defining $\mathscr{Y}$, and the functor $\overline{j}^*$ is an equivalence by Proposition 2.5.3. The functor $D^-_{\mathrm{coh}}(\mathscr{X}) \to D^-_{\mathrm{coh}}(\mathscr{X}, \widehat{\mathscr{O}}_{\mathscr{X}})$ is given by extending scalars $P \mapsto P \otimes_{\mathscr{O}_{\mathscr{X}}} \widehat{\mathscr{O}}_{\mathscr{X}}$. Similarly, $D^-_{\mathrm{coh}}(\mathscr{X}, \widehat{\mathscr{O}}_{\mathscr{X}}) \to D^-_{\mathrm{coh}}(\mathscr{X}, j_*\mathscr{O}_{\mathscr{Y}})$ is given by $Q \mapsto Q \otimes^{\mathrm{L}}_{\widehat{\mathscr{O}}_{\mathscr{X}}} j_*\mathscr{O}_{\mathscr{Y}}$.

Proof. Pick a a diagram as in (2.5.2). Then we know that $L(j_{\mathrm{qc}})^*$ is given locally as $j^+_{\bullet,\text{ét}}\colon U^+_{\bullet,\text{ét}} \to V^+_{\bullet,\text{ét}}$. As observed in Proposition 2.5.3, this factors as

$$U^+_{\bullet,\text{ét}} \xrightarrow{\overline{j}^+_{\bullet,\text{ét}}} (V^+_{\bullet,\text{ét}}, (j^+_{\bullet,\text{ét}})_*\mathscr{O}_{U^+_{\bullet,\text{ét}}}) \xrightarrow{k^+_{\bullet,\text{ét}}} V^+_{\bullet,\text{ét}}.$$

The result follows. ◻

Theorem 2.5.5 (Grothendieck existence for pseudo-coherent complexes)**.** *Let $\mathscr{X}$ that be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Let $P_n \in D^-_{\mathrm{coh}}(\mathscr{X}_n)$ be an adic system of pseudo-coherent complexes. Then there exists a pseudo-coherent complex $P \in D^-_{\mathrm{coh}}(\mathscr{X})$ such that $L(j_{n,\mathrm{qc}})^* P \simeq P_n$.*

Proof. By formal GAGA (Theorem 2.4.10), Proposition 2.5.3 and Proposition 2.5.4, we reduce to showing the following. Let $Q_n \in D^-_{\mathrm{coh}}(\mathscr{X}, j_{n,*}\mathscr{O}_{\mathscr{X}_n})$ be an adic system of pseudo-coherent complexes. Then there exists a pseudo-coherent complex Q on $D^-_{\mathrm{coh}}(\widehat{\mathscr{X}})$ such that

$$Q \otimes^{\mathrm{L}}_{\widehat{\mathscr{O}}_{\mathscr{X}}} j_{n,*}\mathscr{O}_{\mathscr{X}_n} = Q_n.$$

Motivated by Remark 2.4.9, we construct Q as the derived limit $R\lim Q_k$. First, we show that $R\lim Q_k$ is bounded above with coherent cohomology as follows. As the question of coherence is local, we may assume that $\mathscr{X} = \operatorname{Spec} B$ equipped with the lisse-étale topology. By Proposition A.0.5, Corollary A.0.6 and Remark A.0.7, we may assume that each Q_k is a pseudo-coherent object on the ringed site $(\operatorname{Spec} B, j_{k,*}\mathscr{O}_{\operatorname{Spec} B_k})$ in the Zariski topology. In fact, since $\mathrm{Coh}(\operatorname{Spec} B, j_{k,*}\mathscr{O}_{\operatorname{Spec} B_k}) \simeq \mathrm{Coh}(\operatorname{Spec} B_k)$, we may reduce to the following situation. We have an adic system of pseudo-coherent objects Q_k on $\operatorname{Spec} B_k$ and we must prove that $R\lim Q_k$ is pseudo-coherent on $(\operatorname{Spec} B, \widehat{\mathscr{O}}_{\operatorname{Spec} B})$.

Choose a bounded above complex of finite free $\mathscr{O}_{B_0}$-modules $F_0^\bullet$ and a quasi-isomorphism $F_0^\bullet \to Q_0$. By [Stacks, Tag 0BCB] there is a bounded above complex of finite free $\mathscr{O}_{B_1}$-modules $F_1^\bullet$, a quasi-isomorphism $F_1^\bullet \to Q_1$ and a map of complexes(!) $F_1^\bullet \to F_0^\bullet$ lying over $Q_1 \to Q_0$ such that $F_1^\bullet \otimes_{B_1} B_0 = F_0^\bullet$. Continuing this procedure, we get an adic system of complexes $\{F_k^\bullet\}$ such that

$$R\lim Q_k \xrightarrow{\sim} R\lim F_k^\bullet$$

in $D(\operatorname{Spec} B, \widehat{\mathscr{O}}_{\operatorname{Spec} B})$. For a fixed index i, the proof of Lemma A.0.2 shows that $R\lim F_k^i = \lim F_k^i$ and therefore $R\lim Q_k = \lim F_k^\bullet$. Now $F_0^\bullet$ is bounded above and by Nakayama's lemma, the cohomology of each $F_k^\bullet$ lives in the same interval as $F_0^\bullet$. This proves that $\lim F_k^\bullet$ is bounded above. Furthermore for each i, $\{F_k^i\}$ is adic and therefore $\lim F_k^i$ is a coherent sheaf on $(\operatorname{Spec} B, \widehat{\mathscr{O}}_{\operatorname{Spec} B})$ by the theory of formal schemes. It follows that $\lim F_k^\bullet$ has coherent cohomology sheaves and therefore $R\lim Q_k$ is pseudo-coherent.

Finally, we must show that $R\lim Q_k \otimes^{\mathrm{L}}_{\widehat{\mathscr{O}}_{\mathscr{X}}} j_{n,*}\mathscr{O}_{\mathscr{X}_n} \xrightarrow{\sim} Q_n$. By Corollary B.0.5 it is enough to prove for every smooth morphism $\operatorname{Spec} B \to \mathscr{X}$ that

$$R\Gamma(\operatorname{Spec} B, R\lim Q_k) \otimes^{\mathrm{L}}_{\widehat{B}} B_n \xrightarrow{\sim} R\Gamma(\operatorname{Spec} B, Q_n).$$

Now $R\Gamma$ commutes with $R\lim$ and therefore the result follows from [Stacks, Tag 0CQF]. □

2.6 Formal GAGA and Grothendieck's existence theorem for perfect complexes

2.6.1 Preliminary definitions

Definition 2.6.1. Let R be a Noetherian ring. A complex of R-modules $P^\bullet$ is said to be *strictly perfect* if it is bounded with each P^i a finite projective module. Furthermore, we say that an object $P \in D(R)$ is *perfect* if it is quasi-isomorphic to a strictly perfect complex.

Definition 2.6.2. Let R be a ring. An object $K \in D(R)$, is said to have finite Tor dimension if it has Tor-amplitude in some interval $[a, b]$. That is, for any R-module M, $H^i(K \otimes_R^{\mathrm{L}} M) = 0$ for all $i \notin [a, b]$.

Remark 2.6.3. An object $P \in D(R)$ is perfect if and only if it is pseudo-coherent and of finite Tor dimension [Stacks, Tag 0658].

As in the case of pseudo-coherence, the notions of being perfect and having finite Tor-dimension generalize to any ringed site $(C, \mathscr{O}_C)$. A complex $P^\bullet$ of $\mathscr{O}_C$-modules is said to be *perfect* if for any object U there is a cover $\{U_i \to U\}$, strictly perfect complexes $P_i^\bullet$ and quasi-isomorphisms of complexes $\alpha_i \colon P_i^\bullet \to P^\bullet|_{U_i}$. An object $P \in D(\mathscr{O}_C)$ is perfect if it is quasi-isomorphic to a perfect complex. Similarly, an object $K \in D(\mathscr{O}_C)$ has finite Tor-dimension if there is an interval $[a, b]$ such that $H^i(K \otimes_{\mathscr{O}_C}^{\mathrm{L}} F) = 0$ for all $\mathscr{O}_C$-modules F.

Remark 2.6.4. In the case that C is the Zariski site of an affine scheme, and K an object of $D_{\mathrm{qcoh}}(\mathscr{O}_C)$, the two notions of being of finite Tor dimension (rings and ringed sites) agree [Stacks, Tag 08EA]. This is not a priori obvious because in the definition of finite Tor dimension for sites, the test module F is *any* $\mathscr{O}_C$-module, not necessarily quasi-coherent.

2.6.2 Affine formal GAGA for perfect complexes

The goal of this subsection is to prove a theorem comparing perfect complexes on the Zariski and formal spectra of a J-adically complete Noetherian ring B. This will be used in the next subsection where we prove the formal GAGA theorem for perfect complexes on a proper algebraic stack $\mathscr{X}$. In order to state this theorem precisely, we need to recall some facts about coherent sheaves on affine formal schemes. Let B be a Noetherian ring, J an ideal of B and $\widehat{B}$ the J-adic completion of B. It is a fact that the canonical flat map of ringed spaces $i \colon \operatorname{Spf} \widehat{B} \to \operatorname{Spec} \widehat{B}$ gives rise to an equivalence of categories

$$i^* \colon \operatorname{Coh}(\operatorname{Spec} \widehat{B}) \xrightarrow{\sim} \operatorname{Coh}(\operatorname{Spf} \widehat{B}).$$

Furthermore, the inverse to i^* is given by the global sections functor $\Gamma(\operatorname{Spf} \widehat{B}, -)$ (by abuse of notation we identify $\operatorname{Coh}(\operatorname{Spec} \widehat{B})$ with finitely generated $\widehat{B}$-modules). Now given coherent sheaves

F, F' on $\operatorname{Spec} \widehat{B}$, the canonical map

$$\operatorname{Ext}^n_{\operatorname{Spec} \widehat{B}}(F, F') \xrightarrow{\sim} \operatorname{Ext}^n_{\operatorname{Spf} \widehat{B}}(i^* F, i^* F')$$

is an isomorphism for all $n \geq 0$ by [EGA, III, 4.5.2]. Therefore, the functor

$$i^* \colon D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\operatorname{Spf} \widehat{B}) \tag{2.6.1}$$

is an equivalence of categories following the method of proof of Theorem 2.4.10.

Theorem 2.6.5 (Affine formal GAGA for perfect complexes)**.** *The equivalence of categories (2.6.1) restricts to an equivalence of categories*

$$i^* \colon \operatorname{Perf}(\operatorname{Spec} \widehat{B}) \xrightarrow{\sim} \operatorname{Perf}(\operatorname{Spf} \widehat{B}).$$

The theorem above will follow from Corollary 2.6.8 below, which morally asserts that the property of being perfect can be detected by base change to the residue field. Note it is important that we work with perfect complexes rather than just vector bundles, for otherwise the result is false. Indeed, over a Noetherian local ring R, the (ordinary) base change of any finite R-module M to the residue field is finite free, but obviously not every finite module is projective. Corollary 2.6.8 follows from [Stacks, Tag 068V] in the non-Noetherian case. Nonetheless, since we are in the Noetherian situation, we feel compelled to give a more intuitive proof, in the sense that we use standard dévissage techniques.

Lemma 2.6.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and P an object of $D(R)$. If the Tor amplitude of $P \otimes^{\mathrm{L}}_R R/\mathfrak{m}$ (as an object of $D(R/\mathfrak{m})$) lies in $[a, b]$, then for any R-module N of finite length,*

$$H^i(P \otimes^{\mathrm{L}}_R N) = 0$$

for all $i \notin [a, b]$.

Proof. We will induct on the length of N, denoted $\ell(N)$. When $\ell(N) = 1$, necessarily $N \cong R/\mathfrak{m}$ and the result is clear. When $\ell(N) > 1$, we can find a submodule $N_0 \subseteq N$ such that $\ell(N_0) < \ell(N)$ and $\ell(N/N_0) = 1$. The result follows by considering the cohomology of the exact triangle

$$P \otimes^{\mathrm{L}}_R N_0 \to P \otimes^{\mathrm{L}}_R N \to P \otimes^{\mathrm{L}}_R N/N_0.$$

□

Proposition 2.6.7. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and P a pseudo-coherent object in $D(R)$. Suppose that $P \otimes^{\mathrm{L}}_R R/\mathfrak{m}$ has Tor amplitude in $[a, b]$. Then P has Tor amplitude in $[a, b]$.*

Proof. Choose a quasi-isomorphism $F^\bullet \to P$ where $F^\bullet$ is a bounded above complex of finite free modules (such a $F^\bullet$ exists because P is pseudo-coherent). The complex $F^\bullet$ is K-flat (in the sense of [Spa88]) and therefore can be used to compute derived tensor products. In other words, for any R-module M,

$$P \otimes_R^{\mathbf{L}} M = F^\bullet \otimes_R M$$

in $D(R)$. We must now show that $H^i(F^\bullet \otimes_R M) = 0$ for all $i \notin [a, b]$. The formation of the (underived) tensor product commutes with direct limits and similarly for cohomology. Therefore, we may suppose that M is finite, a fortiori complete and so

$$F^\bullet \otimes_R M = F^\bullet \otimes_R \lim(M/\mathfrak{m}^k M) = \lim(F^\bullet \otimes_R M/\mathfrak{m}^k M).$$

On the other hand, by [Stacks, Tag 091D] we have

$$R\lim(F^\bullet \otimes_R M/\mathfrak{m}^k M) = \lim(F^\bullet \otimes_R M/\mathfrak{m}^k M)$$

and so we can consider the Milnor exact sequence

$$0 \to R^1 \lim H^{i-1}(F^\bullet \otimes_R M/\mathfrak{m}^k M) \to H^i(\lim F^\bullet \otimes_R M/\mathfrak{m}^k M) \to \lim H^i(F^\bullet \otimes_R M/\mathfrak{m}^k M) \to 0.$$

For every k, $M/\mathfrak{m}^k M$ is a finite length R-module and therefore the previous lemma implies the right term vanishes for all $i \notin [a, b]$ *independent of* k *and the module* M. The same is true of the left term for $i \notin [a+1, b+1]$.

To finish the proof, we have to show that $R^1 \lim H^b(F^\bullet \otimes_R M/\mathfrak{m}^k M) = 0$. There is a short exact sequence of complexes

$$0 \to F^\bullet \otimes_R \mathfrak{m}^{k-1} M/\mathfrak{m}^k M \to F^\bullet \otimes_R M/\mathfrak{m}^k M \to F^\bullet \otimes_R M/\mathfrak{m}^{k-1} M \to 0$$

which upon taking cohomology gives a short exact sequence

$$H^b(F^\bullet \otimes_R M/\mathfrak{m}^k M) \longrightarrow H^b(F^\bullet \otimes_R M/\mathfrak{m}^{k-1} M) \longrightarrow H^{b+1}(F^\bullet \otimes_R \mathfrak{m}^{k-1} M/\mathfrak{m}^k M).$$

The right term is zero by the previous lemma since $\mathfrak{m}^{k-1} M/\mathfrak{m}^k M$ is an R-module of finite length. We have shown the inverse system $\{H^b(F^\bullet \otimes_R M/\mathfrak{m}^k M)\}_k$ has all transition maps surjective and therefore

$$R^1 \lim H^b(F^\bullet \otimes_R M/\mathfrak{m}^k M) = 0.$$

□

Corollary 2.6.8. *Let (R, I) be a Zariski pair, i.e., a Noetherian ring R and an ideal I contained in the Jacobson radical of R. Let P a pseudo-coherent object of $D(R)$. If $P \otimes^{\mathrm{L}}_R R/I$ has Tor amplitude in $[a, b]$ then P has Tor amplitude in $[a, b]$.*

Proof. The assumption on the Tor amplitude of $P \otimes^{\mathrm{L}}_R R/I$ implies for any maximal ideal $\mathfrak{m}$ of R, the Tor amplitude of $(P \otimes^{\mathrm{L}}_R R/I)_{\mathfrak{m}} = P_{\mathfrak{m}} \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/I_{\mathfrak{m}}$ is in $[a, b]$. Therefore, the same is true of

$$(P_{\mathfrak{m}} \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/I_{\mathfrak{m}}) \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}/I_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} = P_{\mathfrak{m}} \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$$

and hence also of

$$P_{\mathfrak{m}} \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}} = (P_{\mathfrak{m}} \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}) \otimes^{\mathrm{L}}_{\widehat{R}_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}},$$

where $\widehat{R}_{\mathfrak{m}}$ is the $\mathfrak{m}$-adic completion of $R_{\mathfrak{m}}$. By Proposition 2.6.7 we conclude that $P_{\mathfrak{m}} \otimes^{\mathrm{L}}_{R_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}$ has Tor amplitude in $[a, b]$. By faithful flatness, $P_{\mathfrak{m}}$ has Tor amplitude in $[a, b]$. Since $\mathfrak{m}$ was any maximal ideal of R, we are done. □

Proof of Theorem 2.6.5. It is clear that

$$i^*\colon \operatorname{Perf}(\operatorname{Spec} \widehat{B}) \to \operatorname{Perf}(\operatorname{Spf} \widehat{B})$$

is fully faithful. To prove essential surjectivity, by equivalence (2.6.1) it is enough to show the following. Let P be an object of $D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B})$. If i^*P is perfect, then P is perfect. To this end, consider the 2-commutative diagram

$$\begin{array}{ccc} D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B}) & \longrightarrow & D^-_{\mathrm{coh}}(\operatorname{Spf} \widehat{B}) \\ \downarrow & & \downarrow \\ D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B}/\widehat{J}) & \longrightarrow & D^-_{\mathrm{coh}}(\operatorname{Spf} \widehat{B}/\widehat{J}). \end{array}$$

Notice that $\operatorname{Spec} \widehat{B}/\widehat{J}$ and $\operatorname{Spf} \widehat{B}/\widehat{J}$ are *canonically isomorphic* as ringed spaces. Therefore, it is enough to show that if P is a pseudo-coherent object of $D(\operatorname{Spec} \widehat{B})$ such that $P \otimes^{\mathrm{L}}_{\widehat{B}} \widehat{B}/\widehat{J}$ is perfect, then P is perfect. Since $(\widehat{B}, \widehat{J})$ is a Zariski pair, Corollary 2.6.8 applies and we win. □

Finally, we record a consequence of Theorem 2.6.5 that will be used in the next subsection. Keeping with the notation above, let B be a Noetherian ring, $J \subseteq B$ an ideal and $\widehat{B}$ the J-adic completion of B. There are canonical flat morphisms of ringed spaces $i\colon \operatorname{Spf} \widehat{B} \to \operatorname{Spec} \widehat{B}$ and $j\colon \operatorname{Spec} \widehat{B} \to \operatorname{Spec} B$ which induce pullback maps

$$i^*\colon D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B}) \to D^-_{\mathrm{coh}}(\operatorname{Spf} \widehat{B}) \qquad \text{and} \qquad j^*\colon D^-_{\mathrm{coh}}(\operatorname{Spec} B) \to D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B}).$$

Corollary 2.6.9. *Let P be an object of $D^-_{\mathrm{coh}}(\operatorname{Spec} B)$ and define ι to be the composition*

$$\iota \equiv j \circ i\colon \operatorname{Spf} \widehat{B} \to \operatorname{Spec} \widehat{B} \to \operatorname{Spec} B.$$

Then $\iota^* P$ is perfect if and only if $j^* P$ is perfect.

Proof. The non-trivial direction to prove is $\iota^* P$ perfect implies that $j^* P$ is perfect. However, $\iota^* P = i^*(j^* P)$ and therefore the result follows immediately from the proof of Theorem 2.6.5 above. □

2.6.3 Formal GAGA for perfect complexes

Theorem 2.6.10. *Let $\mathscr{X} \to \operatorname{Spec} A$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. The equivalence*

$$\iota^*\colon D^-_{\mathrm{coh}}(\mathscr{X}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\widehat{\mathscr{X}})$$

of Theorem 2.4.10 restricts to an equivalence

$$\iota^*\colon \operatorname{Perf}(\mathscr{X}) \xrightarrow{\sim} \operatorname{Perf}(\widehat{\mathscr{X}}).$$

Proof. It is clear that the restriction $\iota^*\colon \operatorname{Perf}(\mathscr{X}) \to \operatorname{Perf}(\widehat{\mathscr{X}})$ is fully faithful. To show essential surjectivity, by Theorem 2.4.10 this amounts to showing that if an object P of $D^-_{\mathrm{coh}}(\mathscr{X})$ maps into $\operatorname{Perf}(\widehat{\mathscr{X}})$, then P is perfect. Without loss of generality, we may assume that the pullback of I to $\mathscr{X}$ is not the unit ideal, for otherwise $\mathscr{X} = \widehat{\mathscr{X}}$ and there is nothing to prove.

Let $U = \operatorname{Spec} B \to \mathscr{X}$ be a smooth surjection from an affine scheme. First we will show for any maximal ideal $\mathfrak{m}$ containing IB that the restriction of P to $\operatorname{Spec} B_{\mathfrak{m}}$ *in the Zariski topology* is perfect (The reader that prefers a more deformation-theoretic viewpoint may interpret this as saying for any closed point x_0 in $\mathscr{X}$, the restriction of P to a versal ring of $\mathscr{X}$ at x_0 is perfect). The object $\iota^* P$ is perfect on $\widehat{\mathscr{X}}$ and therefore $\iota^*(P|_{\operatorname{Spec} B_{\mathfrak{m}},\text{lis-ét}})$ is a perfect object of $D^-_{\mathrm{coh}}(\widehat{\operatorname{Spec} B}_{\mathfrak{m},\text{lis-ét}})$. On the other hand, the equivalence

$$(\widehat{\epsilon} \circ \widehat{\delta})^*\colon D^-_{\mathrm{coh}}(\widehat{\operatorname{Spec} B}_{\mathfrak{m},\mathrm{Zar}}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\widehat{\operatorname{Spec} B}_{\mathfrak{m},\text{lis-ét}})$$

given by Proposition A.0.5 restricts to the full subcategories of perfect complexes by Proposition A.0.8. Therefore, we may work with the Zariski topology and must show that if $Q \in D^-_{\mathrm{coh}}(\operatorname{Spec} B_{\mathfrak{m}})$ satisfies the property that $\iota^* Q$ is perfect, then Q is perfect.

By Corollary 2.6.9, we deduce that $j^* Q$ is perfect where (keeping with the notation in the previous subsection) j^* is the pullback

$$j^*\colon D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B_{\mathfrak{m}}}) \to D^-_{\mathrm{coh}}(\operatorname{Spec} B_{\mathfrak{m}}).$$

Now the ideal IB is contained in the maximal ideal $\mathfrak{m}$ and therefore the map $B_{\mathfrak{m}}$ to the I-adic completion $\widehat{B_{\mathfrak{m}}}$ is faithfully flat. We conclude that Q is perfect by [Stacks, Tag 068T]. This completes the claim that $P|_{\operatorname{Spec} B_{\mathfrak{m}}}$ is perfect in the Zariski topology.

The preceding argument shows that the restriction of P to the (Zariski) local ring at any closed point of $U_0 \coloneqq U \times_A A/I$ is perfect. By spreading out, there is a Zariski open V containing U_0 such that $P|_V$ is perfect. Therefore to show that P is perfect, it is enough to show that the composition $V \to U \to \mathscr{X}$ is surjective. Let W denote the image of this morphism. It is an open substack of $\mathscr{X}$ containing $\mathscr{X}_0$. If the complement $|\mathscr{X}| - |W|$ is non-empty, by properness its image in $\operatorname{Spec} A$ is a non-empty closed set disjoint from $\operatorname{Spec} A/I$, contradicting the adic property of A. Therefore $|W| = |\mathscr{X}|$ and so $V \to \mathscr{X}$ is surjective by [Stacks, Tag 04XI]. □

2.6.4 Grothendieck's existence theorem for perfect complexes

Theorem 2.6.11 (Grothendieck Existence)**.** *Let $\mathscr{X}$ be an algebraic stack that is proper over an I-adically complete Noetherian ring A. Define $\mathscr{X}_n \coloneqq \mathscr{X} \times_A A/I^{n+1}$. Let $\{P_n\}$ be an adic system of perfect complexes on $\mathscr{X}_n$. Then there exists a perfect complex $P \in \operatorname{Perf}(\mathscr{X})$ such that $L(j_{n,\mathrm{qc}})^* P \cong P_n$.*

Proof. By formal GAGA for perfect complexes (Theorem 2.6.10) and arguing as in Theorem 2.5.5, it is enough to show the following. Let Q_k be an adic system of perfect complexes with each $Q_k \in \operatorname{Perf}(\mathscr{X}, j_{k,*}\mathscr{O}_{\mathscr{X}_k})$. Then

$$Q \coloneqq R\lim Q_k$$

is perfect on $\widehat{\mathscr{X}}$.

As the question of being perfect is local, just as in the proof of Theorem 2.5.5 we may reduce to the situation that $\mathscr{X} = \operatorname{Spec} B$ and Q_k are perfect on $(\operatorname{Spec} B_k, \mathscr{O}_{\operatorname{Spec} B_k})$ in the Zariski topology. Furthermore, by the proof of Theorem 2.5.5, we know that $Q \coloneqq R\lim Q_k$ is an object of $D^-_{\mathrm{coh}}(\operatorname{Spec} B, \widehat{\mathscr{O}}_{\operatorname{Spec} B}) \simeq D^-_{\mathrm{coh}}(\operatorname{Spf} \widehat{B})$ such that its pullback to $\operatorname{Spec} B_0$ is isomorphic to Q_0. Now recall by (2.6.1) that there is an equivalence of categories

$$\iota^* \colon D^-_{\mathrm{coh}}(\operatorname{Spec} \widehat{B}) \to D^-_{\mathrm{coh}}(\operatorname{Spf} \widehat{B}).$$

It follows that Q is the image of a pseudo-coherent object Q' on $\operatorname{Spec} \widehat{B}$ with the property that $Q' \otimes^{\mathrm{L}}_{\widehat{B}} \widehat{B}/I\widehat{B}$ is perfect. By Corollary 2.6.8, Q' is perfect and therefore Q is perfect. □

2.7 Grothendieck's existence theorem for relatively perfect complexes

Definition 2.7.1. Let S be an algebraic space and $f\colon \mathscr{X} \to S$ an algebraic stack over S that is flat and locally of finite presentation. An object $P \in D(\mathscr{X}_{\text{lis-ét}})$ is said to be *relatively perfect* over S if it is pseudo-coherent as an object of $D(\mathscr{X}_{\text{lis-ét}})$ and locally has finite Tor dimension as an object of $D(f^{-1}\mathscr{O}_{S_{\text{lis-ét}}})$.

Remark 2.7.2. Let $X \to \operatorname{Spec} k$ be the nodal cubic curve. Then the structure sheaf of the node is relatively perfect over $\operatorname{Spec} k$, but is *not* perfect on X. This shows that the notion of being relatively perfect does not mean perfect on fibers.

Proposition 2.7.3. *Let $\mathscr{X}$ be an algebraic stack that is proper and flat over an I-adically complete Noetherian ring A. Define $\mathscr{X}_n := \mathscr{X} \times_A A/I^{n+1}$ and let $j_n\colon \mathscr{X}_n \hookrightarrow \mathscr{X}$ denote the canonical closed immersion of stacks. Let $P_n \in D(\mathscr{X}_n)$ be an adic system of relatively perfect objects over* $\operatorname{Spec} A/I^{n+1}$. *Then there is an object $P \in D(\mathscr{X})$ that is relatively perfect over* $\operatorname{Spec} A$ *such that* $L(j_{n,\text{qc}})^*P = P_n$.

Proof. Let $\operatorname{Spec} B \to \mathscr{X}$ be a smooth cover by an affine scheme. The restriction of $L(j_{n,qc})^*$ to the (small) étale site of $\operatorname{Spec} B$ agrees with the étale pullback $L(j_{n,\text{ét}})^*$. Therefore, as the question of being relatively perfect over A is local on $\mathscr{X}$, we may assume that $\mathscr{X} = \operatorname{Spec} B$ with the étale topology. Furthermore, just as in the proof of Theorem 2.6.10, by [Stacks, Tag 0DK7] and [Stacks, Tag 0DHY] we may work in the Zariski topology and prove the following. Let $\operatorname{Spec} B \to \operatorname{Spec} A$ be a flat morphism of finite type, with A an I-adically complete Noetherian ring A. Let P be a pseudo-coherent object of $D(\operatorname{Spec} B)$ and suppose that $P \otimes_B^{\mathrm{L}} B/IB$ has finite Tor dimension as an A/I-module. Then P has finite Tor dimension as an A-module. To prove this, observe by flatness that the natural map

$$B \otimes_A^{\mathrm{L}} A/I \to B/IB$$

is a quasi-isomorphism. Therefore,

$$\begin{aligned} P \otimes_B^{\mathrm{L}} B/IB &= P \otimes_B^{\mathrm{L}} (B \otimes_A^{\mathrm{L}} A/I) \\ &= P \otimes_A^{\mathrm{L}} A/I. \end{aligned}$$

Since $A \to B$ is finite type, P is a pseudo-coherent object of $D(A)$. The result follows from Corollary 2.6.8. ∎

Appendix A

Comparison between topologies on a scheme

Let X be a locally Noetherian scheme and $X_0 \subseteq X$ a closed subscheme defined by a coherent ideal I. Previously, recall that we defined the ringed site

$$\widehat{X} \coloneqq (X_{\text{lis-ét}}, \widehat{\mathscr{O}}_{X_{\text{lis-ét}}}).$$

Let X_{Zar} and $X_{\text{ét}}$ denote respectively the Zariski and (small) étale sites of X. Analogously, we may define $\widehat{X}_{\text{Zar}}$ and $\widehat{X}_{\text{ét}}$. In this appendix, we will record comparison results between the the categories of pseudo-coherent objects on $\widehat{X}_{\text{Zar}}, \widehat{X}_{\text{ét}}$ and $\widehat{X}$. Similarly, we will also record a result comparing Tor dimensions of pseudo-coherent objects on these three sites. We let $\widehat{\epsilon}\colon \widehat{X}_{\text{ét}} \to \widehat{X}_{\text{Zar}}$ and $\widehat{\delta}\colon \widehat{X}_{\text{lis-ét}} \to \widehat{X}_{\text{ét}}$ the canonical flat morphisms of ringed sites.

Proposition A.0.1. *The pullback maps $\widehat{\epsilon}^*$ and $\widehat{\delta}^*$ induce equivalences of categories*

$$\mathrm{Coh}(\widehat{X}_{\text{Zar}}) \xrightarrow{\widehat{\epsilon}^*} \mathrm{Coh}(\widehat{X}_{\text{ét}}) \xrightarrow{\widehat{\delta}^*} \mathrm{Coh}(\widehat{X}_{\text{lis-ét}}).$$

Proof. See [Con05, Remark 1.6]. □

Lemma A.0.2. *Let X be an affine Noetherian scheme, I a coherent ideal on X. For any coherent sheaf G on $\widehat{X}_{\text{ét}}$, $H^i(\widehat{X}_{\text{ét}}, G) = 0$ for all $i > 0$.*

Proof. Define $G_n \coloneqq G/I^{n+1}G$. First we prove that the natural map $G \to R\lim G_n$ is an isomorphism. Since $G \cong \lim G_n$, it is enough to show $H^i(R\lim G_n) = 0$ for all $i > 0$. Now $H^i(R\lim G_n)$ is the sheafification of the presheaf $\widehat{U} \mapsto H^i(\widehat{U}, R\lim G_n)$ for $\widehat{U} \in \mathrm{Ob}(\widehat{X}_{\text{ét}})$. Therefore we reduce to showing for $Y = \operatorname{Spec} A$ that

$$H^i(\widehat{Y}_{\text{ét}}, R\lim G_n) = 0$$

for all $i > 0$. Define $Y_n \coloneqq \operatorname{Spec} A/I^{n+1}$. The Milnor exact sequence gives

$$0 \to R^1 \lim H^{i-1}(\widehat{Y}_{\text{ét}}, G_n) \to H^i(\widehat{Y}_{\text{ét}}, R\lim G_n) \to \lim H^i(\widehat{Y}_{\text{ét}}, G_n) \to 0. \tag{A.0.1}$$

We have

$$\begin{aligned} H^i(\widehat{Y}_{\text{ét}}, G_n) &= H^i((Y_n)_{\text{ét}}, G_n) \\ &= H^i((Y_n)_{\text{Zar}}, G_n) \end{aligned}$$

by [Stacks, Tag 03DW]. Therefore by (A.0.1),

$$H^i(\widehat{Y}_{\text{ét}}, R\lim G_n) = 0$$

for all $i \geq 2$. To show vanishing when $i = 1$, we must show that

$$R^1 \lim H^0(\widehat{Y}_{\text{ét}}, G_n) = 0. \tag{A.0.2}$$

Now Y is affine and therefore $H^0(\widehat{Y}_{\text{ét}}, G_n) \to H^0(\widehat{Y}_{\text{ét}}, G_{n-1})$ is surjective. Hence (A.0.2) follows and so $G \xrightarrow{\sim} R\lim G_n$. Repeating exactly the same argument, we get that

$$H^i(\widehat{X}_{\text{ét}}, G) = 0$$

for all $i > 0$ as desired. ☐

Corollary A.0.3. *Let X be a locally Noetherian scheme and G a coherent sheaf on $\widehat{X}_{\text{ét}}$. Then $R^i\widehat{\epsilon}_* G = 0$ for all $i > 0$.*

Corollary A.0.4. *Let X be a locally Noetherian scheme and F a coherent sheaf on $\widehat{X}_{\text{Zar}}$. Then*

$$H^i(\widehat{X}_{\text{Zar}}, F) = H^i(\widehat{X}_{\text{ét}}, \widehat{\epsilon}^* F)$$

for all $n \geq 0$.

Proof. We have $H^i(\widehat{X}_{\text{ét}}, \widehat{\epsilon}^* F) = H^i(\widehat{X}_{\text{Zar}}, R\widehat{\epsilon}_* \widehat{\epsilon}^* F)$. By the coherence of $\widehat{\epsilon}^* F$, Corollary A.0.3 and the full-faithfulness of $\widehat{\epsilon}^*$ (Proposition A.0.1) we have $R\widehat{\epsilon}_* \widehat{\epsilon}^* F = \widehat{\epsilon}_* \widehat{\epsilon}^* F = F$. The result follows. ☐

Proposition A.0.5. *We have natural equivalences of categories*

$$D^-_{\text{coh}}(\widehat{X}_{\text{Zar}}) \xrightarrow{\widehat{\epsilon}^*} D^-_{\text{coh}}(\widehat{X}_{\text{ét}}) \xrightarrow{\delta^*} D^-_{\text{coh}}(\widehat{X}_{\text{lis-ét}}).$$

Proof. We first prove that $\widehat{\epsilon}^*$ is an equivalence using Theorem 2.3.7. Condition (i) is Proposition A.0.1 above. Condition (iv) is Lemma A.0.2 above and condition (iii) is true because an open immersion is an étale morphism. It remains to check Condition (ii). Choose $F, F' \in \mathrm{Coh}(\widehat{X}_{\mathrm{Zar}})$. Then following [EGA, 0_{III}, 12.3.3, 12.3.4], one proves that

$$\widehat{\epsilon}^* \underline{\mathrm{Ext}}^n_{\widehat{X}_{\mathrm{Zar}}}(F, F') \xrightarrow{\sim} \underline{\mathrm{Ext}}^n_{\widehat{X}_{\text{ét}}}(\widehat{\epsilon}^* F, \widehat{\epsilon}^* F')$$

is an isomorphism. Now argue using this isomorphism, Corollary A.0.4 and the local-to-global spectral sequence as in [EGA, III_1, 4.5.2] that

$$\mathrm{Ext}^n_{\widehat{X}_{\mathrm{Zar}}}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Ext}^n_{\widehat{X}_{\text{ét}}}(\widehat{\epsilon}^* \mathscr{F}, \widehat{\epsilon}^* \mathscr{G}).$$

All the conditions of Theorem 2.3.7 are satisfied and therefore

$$\widehat{\epsilon}^* \colon D^-_{\mathrm{coh}}(\widehat{X}_{\mathrm{Zar}}) \to D^-_{\mathrm{coh}}(\widehat{X}_{\text{ét}})$$

is an equivalence. Finally, the equivalence

$$\widehat{\delta}^* \colon D^-_{\mathrm{coh}}(\widehat{X}_{\text{ét}}) \xrightarrow{\sim} D^-_{\mathrm{coh}}(\widehat{X}_{\text{lis-ét}}) \tag{A.0.3}$$

is similarly shown. We leave this to the reader; the key point being [LMB00, Proposition 12.7.4] which states that $\widehat{\delta}$ induces an equivalence of ringed topoi

$$(\mathrm{Sh}(X_{\text{ét}}), \widehat{\mathscr{O}}_{X_{\text{ét}}}) \xrightarrow{\sim} (\mathrm{Sh}(X_{\text{lis-ét}}), \widehat{\mathscr{O}}_{X_{\text{lis-ét}}}).$$

□

Corollary A.0.6. *Let P_n be an inverse system of objects in $D^-_{\mathrm{coh}}(\widehat{X}_{\mathrm{Zar}})$ such that $R\lim P_n \in D^-_{\mathrm{coh}}(\widehat{X}_{\mathrm{Zar}})$. Then the canonical morphism*

$$\widehat{\epsilon}^* R\lim P_n \to R\lim \widehat{\epsilon}^* P_n$$

is an isomorphism in $D(\widehat{X}_{\text{ét}})$.

Proof. Let $\widehat{U}_{\text{ét}}$ be any object of $\widehat{X}_{\text{ét}}$. Then we have

$$\begin{aligned} R\Gamma(\widehat{U}_{\text{ét}}, \widehat{\epsilon}^* R\lim P_n) &= R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{ét}}}}(\mathscr{O}_{\widehat{U}_{\text{ét}}}, \widehat{\epsilon}^* R\lim P_n) \\ &= R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{ét}}}}(\widehat{\epsilon}^* \mathscr{O}_{\widehat{U}_{\text{Zar}}}, \widehat{\epsilon}^* R\lim P_n) \\ &= R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{Zar}}}}(\mathscr{O}_{\widehat{U}_{\text{Zar}}}, R\lim P_n) && \text{(Proposition A.0.5)} \\ &= R\lim R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{Zar}}}}(\mathscr{O}_{\widehat{U}_{\text{Zar}}}, P_n) \\ &= R\lim R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{ét}}}}(\widehat{\epsilon}^* \mathscr{O}_{\widehat{U}_{\text{Zar}}}, \widehat{\epsilon}^* P_n) && \text{(Proposition A.0.5)} \\ &= R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{ét}}}}(\widehat{\epsilon}^* \mathscr{O}_{\widehat{U}_{\text{Zar}}}, R\lim \widehat{\epsilon}^* P_n) \\ &= R\operatorname{Hom}_{\mathscr{O}_{\widehat{U}_{\text{ét}}}}(\mathscr{O}_{\widehat{U}_{\text{ét}}}, R\lim \widehat{\epsilon}^* P_n) \\ &= R\Gamma(\widehat{U}_{\text{ét}}, R\lim \widehat{\epsilon}^* P_n). \end{aligned}$$

We used the fact that $R\lim P_n \in D^-_{\text{coh}}(\widehat{X}_{\text{Zar}})$ to pass from the second to third equality. Since $\widehat{U}_{\text{ét}}$ was arbitrary, we are done. ☐

Remark A.0.7. Let P_n be an inverse system of objects in $D^-_{\text{coh}}(\widehat{X}_{\text{ét}})$. Consider the morphism of sites $\widehat{\delta}\colon \widehat{X}_{\text{lis-ét}} \to \widehat{X}_{\text{ét}}$. This morphism is cocontinuous by [Stacks, Tag 0788]. It follows the canonical morphism $\widehat{\delta}^* R\lim P_n \to R\lim \widehat{\delta}^* P_n$ is an isomorphism $D(\widehat{X}_{\text{lis-ét}})$.

Proposition A.0.8. *Let X be a locally Noetherian scheme and let $\widehat{\epsilon}\colon \widehat{X}_{\text{ét}} \to \widehat{X}_{\text{Zar}}$, $\widehat{\delta}\colon \widehat{X}_{\text{lis-ét}} \to \widehat{X}_{\text{ét}}$ denote the canonical morphisms of sites. An object $P \in D(\widehat{X}_{\text{ét}})$, has finite Tor dimension if and only if the same is true of $\widehat{\epsilon}^* P$. Similarly, $Q \in D(\widehat{X}_{\text{lis-ét}})$ has finite Tor dimension if and only if the same is true of $\widehat{\delta}^* Q$.*

Proof. Follows analogously as in [Stacks, Tag 08HF]. ☐

Appendix B

Derived tensor product and sheafification

Let $\mathscr{A}$ be an abelian category. Recall that a complex $I^\bullet$ is said to be K*-injective* if for any acyclic complex $A^\bullet$,

$$\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, I^\bullet) = 0,$$

where $K(\mathscr{A})$ is the homotopy category. Recall also that a complex $F^\bullet$ is said to be K*-flat* if for any acyclic complex $A^\bullet$, the complex

$$\mathrm{Tot}(A^\bullet \otimes F^\bullet)$$

is acyclic. It is a fact that if $\mathscr{A} = \mathrm{Mod}(\mathscr{O}_X)$, the category of sheaves of modules on a ringed site $(X, \mathscr{O}_X)$, then every complex has a quasi-isomorphism to a K-injective one [Stacks, Tag 01DU], and a quasi-isomorphism from a K-flat one [Stacks, Tag 06YS]. The same is true for the category of presheaves, $\mathrm{PMod}(\mathscr{O}_X)$.

There is an adjoint pair

$$(-)^\#\colon \mathrm{PMod}(\mathscr{O}_X) \rightleftarrows \mathrm{Mod}(\mathscr{O}_X)\colon \mathrm{Fgt}$$

where Fgt is the forgetful functor and $(-)^\#$ denotes sheafification. The forgetful functor is left exact, while sheafification is exact. Since $\mathrm{Mod}(\mathscr{O}_X)$ is a Grothendieck abelian category, by [Ser03, Corollary 3.14] this extends to an adjoint pair

$$(-)^\#\colon D(\mathrm{PMod}(\mathscr{O}_X)) \rightleftarrows D(\mathrm{Mod}(\mathscr{O}_X))\colon R\,\mathrm{Fgt}\,.$$

Lemma B.0.1. *Let $(X, \mathscr{O}_X)$ be a ringed site and $F^\bullet \to J^\bullet$ a quasi-isomorphism of complexes of*

$\mathscr{O}_X$-modules. Then for any K-injective complex $I^\bullet$, the canonical morphism

$$\underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(J^\bullet), \mathrm{Fgt}(I^\bullet)) \to \underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(F^\bullet), \mathrm{Fgt}(I^\bullet))$$

is a quasi-isomorphism in the homotopy category $K(\mathrm{PMod}(\mathscr{O}_X))$.

Proof. We begin by recalling the following fact. Any additive functor between abelian categories $\mathscr{A} \to \mathscr{B}$ induces an exact functor $K(\mathscr{A}) \to K(\mathscr{B})$. Now let $C^\bullet$ be the cone of $F^\bullet \to J^\bullet$. Applying $\mathrm{Fgt}(-)$ and then $\underline{\mathrm{Hom}}^\bullet(-, \mathrm{Fgt}(I^\bullet))$, we get the following exact triangle in $K(\mathrm{PMod}(\mathscr{O}_X))$:

$$\underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(C^\bullet), \mathrm{Fgt}(I^\bullet)) \to \underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(J^\bullet), \mathrm{Fgt}(I^\bullet)) \to \underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(F^\bullet), \mathrm{Fgt}(I^\bullet)).$$

To prove the lemma, we must show for every object $j\colon U \to X$ and integer $n \in \mathbf{Z}$ that

$$H^n(\Gamma(U, \underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(C^\bullet), \mathrm{Fgt}(I^\bullet)))) = 0. \tag{B.0.1}$$

We have

$$\begin{aligned} H^n(\Gamma(U, \underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(C^\bullet), \mathrm{Fgt}(I^\bullet)))) &= \mathrm{Hom}_{K(\mathrm{PMod}(\mathscr{O}_U))}(\mathrm{Fgt}(j^*C^\bullet), \mathrm{Fgt}(j^*I^\bullet)[n]) \\ &= \mathrm{Hom}_{K(\mathrm{PMod}(\mathscr{O}_U))}(\mathrm{Fgt}(j^*C^\bullet), \mathrm{Fgt}(j^*I^\bullet[n])) \\ &= \mathrm{Hom}_{K(\mathrm{Mod}(\mathscr{O}_U))}(\mathrm{Fgt}(j^*C^\bullet)^\#, (j^*I^\bullet[n])) \\ &= \mathrm{Hom}_{K(\mathrm{Mod}(\mathscr{O}_U))}(j^*C^\bullet, j^*I^\bullet[n]) \\ &= \mathrm{Hom}_{K(\mathrm{Mod}(\mathscr{O}_U))}(j_!j^*C^\bullet, I^\bullet[n]). \end{aligned}$$

We used adjunction between Fgt and $(-)^\#$ in the third equality. In the fourth equality, we used the fact that forgetting and then sheafifiying is the same as the identity! Now $j_!j^*C^\bullet$ is acyclic since $j^*, j_!$ are exact and $C^\bullet$ is acyclic. It follows by definition of being K-injective that

$$\mathrm{Hom}_{K(\mathrm{Mod}(\mathscr{O}_U))}(j_!j^*C^\bullet, I^\bullet[n]) = 0.$$

Therefore (B.0.1) is zero and we win. ∎

Lemma B.0.2. *Let $(X, \mathscr{O}_X)$ be a ringed site and $\mathscr{F}, \mathscr{G}$ any two objects of $D(\mathrm{Mod}(\mathscr{O}_X))$. Then:*

$$R\,\mathrm{Fgt}\,R\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) = R\underline{\mathrm{Hom}}(R\,\mathrm{Fgt}(\mathscr{F}), R\,\mathrm{Fgt}(\mathscr{G}))$$

in $D(\mathrm{PMod}(\mathscr{O}_X))$.

Proof. Let $F^\bullet \to \mathscr{F}$ be a K-flat resolution of $\mathscr{F}$ and $\mathscr{G} \to I^\bullet$ a K-injective resolution of $\mathscr{G}$. Then the object $R\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ can be represented in $D(\mathrm{Mod}(\mathscr{O}_X))$ by the complex $\underline{\mathrm{Hom}}^\bullet(F^\bullet, I^\bullet)$, with

n-th term of the form

$$\underline{\mathrm{Hom}}^n(F^\bullet, I^\bullet) = \prod_{n=p+q} \underline{\mathrm{Hom}}(F^{-q}, I^p)$$

with differential given by $d(f) = d_F \circ f - (-1)^n f \circ d_I$. Furthermore, given any acyclic complex $A^\bullet$, we have

$$\begin{aligned}
\mathrm{Hom}_{K(\mathrm{Mod}(\mathscr{O}_X))}(A^\bullet, \underline{\mathrm{Hom}}^\bullet(F^\bullet, I^\bullet)) &= \mathrm{Hom}_{K(\mathrm{Mod}(\mathscr{O}_X))}(\mathrm{Tot}(A^\bullet \otimes F^\bullet), I^\bullet) \\
&= 0
\end{aligned}$$

since $F^\bullet$ is K-flat and $I^\bullet$ is K-injective. The upshot is that $\underline{\mathrm{Hom}}^\bullet(F^\bullet, I^\bullet)$ is K-injective and therefore

$$R\,\mathrm{Fgt}\, R\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) = \mathrm{Fgt}\,(\underline{\mathrm{Hom}}^\bullet(F^\bullet, I^\bullet)) \tag{B.0.2}$$

in $D(\mathrm{PMod}(\mathscr{O}_X))$.

On the other hand, let $F^\bullet \to J^\bullet$ be a K-injective resolution of $F^\bullet$. By the previous lemma, we have

$$\underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(F^\bullet), \mathrm{Fgt}(I^\bullet)) = \underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(J^\bullet), \mathrm{Fgt}(I^\bullet))$$

in the *derived category* $D(\mathrm{PMod}(\mathscr{O}_X))$. Now recall that Fgt is right adjoint to the exact functor $(-)^\#$ and thus commutes with arbitrary limits, hence

$$\mathrm{Fgt}\,(\underline{\mathrm{Hom}}^\bullet(F^\bullet, I^\bullet)) = \mathrm{Fgt}\,(\underline{\mathrm{Hom}}^\bullet(J^\bullet, I^\bullet))\,. \tag{B.0.3}$$

By assumption that $I^\bullet, J^\bullet$ are K-injective, we have

$$\begin{aligned}
R\,\mathrm{Fgt}(\mathscr{G}) &= \mathrm{Fgt}(I^\bullet), \\
R\,\mathrm{Fgt}(\mathscr{F}) &= R\,\mathrm{Fgt}(F^\bullet) = \mathrm{Fgt}(J^\bullet)
\end{aligned}$$

in $D(\mathrm{PMod}(\mathscr{O}_X))$ [1]. Furthermore, $R\,\mathrm{Fgt}(I^\bullet)$ is K-injective because Fgt preserves K-injectives and therefore

$$\underline{\mathrm{Hom}}^\bullet(\mathrm{Fgt}(J^\bullet), \mathrm{Fgt}(I^\bullet)) = R\underline{\mathrm{Hom}}(R\,\mathrm{Fgt}(\mathscr{F}), R\,\mathrm{Fgt}(\mathscr{G})) \tag{B.0.4}$$

in $D(\mathrm{PMod}(\mathscr{O}_X))$. Now combine (B.0.2), (B.0.3) and (B.0.4) to deduce the result. □

Remark B.0.3. The reader may question the necessity of Lemma B.0.1 to conclude that (B.0.3) is true. Indeed, it is tempting to argue that because $\underline{\mathrm{Hom}}^\bullet(C^\bullet, I^\bullet)$ is quasi-isomorphic to zero that the same is true of $\mathrm{Fgt}(\underline{\mathrm{Hom}}^\bullet(C^\bullet, I^\bullet))$. This however is absolutely false because an exact sequence of sheaves need not be exact on sections.

[1] The careful reader will note it is important these equalities are in the derived category and *not* the homotopy category.

Proposition B.0.4. *Let $\mathscr{F},\mathscr{G}$ be any two objects of $D(\mathrm{Mod}(\mathscr{O}_X))$. Then*

$$\mathscr{F}\otimes^{\mathrm{L}}\mathscr{G}=(R\,\mathrm{Fgt}(\mathscr{F})\otimes^{\mathrm{L}}R\,\mathrm{Fgt}(\mathscr{G}))^{\#}.$$

Proof. Let $\mathscr{H}$ be any object of $D(\mathrm{Mod}(\mathscr{O}_X))$. Then we have

$$\begin{aligned}
\mathrm{Hom}_{D(\mathrm{Mod}(\mathscr{O}_X))}((R\,\mathrm{Fgt}(\mathscr{F})\otimes^{\mathrm{L}}R\,\mathrm{Fgt}(\mathscr{G}))^{\#},\mathscr{H}) &= \mathrm{Hom}_{D(\mathrm{PMod}(\mathscr{O}_X))}(R\,\mathrm{Fgt}(\mathscr{F})\otimes^{\mathrm{L}}R\,\mathrm{Fgt}(\mathscr{G}),R\,\mathrm{Fgt}(\mathscr{H}))\\
&= \mathrm{Hom}_{D(\mathrm{PMod}(\mathscr{O}_X))}(R\,\mathrm{Fgt}(\mathscr{F}),R\underline{\mathrm{Hom}}(R\,\mathrm{Fgt}(\mathscr{G}),R\,\mathrm{Fgt}(\mathscr{H})))\\
&= \mathrm{Hom}_{D(\mathrm{PMod}(\mathscr{O}_X))}(R\,\mathrm{Fgt}(\mathscr{F}),R\,\mathrm{Fgt}\,R\underline{\mathrm{Hom}}(\mathscr{G},\mathscr{H}))\\
&= H^0(\Gamma(X,R\underline{\mathrm{Hom}}(R\,\mathrm{Fgt}(\mathscr{F}),R\,\mathrm{Fgt}\,R\underline{\mathrm{Hom}}(\mathscr{G},\mathscr{H}))))\\
&= H^0(\Gamma(X,R\,\mathrm{Fgt}\,R\underline{\mathrm{Hom}}(\mathscr{F},R\underline{\mathrm{Hom}}(\mathscr{G},\mathscr{H}))))\\
&= H^0(R\Gamma(X,R\underline{\mathrm{Hom}}(\mathscr{F},R\underline{\mathrm{Hom}}(\mathscr{G},\mathscr{H}))))\\
&= \mathrm{Hom}_{D(\mathrm{Mod}(\mathscr{O}_X))}(\mathscr{F},R\underline{\mathrm{Hom}}(\mathscr{G},\mathscr{H}))\\
&= \mathrm{Hom}_{D(\mathrm{Mod}(\mathscr{O}_X))}(\mathscr{F}\otimes^{\mathrm{L}}\mathscr{G},\mathscr{H}).
\end{aligned}$$

By Yoneda's Lemma, $\mathscr{F}\otimes^{\mathrm{L}}\mathscr{G}=(R\,\mathrm{Fgt}(\mathscr{F})\otimes^{\mathrm{L}}R\,\mathrm{Fgt}(\mathscr{G}))^{\#}$. □

Corollary B.0.5. *For any $\mathscr{F},\mathscr{G}\in D(\mathrm{Mod}(\mathscr{O}_X))$, $\mathscr{F}\otimes^{\mathrm{L}}\mathscr{G}$ is the (derived) sheafification of the (derived) presheaf*

$$U\mapsto R\Gamma(U,\mathscr{F})\otimes^{\mathrm{L}}_{\mathscr{O}_X(U)}R\Gamma(U,\mathscr{G}).$$

Proof. By Proposition B.0.4, it is enough to note that the diagram

$$\begin{array}{ccc}
D(\mathrm{Mod}(\mathscr{O}_X)) & \xrightarrow{R\,\mathrm{Fgt}} & D(\mathrm{PMod}(\mathscr{O}_X))\\
& \searrow^{R\Gamma} & \downarrow{\scriptstyle\Gamma}\\
& & D(\mathrm{Ab})
\end{array}$$

is commutative, and to prove the following fact. For any $\mathcal{F},\mathcal{G}\in D(\mathrm{PMod}(\mathscr{O}_X))$, the derived category of *presheaves*,

$$\Gamma(U,\mathcal{F}\otimes^{\mathrm{L}}\mathcal{G})=\Gamma(U,\mathcal{F})\otimes^{\mathrm{L}}\Gamma(U,\mathcal{G}).$$

Let $K^\bullet$ be a K-flat complex that is quasi-isomorphic to $\mathcal{F}$, and $G^\bullet$ any complex representing $\mathcal{G}$. The complex $\mathrm{Tot}(K^\bullet\otimes G^\bullet)$ represents $\mathcal{F}\otimes^{\mathrm{L}}\mathcal{G}$ in $D(\mathrm{PMod}(\mathscr{O}_X))$. Note that the formation of the total complex commutes with taking sections in the category of presheaves. Therefore,

$$\begin{aligned}
\Gamma(U,\mathcal{F}\otimes^{\mathrm{L}}\mathcal{G}) &= \Gamma(U,\mathrm{Tot}(K^\bullet\otimes G^\bullet))\\
&= \mathrm{Tot}(\Gamma(U,K^\bullet\otimes G^\bullet))\\
&= \mathrm{Tot}(\Gamma(U,K^\bullet)\otimes\Gamma(U,G^\bullet)).
\end{aligned}$$

To complete the proof, we must show that if $K^\bullet$ is a K-flat complex of presheaves, then $\Gamma(U, K^\bullet)$ is a K-flat complex of modules. To this end, let $M^\bullet$ be an acyclic complex of modules, and $\underline{M}^\bullet$ the constant complex of presheaves associated to $M^\bullet$. Then

$$\begin{aligned} \mathrm{Tot}(\Gamma(U, K^\bullet) \otimes M^\bullet) &= \mathrm{Tot}(\Gamma(U, K^\bullet \otimes \underline{M}^\bullet) && \text{(B.0.5)} \\ &= \Gamma(U, \mathrm{Tot}(K^\bullet \otimes \underline{M}^\bullet)). && \text{(B.0.6)} \end{aligned}$$

Since $M^\bullet$ is acyclic, $\underline{M}^\bullet$ is acyclic. By assumption that $K^\bullet$ is K-flat, $\mathrm{Tot}(K^\bullet \otimes \underline{M}^\bullet)$ is acyclic. The functor $\Gamma(U, -)$ is exact (in the category of presheaves!) and thus commutes with cohomology. Therefore (B.0.6) is acyclic which shows that $\Gamma(U, K^\bullet)$ is a K-flat complex of modules. □

Chapter 3

Coherent Completeness in Positive Characteristic

The results in this chapter are joint work with Jack Hall. In forthcoming work with Jarod Alper and Jack Hall, we generalize the results of this chapter to arbitrary $[G\backslash \operatorname{Spec} A]$ with G reductive.

3.1 Introduction

Let $\pi : X \to \operatorname{Spec} R$ be a proper morphism where X is a scheme and R an I-adically complete Noetherian R. There are three important theorems that govern the formal geometry of X, analogous to Serre's GAGA theorems in the setting of complex analytic geometry:

- (Finiteness of Cohomology). For any $\mathscr{F} \in \operatorname{Coh}(X)$ and non-negative integer n, $H^n(X, \mathscr{F})$ is a finite R-module.
- (Formal Functions). Define $Z \coloneqq \pi^{-1}(\operatorname{Spec} R/I)$. For any $\mathscr{F} \in \operatorname{Coh}(X)$ and non-negative integer n,
$$H^n(X, \mathscr{F}) = H^n(\widehat{X}_Z, \iota^*_{\widehat{X}_Z}\mathscr{F}).$$
- (Formal GAGA). Define $Z \coloneqq \pi^{-1}(\operatorname{Spec} R/I)$. There is an equivalence of categories
$$\iota^*_{\widehat{X}_Z} \colon \operatorname{Coh}(X) \xrightarrow{\sim} \operatorname{Coh}(\widehat{X}_Z).$$

The notation $\widehat{X}_Z$ refers to the formal scheme obtained by completing X along the closed subscheme $Z \subseteq X$, while $\iota_{\widehat{X}_Z}$ refers to the canonical morphism of ringed spaces from the formal scheme $\widehat{X}_Z$ to X.

Formal GAGA (and its extension to the case of proper algebraic stacks by Conrad [Con05] and Olsson [Ols05]) is an extremely powerful tool in modern algebraic geometry. For example, it is used to answer lifting questions about schemes to characteristic zero, or to prove the proper base change theorem in étale cohomology. In the context of moduli theory, the work of Artin [Art69] and [Art70] has also shown that a necessary step to proving algebraicity of moduli stacks is to establish algebraization for the objects (schemes, sheaves, etc) in question. In practice, this reduces to checking some incarnation of formal GAGA above.

In recent years, the work of Alper-Hall-Rydh [AHR20] and [AHR19] on the étale local structure of algebraic stacks has crucially relied on an algebraization theorem for coherent sheaves on quotient stacks. Their theorem is very similar in spirit to the formal GAGA theorems of Conrad and Olsson, albeit with several crucial differences. We state their theorem first and explain these differences after: If A is a finite type k-algebra, where k is a field, and G/k is a linearly reductive affine group scheme acting on $\operatorname{Spec} A$ such that there is a k-point fixed by G and the ring of invariants A^G is a complete local ring, then $[G\backslash \operatorname{Spec} A]$ is coherently complete (Definition 3.2.11 below) along the residual gerbe of its unique closed point [AHR20, Theorem 1.3].

The algebraization theorem of Alper-Hall-Rydh above differs from formal GAGA in two ways. First, when G is not finite (as is the situation considered in [AHR20]), the morphism of algebraic stacks $[G\backslash \operatorname{Spec} A] \to \operatorname{Spec} A^G$ is *not* separated, let alone proper. Second, in formal GAGA, the completion is taken with respect to the pullback of the maximal ideal from the base, whereas the completion in [AHR20] is taken with respect to the *smaller* closed substack defined by the residual gerbe at the unique closed point. To illustrate this difference, consider the standard action of SL_2 on $\mathbf{A}^2 = \operatorname{Spec} \mathbf{C}[x, y]$. The reader may readily verify that $0 \in \mathbf{A}^2$ is the unique closed, $\mathbf{C}$-fixed point, and that that the ring of invariants $\mathbf{C}[x, y]^{\mathrm{SL}_2} = \mathbf{C}$. Since the ring of invariants is a field, completing with respect to the maximal ideal yields no new information. On the other hand, [AHR20, Theorem 1.3] yields the non-trivial result that $[\mathrm{SL}_2\backslash \mathbf{A}^2]$ is coherently complete along the residual gerbe at the origin $0 \in \mathbf{A}^2$.

The main goal of this article will be to extend [AHR20, Theorem 1.3] of Alper-Hall-Rydh to the case of BG, where G is an arbitrary reductive group, and to the case of the quotient stack $[\mathrm{SL}_d\backslash \mathbf{A}^d]$, *without assuming* any hypothesis on the characteristic of the ground field. We remind the reader that in characteristic zero, the notion of being reductive and linearly reductive coincide. However, in characteristic p, it is a theorem of Nagata that a reductive group G over an (algebraically closed field) k is linearly reductive if and only if G° is a torus and $|G/G^\circ|$ has order prime to p [Nag62]. Therefore, our result only differs from [AHR20, Theorem 1.3] in the situation where the base field is of characteristic p. We state our results precisely in the next subsection below.

3.1.1 Results

In order to state our results, we will need to work in the more general situation where the completion is taken with respect to an arbitrary closed substack, and not just the closed fiber as in formal GAGA. Recall the following notions (compare with Definitions 3.2.7, 3.2.9 and 3.2.11 respectively):

1. Let X be a Noetherian algebraic stack over a Noetherian ring R. We say that X is *cohomologically proper* over $\operatorname{Spec} R$ if for every $F \in \operatorname{Coh}(X)$, $\mathsf{R}\Gamma(X, F) \in D^+_{\mathrm{Coh}}(R)$.

2. Let X be a Noetherian algebraic stack and $Z \subseteq X$ a closed substack. We say that the pair (X, Z) satisfies *formal functions* if for every $F \in \operatorname{Coh}(X)$,

$$\mathsf{R}\Gamma(X, F) = \mathsf{R}\Gamma(\widehat{X}_Z, \widehat{F}_Z).$$

3. Let X be a Noetherian algebraic stack and $Z \subseteq X$ a closed substack. We say that the pair (X, Z) is *coherently complete* if

$$\iota^*_{\widehat{X}_Z} : \operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X}_Z)$$

is an equivalence of categories.

Theorem 3.1.1 (Cohomological Properness for BG)**.** *Let k be a field and R a Noetherian k-algebra R. Then BG_R is cohomologically proper over* $\operatorname{Spec} R$*.*

Theorem 3.1.2 (Formal Functions for BG)**.** *Let k be a field and R an I-adically complete Noetherian k-algebra. Then the pair (BG_R, Z) with $Z := \pi_R^{-1}(\operatorname{Spec} R/I)$ satisfies formal functions.*

Theorem 3.1.3 (Coherent Completeness for BG)**.** *Let k be a field and R an I-adically complete Noetherian k-algebra. Then the pair (BG_R, Z) with $Z := \pi_R^{-1}(\operatorname{Spec} R/I)$ is coherently complete.*

Theorems 3.1.1, 3.1.2, 3.1.3 appear in [HP14] in the setting of derived algebraic geometry. Nonetheless, our method is different as we do not use any derived techniques. Furthermore, consider the quotient stack $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ where the action of SL_d on $\mathbf{A}^d$ is the standard one. By constructing a suitable proper covering of this stack, we prove cohomological properness, formal functions and coherent completeness, previously only known in characteristic 0 by [GZB15] and [AHR20]. We believe our result is the first in the literature on the geometry of non-trivial quotient stacks with reductive stabilizers in characteristic p.

Theorem 3.1.4 (Cohomological Properness for $[\mathrm{SL}_d \backslash \mathbf{A}^d]$)**.** *Let k be a field. Then $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ is cohomologically proper over* $\operatorname{Spec} k$*.*

Theorem 3.1.5 (Formal Functions for $[\mathrm{SL}_d \backslash \mathbf{A}^d]$)**.** *Let Z be the closed substack of $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ defined by the origin. Then the pair $([\mathrm{SL}_d \backslash \mathbf{A}^d], Z)$ satisfies formal functions.*

Theorem 3.1.6 (Coherent completeness for $[\mathrm{SL}_d\backslash\mathbf{A}^d]$)**.** *Let Z be the closed substack of $[\mathrm{SL}_d\backslash\mathbf{A}^d]$ defined by the origin. Then the pair $([\mathrm{SL}_d\backslash\mathbf{A}^d], Z)$ is coherently complete.*

3.1.2 Strategy of proof

We give a sketch for the proof of Theorem 3.1.1. As in [HP14], the strategy is to use the fact that a split reductive group G has a Borel subgroup B that contains a split maximal torus T. On the level of classifying stacks, this gives representable morphisms

$$BT \overset{(2)}{\to} BB \overset{(1)}{\to} BG.$$

The morphism (1) is smooth and proper. Therefore, for every $n \geq 1$, the n-fold fiber product $(BB/BG)^n = BB \times_{BG} \ldots \times_{BG} BB$ is proper over BB, and thus cohomological properness for BG reduces to the case of BB by Theorem 3.3.1. Now cohomological properness is known for BT as $BT \to \operatorname{Spec} k$ is a good moduli space. However, unlike (1), the morphism (2) is *not* proper. Nonetheless, the key observation of ours is that for any $n \geq 1$, the n-fold fiber product $(BT/BB)^n$ is a good moduli space, and therefore satisfies cohomological properness. The result follows from Theorem 3.3.1 again.

Given that this observation is crucial to our proof, we show why in the simple case that $G = \mathrm{SL}_2$, the 2-fiber product $BT \times_{BB} BT$ is a good moduli space. We take B and T to be the standard subgroups of upper triangular and diagonal matrices respectively. In coordinates,

$$\begin{aligned} B &= \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \\ T &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \end{aligned}$$

To show that $BT \times_{BB} BT$ is a good moduli space, observe that we may write

$$BT \times_{BB} BT \simeq [T\backslash B/T].$$

We must show that the GIT quotient $T \mathbin{\backslash\!\backslash} (B \mathbin{/\!/} T)$ is isomorphic to $\operatorname{Spec} k$. The right and left actions of T on B are given respectively by

$$t \cdot (a, b) = (at, bt^{-1}) \quad \text{(right action)} \tag{3.1.1}$$

$$t \cdot (a, b) = (at, bt) \quad \text{(left action).} \tag{3.1.2}$$

Therefore,

$$B /\!\!/ T \simeq \operatorname{Spec} k[a,b,a^{-1}]^{\langle t \rangle} \simeq \operatorname{Spec} k[ab].$$

On the other hand, by (3.1.2) above, the left T-action on $B /\!\!/ T \simeq \operatorname{Spec} k[ab] \simeq k[x]$ is given by $t \cdot x = t^2 x$. Therefore,

$$T \backslash\!\!\backslash (B /\!\!/ T) \simeq \operatorname{Spec} k[x]^{\langle t \rangle} = \operatorname{Spec} k$$

as desired.

3.1.3 Acknowledgments

We would like to thank Dmitry Kubrak, Zev Rosengarten and Bogdan Zavyalov for very helpful conversations. In particular, we are very grateful to Dmitry Kubrak for first suggesting the double quotient description of the 2-fiber product $BT \times_{BB} BT$.

3.2 Preparations

We recall some preliminary material concerning algebraic groups, good moduli spaces and completions of algebraic stacks.

3.2.1 Algebraic groups

Let k be a field. By a k-group G, we mean a (possibly disconnected) k-group scheme of finite type. In this article, we will only be concerned with those k-groups G that are *affine*. There are two important classes of affine k-groups, namely the linearly reductive and reductive ones.

Definition 3.2.1. Let G be a smooth affine k-group. It is a fact that there exists a unique maximal unipotent, normal, smooth and connected k-subgroup of G, namely the unipotent radical $R_{u,k}(G)$. We say that G is *reductive* if the base change $R_{u,\overline{k}}(G_{\overline{k}}) = 1$.

Definition 3.2.2. Let G be a smooth affine k-group. We say that G is linearly reductive if every finite-dimensional representation of G is completely reducible.

In general, a linearly reductive group G is reductive. In characteristic zero, it turns out that any connected reductive group G is also linearly reductive. However, in characteristic p there are very few linearly reductive groups, in the sense that if k is algebraically closed, G is linearly reductive if and only if G^0 is a torus and $|G/G^0|$ has order prime to p [Nag62]. However, all of the classical groups ($\mathrm{GL}_n, \mathrm{SL}_n$, etc) that are linearly reductive in characteristic zero are still reductive in characteristic p.

3.2.2 Good moduli spaces

The following definition was introduced in the thesis of Jarod Alper [Alp13]. It is a replacement for the notion of a coarse moduli space when the moduli problem in question has infinite, linearly reductive stabilizers.

Definition 3.2.3. Let $\pi\colon X \to Y$ be a morphism from an algebraic stack to an algebraic space. We say that f is a *good moduli space morphism* if:

1. The morphism f is cohomologically affine, i.e. is quasi-compact, quasi-separated, and the induced pushforward $\pi_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is exact.

2. The map on sheaves $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is an isomorphism.

Example 3.2.4. Let k be a field and G a linearly reductive k-group. Then $\pi\colon BG \to \operatorname{Spec} k$ is a good moduli space.

Example 3.2.5. More generally, let X be an affine k-scheme of finite type with the action of a linearly reductive k-group G. Then $\pi\colon [G\backslash X] \to \Gamma(X, \mathscr{O}_X)^G$ is a good moduli space.

3.2.3 Completions

Definition 3.2.6. Let X be a Noetherian algebraic stack and $Z \subseteq X$ a closed immersion defined by a coherent sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_X$. The completion of the pair (X, Z) is the ringed site $\widehat{X}_Z \coloneqq (X_{\text{lis-ét}}, \widehat{\mathscr{O}}_{X,Z})$, where

$$\widehat{\mathscr{O}}_{X,Z} \coloneqq \varprojlim_n \mathscr{O}_X/\mathscr{I}^{n+1}.$$

The limit is taken in the category of lisse-étale modules. For any such X and coherent ideal $\mathscr{I} \subseteq \mathscr{O}_X$, there is a canonical morphism of ringed sites $\iota_{\widehat{X}_Z} : \widehat{X}_Z \to X$. The morphism $\iota_{\widehat{X}_Z}$ is flat [GZB15, Lemma 3.3] and $\iota^*_{\widehat{X}_Z}$ preserves coherence. We leave the reader to verify that if $\mathscr{F}$ is a coherent sheaf, this is consistent with the usual notion of completion using inverse limits, i.e.

$$\widehat{\mathscr{F}}_X \simeq \varprojlim_n \mathscr{F}/\mathscr{I}^n\mathscr{F}.$$

3.2.4 Cohomological properness, formal functions and coherent completeness

The following definition is a variant of what appeared in [HP14].

Definition 3.2.7. Let $\pi : X \to Y$ be a finite type morphism of Noetherian algebraic stacks. We say that π is

1. *cohomologically proper* if for all $\mathscr{F} \in D^+_{\mathrm{Coh}}(X)$, $\mathsf{R}\pi_*\mathscr{F} \in D^+_{\mathrm{Coh}}(Y)$.
2. *uniformly* cohomologically proper if for every flat morphism $Y' \to Y$, where Y' is Noetherian, the induced morphism $\pi' : X \times_Y Y' \to Y'$ is cohomologically proper; and
3. *universally* cohomologically proper if for every morphism $Y' \to Y$, where Y' is Noetherian, the induced morphism $\pi' : X \times_Y Y' \to Y'$ is cohomologically proper.

The following lemma is sometimes useful.

Lemma 3.2.8. *Let $\pi : X \to Y$ be a finite type morphism of Noetherian algebraic stacks. The following conditions are equivalent:*

1. *π is cohomologically proper;*
2. *for all $\mathscr{F} \in \mathrm{Coh}(X)$, $\mathsf{R}^i\pi_*\mathscr{F} \in \mathrm{Coh}(X)$ for all $i \geq 0$.*

Proof. The equivalence is trivial from the convergent hypercohomology spectral sequence:

$$\mathsf{R}^i\pi_*\mathscr{H}^j(\mathscr{F}) \Rightarrow \mathscr{H}^{i+j}(\mathsf{R}\pi_*\mathscr{F}). \qquad \square$$

Another key definition is the following.

Definition 3.2.9. Let X be a Noetherian algebraic stack and $Z \subseteq X$ a closed immersion. We say that the pair (X, Z) satisfies *formal functions* if for any $\mathscr{F} \in \mathrm{Coh}(X)$,

$$\mathsf{R}\Gamma(X, \mathscr{F}) = \mathsf{R}\Gamma(\widehat{X}_Z, \widehat{\mathscr{F}_Z}).$$

The following lemma describes an expected compatibility between formal functions and cohomological properness.

Lemma 3.2.10. *Let Y be a Noetherian algebraic stack. Let $W \subseteq Y$ be a closed substack. Let $\pi\colon X \to Y$ be a morphism of algebraic stacks with X Noetherian. Let $Z = \pi^{-1}(W)$. There is a natural comparison morphism for every $\mathscr{F} \in D(X)$:*

$$\iota^*_{\widehat{Y}_W}\mathsf{R}\pi_*\mathscr{F} \to \mathsf{R}\widehat{\pi}_*\iota^*_{\widehat{X}_Z}\mathscr{F},$$

where $\mathsf{R}\widehat{\pi}_*$ is the derived functor of $\widehat{\pi}_*\colon \mathrm{Mod}(\widehat{X}_Z) \to \mathrm{Mod}(\widehat{Y}_W)$. If π is uniformly cohomologically proper, then

1. *this morphism is a quasi-isomorphism whenever $\mathscr{F} \in D^+_{\mathrm{Coh}}(X)$; and*
2. *if, in addition, the pair (Y, W) satisfies formal functions, then so does the pair (X, Z).*

Proof. Claim (2) follows from (1): Indeed, if $\mathscr{F} \in D^+_{\mathrm{Coh}}(X)$, then

$$\mathsf{R}\Gamma(X, \mathscr{F}) \simeq \mathsf{R}\Gamma(Y, \mathsf{R}\pi_*\mathscr{F}) \simeq \mathsf{R}\Gamma(\widehat{Y}_W, \widehat{(\mathsf{R}\pi_*\mathscr{F})}_W) \simeq \mathsf{R}\Gamma(\widehat{Y}_W, \mathsf{R}\widehat{\pi}_*\iota_{\widehat{X}_Z}\mathscr{F}) \simeq \mathsf{R}\Gamma(\widehat{X}_Z, \widehat{\mathscr{F}_Z}).$$

We now treat (1): Let $\operatorname{Spec} B \to Y$ be a smooth morphism, $J \subseteq B$ the ideal defined by a closed immersion $W \subseteq Y$, and $\widehat{B}$ the J-adic completion of B. Let $\pi_B \colon X_B \to B$ be the induced morphism, where $X_B = X \times_Y \operatorname{Spec} B$. If $i \in \mathbf{Z}$, then $\mathscr{H}^i(\iota^*_{\widehat{Y}_W} \mathsf{R}\pi_*\mathscr{F})$ is the sheafification of the presheaf:

$$(\operatorname{Spec} B \to Y) \mapsto \mathrm{H}^i(X_B, \mathscr{F}_{X_B}) \otimes_B \widehat{B}.$$

One similarly finds that $\mathscr{H}^i(\mathsf{R}\widehat{\pi}_*\iota^*_{\widehat{X}_Z}\mathscr{F})$ is the sheafification of the presheaf:

$$(\operatorname{Spec} B \to Y) \mapsto \mathrm{H}^i(\widehat{(X_B)}_{Z_B}, \mathscr{F}_{\widehat{(X_B)}_{Z_B}}).$$

Hence, we may reduce to the situation where $Y = \operatorname{Spec} B$, $\pi : X \to \operatorname{Spec} B$ is cohomologically proper, and $W = \operatorname{Spec} B/J$. In this case, the result follows immediately from an argument identical to that provided in [Stacks, Tag 0A0M]. □

The following was a key definition in [AHR20], and was an essential result in establishing the local structure theorem of stacks with linearly reductive stabilizers.

Definition 3.2.11. Let X be a Noetherian algebraic stack and $Z \subseteq X$ a closed immersion. We say that the pair (X, Z) is *coherently complete* if the functor

$$\iota^*_{\widehat{X}_Z} : \mathrm{Coh}(X) \to \mathrm{Coh}(\widehat{X}_Z)$$

is an equivalence.

We have the following standard lemma.

Lemma 3.2.12. *Let X be a Noetherian algebraic stack and $Z \subseteq X$ a closed immersion. If formal functions holds for the pair (X, Z), then the functor*

$$\iota^*_{\widehat{X}_Z} : \mathrm{Coh}(X) \to \mathrm{Coh}(\widehat{X}_Z)$$

is fully faithful, with essential image stable under kernels, cokernels and extensions.

Proof. This is immediate from the following observations (see [Ols05]):

1. $\iota_{\widehat{X}_Z}$ is flat;
2. if $(V, \mathscr{O}_V)$ is a ringed topos, then

$$\mathsf{R}\mathrm{Hom}_{\mathscr{O}_V}(\mathscr{F}, \mathscr{G}) \simeq \mathsf{R}\Gamma(V, \mathsf{R}\mathscr{H}om_{\mathscr{O}_V}(\mathscr{F}, \mathscr{G}))$$

for any $\mathscr{F}, \mathscr{G} \in D(V)$;

3. if $(V, \mathscr{O}_V)$ is a ringed topos with $\mathscr{O}_V$ coherent, then

$$\mathsf{R}\mathscr{H}om_{\mathscr{O}_V}(\mathscr{F}, \mathscr{G}) \in D^+_{\mathrm{Coh}}(V),$$

whenever $\mathscr{F} \in D^-_{\mathrm{Coh}}(V)$ and $\mathscr{G} \in D^+_{\mathrm{Coh}}(V)$; and

4. if $c : (W, \mathscr{O}_W) \to (V, \mathscr{O}_V)$ is a flat morphism of ringed topoi, where both $\mathscr{O}_W$ and $\mathscr{O}_V$ are coherent; and $\mathscr{F} \in D^-_{\mathrm{Coh}}(V)$ and $\mathscr{G} \in D^+_{\mathrm{Coh}}(V)$, then

$$c^* \mathsf{R}\mathscr{H}om_{\mathscr{O}_V}(\mathscr{F}, \mathscr{G}) \simeq \mathsf{R}\mathscr{H}om_{\mathscr{O}_W}(c^*\mathscr{F}, c^*\mathscr{G}). \qquad \square$$

The following are examples of algebraic stacks X that satisfy all three definitions above, namely cohomological properness, formal functions and coherent completeness.

Example 3.2.13. Let $\pi\colon X \to \operatorname{Spec} R$ be a proper morphism from an algebraic stack X to a Noetherian ring R. Then π is universally cohomologically proper [Ols05, Theorem 1.2]. Moreover, if $I \subseteq R$ is an ideal and R is I-adically complete, then the pair $(X, \pi^{-1}(\operatorname{Spec} R/I))$ satisfies formal functions [Con05, Corollary 3.2] and is coherently complete [Con05, Theorem 4.1]

Example 3.2.14. Let $\pi\colon X \to \operatorname{Spec} R$ be a good moduli space morphism, where X is a Noetherian algebraic stack with affine diagonal [Alp13]. Then R is Noetherian, π is of finite type [AHR20, Theorem A.1], and is universally cohomologically proper (combine [Alp13, Thm 4.16 (x)] with [Alp13, Proposition 4.7 (i)]). Let $Z \subseteq X$ be a closed immersion defined by a coherent ideal $\mathscr{I} \subseteq \mathscr{O}_X$. Let $I = \Gamma(X, \mathscr{I})$, which is an ideal of R. If Z has the resolution property and R is I-adically complete, then the pair (X, Z) satisfies formal functions and is coherently complete. This is one of the main results of [AHR20]. When X has the resolution property and $Z = \pi^{-1}(\operatorname{Spec} R/I)$, then this was the main result of [GZB15]. If R has equicharacteristic, X has the resolution property, and Z is a closed point of X, then this is [AHR20, Theorem 1.3].

Example 3.2.15. Let $R = \mathbf{C}[\![x]\!]$ and $I = (x)$, Take $X = B\mathbf{G}_{a,R}$. Then $\pi\colon X \to \operatorname{Spec} R$ is cohomologically proper, but the pair $(X, \pi^{-1}(R/I))$ is not coherently complete (see [HP14]).

3.3 Descent of cohomological properness along universally submersive morphisms

The main result of this section is a criterion for descending cohomological properness along a universally submersive morphism of algebraic stacks $p : X \to Y$. If p is proper, then this is standard. The novelty here is the replacement of properness of the map p with a global condition on the n-fold fiber products $(X/Y)^n$.

Theorem 3.3.1. *Let $p : X \to Y$ be a universally submersive morphism of Noetherian algebraic stacks of finite type over a Noetherian ring R. If for each $n \geq 1$ the n-fold fiber product $(X/Y)^n = X \times_Y \ldots \times_Y X$ is cohomologically proper over* $\operatorname{Spec} R$*, then Y is cohomologically proper over* $\operatorname{Spec} R$.

Proof. We argue by dévissage on the abelian category $\operatorname{Coh}(Y)$. Hence, we assume the result true for all closed proper substacks $Y_0 \subseteq Y$. By a standard argument, we may further reduce to the situation where Y is reduced. By generic flatness, there is now a dense open $U \subseteq Y$ such that $p^{-1}(U) \to U$ is flat. By Rydh's [Ryd16] extension of Raynaud-Gruson's flatification by blow-ups results to stacks, there is a 2-commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f_X} & X \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{Y} & \xrightarrow{f} & Y, \end{array}$$

where f is a blow-up away from U, $\tilde{p}$ is faithfully flat, and $\tilde{X}$ is the strict transform of X along f; in particular, f_X is proper and representable.

Let $n \geq 1$. Since f is proper and representable, it is separated, and so the induced morphism $(\tilde{X}/\tilde{Y})^n \to (\tilde{X}/Y)^n$ is a closed immersion. Since f_X is proper, it now follows by successive projections that $(\tilde{X}/Y)^n \to (X/Y)^n$ is proper. Hence, $(\tilde{X}/\tilde{Y})^n \to (X/Y)^n$ is proper and so $(\tilde{X}/\tilde{Y})^n$ is cohomologically proper over $\operatorname{Spec} R$.

Since $\tilde{p} : \tilde{X} \to \tilde{Y}$ is faithfully flat, by cohomological descent it follows that if $\mathscr{M} \in \operatorname{Coh}(\tilde{Y})$, then there is a convergent spectral sequence:

$$\mathrm{H}^i((\tilde{X}/\tilde{Y})^j, \mathscr{M}_{(\tilde{X}/\tilde{Y})^j}) \Rightarrow \mathrm{H}^{i+j}(\tilde{Y}, \mathscr{M}).$$

It follows immediately that $\tilde{Y}$ is cohomologically proper over $\operatorname{Spec} R$.

Now let $\mathscr{F} \in \operatorname{Coh}(Y)$ and consider the distinguished triangle:

$$\mathscr{F} \to \mathsf{R}f_* f^* \mathscr{F} \to \mathscr{C} \to \mathscr{F}[1]. \tag{3.3.1}$$

Since f is proper and representable, $\mathsf{R}f_* f^* \mathscr{F}, \mathscr{C} \in D^b_{\operatorname{Coh}}(Y)$. But $\tilde{Y}$ is cohomologically proper, so

$$\mathsf{R}\Gamma(Y, \mathsf{R}f_* f^* \mathscr{F}) \simeq \mathsf{R}\Gamma(\tilde{Y}, f^* \mathscr{F}) \in D^+_{\operatorname{Coh}}(R).$$

Now let $j : U \subseteq Y$ be the open immersion; then f is birational over U and so $j^* \mathscr{C} \simeq 0$. Thus, by the distinguished triangle (3.3.1) it remains to prove that if $\mathscr{E} \in D^b_{\operatorname{Coh}}(Y)$ and $j^* \mathscr{E} \simeq 0$, then $\mathsf{R}\Gamma(Y, \mathscr{E}) \in D^+_{\operatorname{Coh}}(R)$. This we can prove by induction on the number of non-zero cohomology groups of $\mathscr{E}$, and it is easy to see that it suffices to prove the result when $\mathscr{E}$ is a sheaf supported in cohomological degree 0. Now $j^* \mathscr{E} = 0$ implies that $\mathscr{J} \mathscr{E} = 0$ for some coherent $\mathscr{O}_Y$-ideal $\mathscr{J} \subseteq \mathscr{O}_Y$

with $V(\mathscr{J}) \cap U = \emptyset$. It follows that $\mathscr{E} = i_*\mathscr{E}_0$, where $Y_0 = V(\mathscr{J})$ and $i : Y_0 \hookrightarrow Y$ is the resulting closed immersion. By dévissage, the result follows. ☐

3.4 Descent of formal functions along universally submersive morphisms

Theorem 3.4.1. *Let $p : X \to Y$ be a universally submersive morphism of finite type of Noetherian algebraic stacks. Let $W \subseteq Y$ be a closed substack. If for every $n \geq 1$, the pair $((X/Y)^n, p_n^{-1}(W))$ satisfies formal functions, then (Y, W) satisfies formal functions.*

Proof. As in the proof of Theorem 3.3.1, we do dévissage on the abelian category $\mathrm{Coh}(Y)$. Hence, we assume for every proper closed substack $Y_0 \subseteq Y$ that the pair $(Y_0, W \cap Y_0)$ satisfies formal functions. A standard argument reduces to the situation where Y is reduced. Then, just as in the proof of Theorem 3.3.1, we arrive at a dense open $U \subseteq Y$ and a 2-commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f_X} & X \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{Y} & \xrightarrow{f} & Y, \end{array}$$

where f is a blow-up away from U, $\tilde{p}$ is faithfully flat, and $\tilde{X}$ is the strict transform of X along f; in particular, f_X is proper and representable.

Let $n \geq 1$. Arguing just as in the proof of Theorem 3.3.1, the morphisms $(\tilde{X}/\tilde{Y})^n \to (X/Y)^n$ are proper and representable for all $n \geq 1$. Hence, they are universally cohomologically proper (Example 3.2.13) and the pairs $((\tilde{X}/\tilde{Y})^n, \tilde{p}_n^{-1} f^{-1}(W))$ satisfy formal functions (Lemma 3.2.10).

Since $\tilde{p} : \tilde{X} \to \tilde{Y}$ is faithfully flat, by cohomological descent it follows that if $\mathscr{M} \in \mathrm{Coh}(\tilde{Y})$, then we have a diagram

$$\begin{array}{ccc} \mathrm{H}^i((\tilde{X}/\tilde{Y})^j, \mathscr{M}_{(\tilde{X}/\tilde{Y})^j}) & \Longrightarrow & \mathrm{H}^{i+j}(\tilde{Y}, \mathscr{M}) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(\widehat{(\tilde{X}/\tilde{Y})^j}, \widehat{\mathscr{M}_{(\tilde{X}/\tilde{Y})^j}}) & \Longrightarrow & \mathrm{H}^{i+j}(\widehat{\tilde{Y}}, \widehat{\mathscr{M}}) \end{array}$$

where the top and bottom are convergent rows spectral sequences, and the vertical arrows are maps of spectral sequences. In the bottom spectral sequence, the completion on the left is taken with respect to $\tilde{p}_j^{-1} f^{-1}(W)$, and on the right with respect to $f^{-1}(W)$. It follows immediately that $(\tilde{Y}, f^{-1}(W))$ satisfies formal functions.

Now let $\mathscr{F} \in \mathrm{Coh}(Y)$ and consider the distinguished triangle:

$$\mathscr{F} \to \mathsf{R}f_* f^* \mathscr{F} \to \mathscr{C} \to \mathscr{F}[1].$$

This gives rise to a commutative diagram with top and bottom rows exact triangles:

$$\begin{array}{ccccccc} \mathsf{R}\Gamma(Y,\mathscr{F}) & \longrightarrow & \mathsf{R}\Gamma(Y,\mathsf{R}f_*f^*\mathscr{F}) & \longrightarrow & \mathsf{R}\Gamma(Y,\mathscr{C}) & \longrightarrow & \mathsf{R}\Gamma(Y,\mathscr{F})[1] \\ \downarrow & & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow \\ \mathsf{R}\Gamma(\widehat{Y}_W,\widehat{\mathscr{F}}_W) & \longrightarrow & \mathsf{R}\Gamma(\widehat{Y}_W,(\widehat{\mathsf{R}f_*f^*\mathscr{F}})_W) & \longrightarrow & \mathsf{R}\Gamma(\widehat{Y}_W,(\widehat{\mathscr{C}})_W) & \longrightarrow & \mathsf{R}\Gamma(\widehat{Y}_W,(\widehat{\mathscr{F}})_W)[1]. \end{array}$$

Since f is proper and representable, it is universally cohomologically proper (Example 3.2.13). Thus by Lemma 3.2.10 and formal functions for the pair $(\tilde{Y}, f^{-1}(W))$, we deduce that vertical arrow (1) is an isomorphism. Finally, arguing as in Theorem 3.3.1,we may assume that $\mathscr{C}$ is a coherent sheaf concentrated in degree 0 such that $\mathscr{C}|_U \simeq 0$. By induction, (2) is an isomorphism and we are done. ☐

3.5 Descent of coherent completeness along universally submersive morphisms

Theorem 3.5.1. *Let $p : X \to Y$ be a universally submersive morphism of Noetherian algebraic stacks that are both finite type over a Noetherian ring R. Let $W \subseteq Y$ be a closed immersion defined by a coherent $\mathscr{O}_Y$-ideal $\mathscr{I}$ and let $I = \Gamma(Y, \mathscr{I})$, which we view as an ideal of $\Gamma(Y, \mathscr{O}_Y)$. Assume the following:*

1. *R is I-adically complete;*
2. *the pair $(X, p^{-1}(W))$ is coherently complete;*
3. *if $p_n : (X/Y)^n \to Y$ is the projection from the n-fold fiber product to Y, then the pair $((X/Y)^n, p_n^{-1}(W))$ satisfies formal functions.*

If formal functions holds for the pair (Y, W), then the pair (Y, W) is coherently complete.

Proof. As in the proof of Theorem 3.3.1, we do dévissage on the abelian category $\mathrm{Coh}(\widehat{Y}_W)$. By that, we mean that if $\mathfrak{H} \in \mathrm{Coh}(\widehat{Y}_W)$ is annihilated by $\mathscr{K}\mathscr{O}_{\widehat{Y}_W)}$ for some coherent ideal sheaf $\mathscr{K} \subseteq \mathscr{O}_Y$, then $\mathfrak{h}$ belongs to the image of $\iota^*_{\widehat{Y}_W}$ (i.e., is algebraizable).

By Lemma 3.2.12, since formal functions holds for the pair (Y, W), the functor

$$\iota^*_{\widehat{Y}_W} : \mathrm{Coh}(Y) \to \mathrm{Coh}(\widehat{Y}_W)$$

is fully faithful, with image stable under kernels, cokernels and extensions. In particular, a standard argument reduces us to the situation where X is reduced. Arguing exactly as in the proof of Theorem

3.3.1, we arrive at a dense open $U \subseteq Y$ and a 2-commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f_X} & X \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{Y} & \xrightarrow{f} & Y, \end{array}$$

where f is a blow-up away from U, $\tilde{p}$ is faithfully flat, and $\tilde{X}$ is the strict transform of X along f; in particular, f_X is proper and representable.

Now let $\mathfrak{F} \in \mathrm{Coh}(\widehat{Y}_W)$. By coherent completeness of the pair (X, W), there is a unique coherent sheaf $\mathscr{F} \in \mathrm{Coh}(X)$ and an isomorphism $\widehat{\mathscr{F}}_{p^{-1}(W)} \simeq \widehat{p}^*\mathfrak{F}$. Let $\mathscr{G} = f_X^*\mathscr{F}$, which is a coherent sheaf on $\tilde{X}$. Now $\widehat{\tilde{p}}^* \widehat{f}^*\mathfrak{F} \simeq \widehat{\mathscr{G}}$. Let $\tilde{p}_i : \tilde{X} \times_Y \tilde{X} \to \tilde{X}$ be the two projections. Then because $\widehat{\mathscr{G}}$ is formally pulled back from $\tilde{Y}$, there is an isomorphism

$$\vartheta : \widehat{\tilde{p}_1^*\mathscr{G}} \xrightarrow{\sim} \widehat{\tilde{p}_2^*\mathscr{G}}$$

that satisfies the cocycle condition in $\mathrm{Coh}((\widehat{\tilde{X}/\tilde{Y}})^3)$. But f_X is proper and representable, and arguing just as in the proof of Theorem 3.3.1, the morphisms $(\tilde{X}/\tilde{Y})^n \to (X/Y)^n$ are proper and representable for all $n \geq 1$. Hence, they are universally cohomologically proper (Example 3.2.13) and the pairs $((\tilde{X}/\tilde{Y})^n, \tilde{p}_n^{-1} f^{-1}(W))$ satisfy formal functions (Lemma 3.2.10).

It now follows that there is a unique isomorphism $\theta : \tilde{p}_1^*\mathscr{G} \xrightarrow{\sim} \tilde{p}_2^*\mathscr{G}$ in $(\tilde{X}/\tilde{Y})^2$ such that $\widehat{\theta} = \vartheta$ that satisfies the cocycle condition in $\mathrm{Coh}((\tilde{X}/\tilde{Y})^3)$. By faithfully flat descent, there is a unique coherent sheaf $\tilde{F}$ on $\tilde{Y}$ such that $\tilde{p}^*\tilde{F} \simeq \mathscr{G}$ and inducing the given descent datum. In fact, we also have $\widehat{\tilde{\mathscr{F}}} \simeq \widehat{f}^*\mathfrak{F}$ (this is by observing the induced formal descent data).

Next observe that Lemma 3.2.10 implies that:

$$\widehat{f}_*\widehat{f}^*\mathfrak{F} \simeq \widehat{f}_*\widehat{\tilde{\mathscr{F}}} \simeq \widehat{f_*\tilde{\mathscr{F}}}.$$

Let $\operatorname{Spec} B \to Y$ be a morphism, and let $J \subseteq B$ be the ideal induced by $\mathscr{I}$. Let $\widehat{B}$ be the J-adic completion of B. Let $\tilde{Y}_B = \tilde{Y} \times_Y \operatorname{Spec} B$ and let $f_B : \tilde{Y}_B \to \operatorname{Spec} B$ be the induced morphism, which is proper. Now let $\operatorname{Spec} B \to Y$ be a smooth morphism. Then $\mathfrak{F}(\operatorname{Spec} B) = F$ for some finite $\widehat{B}$-module F. It follows that the morphism $\mathfrak{F}(\operatorname{Spec} B) \to (\widehat{f}_*\widehat{f}^*\tilde{\mathscr{F}})(\operatorname{Spec} B)$ corresponds to the morphism of finitely generated $\widehat{B}$-modules:

$$F \to \mathrm{H}^0(\widehat{\tilde{Y}}_B, \widehat{f_B}^* F),$$

where the completion is along $f_B^{-1}(\operatorname{Spec} B/J)$. By the usual theorem of formal functions, this is equivalent to the morphism

$$F \to \mathrm{H}^0(\tilde{Y}_{\widehat{B}}, \tilde{f}_{\widehat{B}}^* F).$$

Now f is birational over $U \subseteq Y$, so let $\mathscr{K} \subseteq \mathscr{O}_Y$ denote a coherent ideal with support exactly $Y\backslash U$. Then the kernel and the cokernel of the above are annihilated by some power of $\mathscr{K}(\operatorname{Spec} B)\widehat{B}$. It now follows that the kernel and the cokernel of the morphism

$$\mathfrak{F} \to \widehat{f}_*\widehat{f}^*\mathfrak{F}$$

are annihilated by some power of $\mathscr{K}\mathscr{O}_{\widehat{Y}_W}$. But we have already seen above that $\widehat{f}_*\widehat{f}^*\mathfrak{F}$ is algebraizable, so it follows by dévissage that $\mathfrak{F}$ is algebraizable. □

3.6 Cohomological properness, formal functions and coherent completeness for BG

Let k be a field, R an I-adically complete Noetherian k-algebra, and G/k a split reductive group. Let $\pi : BG \to \operatorname{Spec} k$ denote the structure morphism. In this section, we prove that the pair $(BG_R, \pi_R^{-1}(\operatorname{Spec} R/I))$ satisfies formal functions and is coherently complete. It is rather unfortunate that we start over a field instead of a more general base. However, this is necessary at the moment as our argument relies on a fact about the representation theory of reductive groups over a field. We do not know if the same fact is true for reductive group schemes over an I-adically complete Noetherian ring R.

The following lemma provides an explicit description of the 2-fiber product of quotient stacks.

Lemma 3.6.1. *Let k be a field, G a smooth affine k-group and $H \subseteq G$ be a closed k-subgroup. Suppose that Y is a k-scheme with a left G-action. Then the diagram*

$$\begin{array}{ccc} [H\backslash(G/H \times Y)] & \longrightarrow & [H\backslash Y] \\ \downarrow & & \downarrow \\ [H\backslash Y] & \longrightarrow & [G\backslash Y] \end{array}$$

is 2-Cartesian.

Proof. It is enough to show for any other k-scheme X with a left G-action that

$$[H\backslash X] \simeq [G\backslash(G/H \times X)], \tag{3.6.1}$$

where G acts on $G/H \times X$ diagonally. For then

$$\begin{aligned}
[H\backslash Y] \times_{[G\backslash Y]} [H\backslash Y] &\simeq \big[G\backslash(G/H \times Y)\big] \times_{[G\backslash Y]} \big[G\backslash(G/H \times Y)\big] \\
&\simeq \big[G\backslash\left((G/H \times Y) \times_Y (G/H \times Y)\right)\big] \\
&\simeq [G\backslash(G/H \times G/H \times Y)] \\
&\simeq [H\backslash(G/H \times Y)].
\end{aligned}$$

The last line follows from (3.6.1) with $X = G/H \times Y$. To prove (3.6.1), it enough to prove that the *prestacks* $[H\backslash X]^{\mathrm{pre}}$ and $[G\backslash((G/H)^{\mathrm{pre}} \times X)]^{\mathrm{pre}}$ are isomorphic. The prestack $[H\backslash X]^{\mathrm{pre}}$ is the pseudo-functor whose S-points are described as follows: Objects of $[H\backslash X]^{\mathrm{pre}}(S)$ are simply elements of $X(S)$. For $x, y \in X(S)$, a morphism $x \to y$ is simply an element $g \in G(S)$ such that $gy = x$. In addition, $(G/H)^{\mathrm{pre}}$ is the presheaf on the (big) fppf site of $\operatorname{Spec} k$ given by $S \mapsto G(S)/H(S)$.

To this end, fix a k-scheme S. We define a functor

$$\Phi(S) \colon [H\backslash X]^{\mathrm{pre}}(S) \to [G\backslash((G/H)^{\mathrm{pre}} \times X)]^{\mathrm{pre}}(S)$$

by sending an element $x \in X(S)$ to $(\overline{e}, x)$ where $\overline{e}$ is the identity element of $G(S)/H(S)$. The action of $\Phi(S)$ on morphisms is as follows. Suppose we have $x, y \in X(S)$ and a morphism $x \to y$. Then there exists $h \in H(S)$ such that $y = hx$. It follows that

$$(\overline{e}, y) = (\overline{e}, hx) = h(\overline{h^{-1}}, x) = h(\overline{e}, x).$$

This defines a morphism $(\overline{e}, x) \to (\overline{e}, y)$. Now define a functor in the opposite direction

$$\Psi(S) \colon [G\backslash((G/H)^{\mathrm{pre}} \times X)]^{\mathrm{pre}}(S) \to [H\backslash X]^{\mathrm{pre}}(S)$$

as follows: We send $(\overline{g}, x) \in G(S)/H(S) \times X(S)$ to $g^{-1}x \in X(S)$. To see that $\Psi(S)$ is well-defined, suppose that $g, g' \in G(S)$ are such that $\overline{g} = \overline{g'}$ in $G(S)/H(S)$. Then there exists $h \in H(S)$ such that $g = g'h$ and hence $g^{-1}x = h^{-1}g'^{-1}x$. In conclusion, the objects $g^{-1}x$ and $g'^{-1}x$ of the groupoid $[H\backslash X]^{\mathrm{pre}}(S)$ are isomorphic and thus $\Psi(S)$ is well-defined. We leave it to the reader to define the action of $\Psi(S)$ on morphisms.

Finally, we check that $\Phi(S)$ and $\Psi(S)$ are mutual inverses. We will only check that the the composition $\Phi(S) \circ \Psi(S)$ is isomorphic to $\mathrm{id}_{[G\backslash((G/H)^{\mathrm{pre}} \times X)]^{\mathrm{pre}}}(S)$, leaving the other composition to the reader. Now the object $(\overline{g}, x) \in [G\backslash((G/H)^{\mathrm{pre}} \times X)]^{\mathrm{pre}}(S)$ is sent to $(\overline{e}, g^{-1}x)$ under $\Phi(S) \circ \Psi(S)$. However,

$$(\overline{e}, g^{-1}x) = g^{-1}(\overline{g}, x)$$

which is isomorphic to $(\overline{g}, x)$, and therefore

$$\Phi(S) \circ \Psi(S) \simeq \mathrm{id}_{[G\backslash((G/H)^{\mathrm{pre}} \times X)]^{\mathrm{pre}}}(S)$$

as desired. ☐

Proposition 3.6.2. *Let k be a field and G a split reductive group over k. Let T be a maximal torus and B a Borel containing T corresponding to a choice of positive roots. For $n \geq 1$, let n-fold fiber product $(BT/BB)^n = BT \times_{BB} \ldots \times_{BB} BT$ is a good moduli space over* $\operatorname{Spec} k$.

Proof. By Lemma 3.6.1, there is an isomorphism of algebraic stacks

$$(BT/BB)^n \simeq [T\backslash(\underbrace{B/T \times \ldots \times B/T}_{n-1 \text{ times}})]$$

where the left T-action on the product $B/T \times \ldots B/T$ is diagonal. Furthermore, by [Alp13, Remark 4.8], the stack quotient above admits a good moduli space morphism

$$[T\backslash(B/T \times \ldots \times B/T)] \to \Gamma(B/T \times \ldots \times B/T, \mathscr{O}_{B/T \times \ldots \times B/T})^T.$$

To complete the proof, we must show that the ring of invariants on the right is k. To this end, identify B/T with the unipotent subgroup U. As a scheme, U is affine space $\mathbf{A}^n_k$ with coordinate ring $\Gamma(U, \mathscr{O}_U) \simeq k[x_1, \ldots, x_n]$. Each x_i corresponds to a positive root α_i. The action of T on a monomial $x_1^{\beta_1} \ldots x_n^{\beta_n}$ is given by $t \cdot x_1^{\beta_1} \ldots x_n^{\beta_n} = \alpha_1(t)^{\beta_1} \ldots \alpha_n(t)^{\beta_n} x_1^{\beta_1} \ldots x_n^{\beta_n}$. Now by choice all the α_i are *positive*, viz. T acts on the graded ring

$$\Gamma(U, \mathscr{O}_U) \simeq \bigoplus_{\beta_1, \ldots, \beta_n} k \cdot x_1^{\beta_1} \ldots x_n^{\beta_n}$$

by positive weights. Therefore,

$$\Gamma(B/T \times \ldots \times B/T, \mathscr{O}_{B/T \times \ldots \times B/T})^T \simeq \left(\Gamma(U, \mathscr{O}_U)^{\otimes(n-1)}\right)^T \simeq \left(\Gamma(U, \mathscr{O}_U)^T\right)^{\otimes(n-1)} \simeq k.$$ ☐

We can now prove that BG satisfies cohomological properness (Theorem 3.1.1), formal functions (Theorem 3.1.2) and coherent completeness (Theorem 3.1.3). For convenience, we restate the theorems below.

Theorem 3.6.3 (Cohomological Properness for BG)**.** *Let k be a field and R a Noetherian k-algebra. Then BG_R is cohomologically proper over* $\operatorname{Spec} R$.

Theorem 3.6.4 (Formal Functions for BG)**.** *Let k be a field and R an I-adically complete Noetherian k-algebra. Then the pair (BG_R, Z) with $Z \coloneqq \pi_R^{-1}(\operatorname{Spec} R/I)$ satisfies formal functions.*

Theorem 3.6.5 (Coherent Completeness for BG)**.** *Let k be a field and R an I-adically complete Noetherian k-algebra. Then the pair (BG_R, Z) with $Z := \pi_R^{-1}(\operatorname{Spec} R/I)$ is coherently complete.*

Proof of Cohomological Properness. Fix a complete Noetherian local k-algebra R, a maximal torus $T \subseteq G$ and a Borel B containing T corresponding to a choice of positive roots. Notice that the fiber product $\operatorname{Spec} k \times_{BG} BB$ is identified with the flag variety G/B and therefore the base change $BB_R \to BG_R$ is proper. Consequently for every $n \geq 1$, the n-fold fiber product $(BB/BG)^n_R$ is proper over BB_R. By (Example 3.2.13), we may apply Theorem 3.3.1, and reduce to proving that BB_R is cohomologically proper over $\operatorname{Spec} R$. Now consider the fppf morphism

$$BT_R \to BB_R.$$

Since good moduli spaces are stable under flat base change, Proposition 3.6.2 shows that for any $n \geq 1$, the n-fold fiber product $(BT/BB)^n_R$ is a good moduli space over $\operatorname{Spec} R$. By Example 3.2.14, each $(BT/BB)^n_R$ is cohomologically proper over $\operatorname{Spec} R$. Thus by Theorem 3.3.1 again, we conclude that BB_R is cohomologically proper over $\operatorname{Spec} R$. □

Proof of formal functions. Mutatis Mutandis the same as the proof of Theorem 3.6.3. We first reduce to the case of BB_R using Theorem 3.4.1, and then to the case of BT_R using *loc. cit.*. □

Proof of coherent completeness. We have just proven formal functions for the pair (BG_R, Z). Therefore, by Theorem 3.5.1, it is enough to check for every $n \geq 1$ that $((BB/BG)^n_R, p_n^{-1}(Z))$ satisfies formal functions. Now each $(BB/BG)^n_R \to BB_R$ is proper and the proof of Theorem 3.6.4 proves formal functions for BB_R. The result follows by Lemma 3.2.10. □

3.7 Cohomological properness, formal functions and coherent completeness for $[\mathrm{SL}_d \backslash \mathbf{A}^d]$

In this section, we will consider a simple example of a non-trivial quotient stack and prove that it is cohomologically proper, satisfies formal functions, and is coherently complete. The stack that we will consider is $[\mathrm{SL}_d \backslash \mathbf{A}^d]$, where the action on $\mathbf{A}^d := \mathbf{A}^d_k$ is the standard one. We believe that the example of $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ illustrates many of the key ideas needed to proving cohomological properness, formal functions, and coherent completeness for the general case of $[G \backslash \operatorname{Spec} A]$. Again, we remind the reader that we do not assume any hypothesis on the characteristic of the ground field.

We begin with a first approximation of how the proof from BG to $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ should generalize. Let T and B the standard torus and Borel subgroup of upper triangular matrices in SL_d. Consider the diagram:

$$\begin{array}{ccccc} [T\backslash \mathbf{A}^d] & \longrightarrow & [B\backslash \mathbf{A}^d] & \longrightarrow & [\mathrm{SL}_d\backslash \mathbf{A}^d] \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(\mathbf{A}^d, \mathscr{O}_{\mathbf{A}^d})^T & \longrightarrow & \Gamma(\mathbf{A}^d, \mathscr{O}_{\mathbf{A}^d})^B & \longrightarrow & \Gamma(\mathbf{A}^d, \mathscr{O}_{\mathbf{A}^d})^{\mathrm{SL}_d}. \end{array}$$

Notice that the morphism $[B\backslash \mathbf{A}^d] \to [\mathrm{SL}_d\backslash \mathbf{A}^d]$ is proper. A simple calculation shows that

$$\Gamma(\mathbf{A}^d, \mathscr{O}_{\mathbf{A}^d})^B = \Gamma(\mathbf{A}^d, \mathscr{O}_{\mathbf{A}^d})^{\mathrm{SL}_d} = k$$

and thus we may reduce to the case of $[B\backslash \mathbf{A}^d]$. However, we now run into a problem: The ring of invariants

$$\Gamma(\mathbf{A}^d, \mathscr{O}_{\mathbf{A}^d})^T \neq k.$$

For instance, when $d = 2$, $\Gamma(\mathbf{A}^2, \mathscr{O}_{\mathbf{A}^2})^T = k[xy]$. This means that $[T\backslash \mathbf{A}^d]$ is *not* a good moduli space over $\operatorname{Spec} k$. In addition, there is also the problem that the T-action on $\mathbf{A}^d$ does not have positive weight, and thus we cannot conclude that the n-fold fiber product $([B\backslash \mathbf{A}^d]/[T\backslash \mathbf{A}^d])^n$ is a good moduli space over $\operatorname{Spec} k$.

In order to solve this problem, we will need to pass to a smaller closed subscheme of $\mathbf{A}^d$ that bypasses this issue of negative weight. We now introduce some terminology. Fix the diagonal torus $T \subseteq \mathrm{SL}_d$. For any cocharacter $\lambda : \mathbf{G}_m \to T$, we can identify λ with a tuple of integers $(a_1, \ldots, a_d) \in \mathbf{Z}^{\oplus d}$ such that $\sum a_i = 0$. We say that λ is *regular* if all the a_i's are distinct and non-zero. In this situation, the parabolic associated to λ,

$$P_G(\lambda) \coloneqq \left\{ g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists } \right\}$$

will be a Borel, and the centralizer of λ,

$$Z_G(\lambda) = \{g \in G : \lambda(t) g = g \lambda(t) \text{ for all } t\},$$

will be a maximal torus (in fact equal to T). For any cocharacter $\lambda : \mathbf{G}_m \to T$, we define the *attractor* locus

$$(\mathbf{A}^d)^+_\lambda \coloneqq \left\{ x \in \mathbf{A}^d : \lim_{t \to 0} \lambda(t) x \text{ exists} \right\}.$$

It is a closed subscheme of $\mathbf{A}^d$ that inherits an action of $P_G(\lambda)$. Under the action of λ, the global sections of the scheme $(\mathbf{A}^d)^+_\lambda$ admits a $\mathbf{Z}$-grading with non-zero terms only in degree ≥ 0.

Now suppose there exists a cocharacter $\lambda : \mathbf{G}_m \to T$ satisfying the following properties:

1. λ is regular
2. $\Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^\lambda = k$.

The regularity of λ implies that $Z_G(\lambda)$ is a maximal torus, in particular linearly reductive. Therefore, by [Alp13, Remark 4.8]

$$[Z_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda] \to \Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{Z_G(\lambda)}$$

is a good moduli space, in fact a good moduli space to $\operatorname{Spec} k$, because

$$\Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{Z_G(\lambda)} \subseteq \Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{\lambda} = k.$$

We will show in the proof of Proposition 3.7.2 below that in fact, for every $n \geq 1$,

$$\big([Z_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda]/[P_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda]\big)^n \to \operatorname{Spec} k$$

is a good moduli space. Finally, if in addition the induced map on quotient stacks

$$[P_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda] \to [\mathrm{SL}_d\backslash\mathbf{A}^d]$$

is *surjective*, then it is universally submersive (since it is already proper), and we can apply Theorems 3.3.1, 3.4.1, 3.5.1.

The following proposition shows that such a cocharacter exists.

Proposition 3.7.1. *Let T be the diagonal torus in SL_d. Identify a cocharacter $\lambda : \mathbf{G}_m \to T$ with a tuple $(a_1, \ldots, a_d) \in \mathbf{Z}^{\oplus d}$ such that $\sum a_i = 0$. Then there exists λ such that:*

1. *λ is regular.*
2. *$\Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{\lambda} = k$.*
3. *$[P_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda] \to [\mathrm{SL}_d\backslash\mathbf{A}^d]$ is surjective.*

Proof. We take λ to be any tuple $(a_1, \ldots, a_d) \in \mathbf{Z}^{\oplus d}$ satisfying:

(a) $\sum_{i=1}^d a_i = 0$.

(b) $a_d < 0 < a_{d-1} < \ldots < a_1$.

We verify that all three conditions of Proposition 3.7.1 are satisfied.

1. This is clear from the definition of regularity.
2. Identify $\mathbf{A}^d$ with $\operatorname{Spec} k[x_1, \ldots, x_d]$. Then a simple calculation shows that $(\mathbf{A}^d)^+_\lambda = V(x_d)$. Suppose a polynomial $f(x_1, \ldots, x_{d-1}) \in k[x_1, \ldots, x_{d-1}]$ is invariant under λ. This means

$$f(x_1, \ldots, x_{d-1}) = f(t^{a_1}x_1, \ldots, t^{a_{d-1}}x_{d-1})$$

for every $t \neq 0$. However, all the $a_1, \ldots, a_{d-1}$ are positive and therefore f must be a constant.

3. Observe that $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ consists of two points, namely the origin 0 and the orbit of $(1, 0, \dots, 0)$. Since both these points are also contained in $(\mathbf{A}^d)^+_\lambda = V(x_d)$, the result follows. □

We are now ready to prove that $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ satisfies cohomological properness (Theorem 3.1.4), formal functions (Theorem 3.1.5) and coherent completeness (Theorem 3.1.6). For convenience, we restate the theorems below.

Theorem 3.7.2 (Cohomological Properness for $[\mathrm{SL}_d \backslash \mathbf{A}^d]$)**.** *Let k be a field. Then $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ is cohomologically proper over* $\operatorname{Spec} k$*.*

Theorem 3.7.3 (Formal Functions for $[\mathrm{SL}_d \backslash \mathbf{A}^d]$)**.** *Let Z denote the closed substack of $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ defined by the origin. Then the pair $([\mathrm{SL}_d \backslash \mathbf{A}^d], Z)$ satisfies formal functions.*

Theorem 3.7.4 (Coherent completeness for $[\mathrm{SL}_d \backslash \mathbf{A}^d]$)**.** *Let Z denote the closed substack of $[\mathrm{SL}_d \backslash \mathbf{A}^d]$ defined by the origin. Then the pair $([\mathrm{SL}_d \backslash \mathbf{A}^d], Z)$ is coherently complete.*

Proof of Cohomological Properness. Let λ be the cocharacter found in Proposition 3.7.1. The morphism $[P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda] \to [\mathrm{SL}_d \backslash \mathbf{A}^d]$ factors as a composition of proper morphisms

$$[P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda] \to [P_G(\lambda) \backslash \mathbf{A}^d] \to [\mathrm{SL}_d \backslash \mathbf{A}^d].$$

Consequently, for every $n \geq 1$, the n-fold fiber product

$$\left([P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda] / [\mathrm{SL}_d \backslash \mathbf{A}^d]\right)^n$$

is proper over $[P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda]$. By Example 3.2.13, we may apply Theorem 3.3.1, and reduce to proving that

$$[P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda]$$

is cohomologically proper over $\operatorname{Spec} k$. Now consider the morphism

$$[Z_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda] \to [P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda].$$

Notice that ring of invariants for each of these quotient stacks is k, precisely because

$$\Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{P_G(\lambda)} \subseteq \Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{Z_G(\lambda)} \subseteq \Gamma((\mathbf{A}^d)^+_\lambda, \mathscr{O}_{(\mathbf{A}^d)^+_\lambda})^{\lambda} = k. \tag{3.7.1}$$

Therefore, by Example 3.2.14 and Theorem 3.3.1 again, it is enough to argue as in the proof of Theorem 3.6.3 and show for every $n \geq 1$ that

$$\left([Z_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda] / [P_G(\lambda) \backslash (\mathbf{A}^d)^+_\lambda]\right)^n$$

is a good moduli space over $\operatorname{Spec} k$.

We now argue *mutatis mutandis* as in the proof of Proposition 3.6.2: Applying Lemma 3.6.1 repeatedly, we find that

$$\left([Z_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda]/[P_G(\lambda\backslash(\mathbf{A}^d)^+_\lambda]\right)^n \simeq \left[Z_G(\lambda)\backslash\left(P_G(\lambda)/Z_G(\lambda)\times\ldots\times P_G(\lambda)/Z_G(\lambda)\times(\mathbf{A}^d)^+_\lambda\right)\right].$$

On the right, there are $n-1$ copies of $P_G(\lambda)/Z_G(\lambda)$ and the $Z_G(\lambda)$-action is diagonal. The assumption that λ is regular implies that the centralizer $Z_G(\lambda)$ is a torus, which is linearly reductive in any characteristic. Therefore, by [Alp13, Remark 4.8], it is sufficient to show that the ring of invariants for the $Z_G(\lambda)$-action on

$$P_G(\lambda)/Z_G(\lambda)\times\ldots\times P_G(\lambda)/Z_G(\lambda)\times(\mathbf{A}^d)^+_\lambda$$

is k. Now the proof of Proposition 3.6.2 shows that the $Z_G(\lambda)$-action on $P_G(\lambda)/Z_G(\lambda)$ has *strictly* positive weight, and by construction, the same is true of $(\mathbf{A}^d)^+_\lambda$. Hence

$$\begin{aligned}\Gamma(P_G(\lambda)/Z_G(\lambda)\times\ldots\times P_G(\lambda)/Z_G(\lambda)\times(\mathbf{A}^d)^+_\lambda)^{Z_G(\lambda)} &\simeq \left(\Gamma(P_G(\lambda)/Z_G(\lambda))^{\otimes(n-1)}\otimes\Gamma((\mathbf{A}^d)^+_\lambda)\right)^{Z_G(\lambda)}\\ &\simeq \left(\Gamma(P_G(\lambda)/Z_G(\lambda))^{\otimes(n-1)}\right)^{Z_G(\lambda)}\otimes\Gamma((\mathbf{A}^d)^+_\lambda)^{Z_G(\lambda)}\\ &\simeq \left(\Gamma(P_G(\lambda)/Z_G(\lambda))^{Z_G(\lambda)}\right)^{\otimes(n-1)}\otimes\Gamma((\mathbf{A}^d)^+_\lambda)^{Z_G(\lambda)}\\ &\simeq k,\end{aligned}$$

where the last step uses the assumption that the $Z_G(\lambda)$-invariants on $(\mathbf{A}^d)^+_\lambda$ are trivial (3.7.1). ☐

Proof of formal functions. We will argue as in the proof of Theorem 3.7.2 and reduce to the good moduli space situation, where the result is known by Example 3.2.14. The difference now is that we will descend formal functions using Theorem 3.4.1 instead. In order to apply the theorem, the only condition we must check is that the sequence of morphisms

$$[Z_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda]\to[P_G(\lambda)\backslash(\mathbf{A}^d)^+_\lambda]\to[\mathrm{SL}_d\backslash\mathbf{A}^d]$$

is *adic*. This follows from the fact that the origin in $\mathbf{A}^d$ is the unique fixed point under the action of either SL_d, $P_G(\lambda)$ or $Z_G(\lambda)$. ☐

Proof of coherent completeness. This follows *mutadis mutandis* from the proof of Theorem 3.7.3, but with Theorem 3.5.1 instead to descend coherent completeness. ☐

Remark 3.7.5. As a final remark, we note that one can alternatively deduce formal functions from cohomological properness using Theorem B.1.4 instead.

B.1 Cohomological properness and formal functions

In this section, we introduce the following key definition.

Definition B.1.1. Let $\pi\colon X \to Y$ be a finite type morphism of Noetherian algebraic stacks. We say that π is *strongly* cohomologically proper if for every affine morphism of finite type $p\colon U \to X$, $\Gamma(U, \mathscr{O}_U)$ is a Noetherian R-algebra and the resulting morphism $U \to \operatorname{Spec}\Gamma(U, \mathscr{O}_U)$ is cohomologically proper.

Remark B.1.2. Strong cohomological properness implies universal cohomological properness when Y has affine diagonal.

Example B.1.3. Let $\pi\colon X \to \operatorname{Spec} R$ be a good moduli space, where π is of finite type and R is noetherian. Then π is strongly cohomologically proper. Indeed, let $U \to X$ be an affine morphism of finite type; then $U \to \operatorname{Spec}\Gamma(U, \mathscr{O}_U)$ is a good moduli space [Alp14, Lemma 5.2.11]. But the composition $U \to X \to \operatorname{Spec} R$ is of finite type, so $\operatorname{Spec}\Gamma(U, \mathscr{O}_U) \to \operatorname{Spec} R$ is of finite type [Alp14, Theorem 6.3.3]. Since R is noetherian, the claim follows. A key result of this paper is that this extends to the adequate situation.

A major result of [HP14] (also see [Stacks, Tag 0A0M]) was that if $\pi\colon X \to \operatorname{Spec} R$ is cohomologically proper and $I \subseteq R$ is an ideal, then the pair $(X, \pi^{-1}(\operatorname{Spec}(R/I))$ satisfies formal functions whenever R is I-adically complete. In the case of a strongly cohomological proper morphism, we can extend this significantly to arbitrary pairs (X, Z).

Theorem B.1.4. *Let $\pi\colon X \to \operatorname{Spec} R$ be strongly cohomologically proper with $R = \Gamma(X, \mathscr{O}_X)$. Let $Z \subseteq X$ be a closed immersion defined by a coherent sheaf of ideals $\mathscr{I}$ and let $I = \Gamma(X, \mathscr{I})$, which is an ideal of R. If R is I-adically complete, then formal functions holds for the pair (X, Z).*

Proof. The proof proceeds in exactly the same way as Serre's argument in [EGA, III.4.1.5], and we follow the exposition of [FGI+05, §8.2] closely. Also see [AHR19, §4] for some related material.

We recall the setup: Let A be a ring, $I \subseteq A$ an ideal, and M an A-module. A filtration $(M_n)_{n\in\mathbf{Z}}$ of M is *I-good* (or *I-stable*) if the following three conditions are satisfied:

1. $M = M_k$ for some $k \in \mathbf{Z}$;
2. $IM_n \subseteq M_{n+1}$ for all $n \in \mathbf{Z}$; and
3. $IM_n = M_{n+1}$ for all $n \gg 0$.

Obviously, the filtration $(I^{n+1}M)_{n\geq -1}$ is I-good; the topology that this filtration defines on M is called the I-adic topology on M. A key observation is that the topology on M defined by any I-good filtration is equivalent to the I-adic topology on M [AM69, Lemma 10.6]. A much deeper fact is that if A is Noetherian and M is finitely generated, then a filtration (M_n) is I-stable if and only if

$M_* = \oplus_{n\in\mathbf{Z}} M_n$ is a finitely generated $A_* = \oplus_{n\geq 0} I^n$-module [AM69, Lem. 10.8]. A key consequence of this whole theory is that if M is a finitely generated A-module and A is I-adically complete, then M is I-adically complete.

Let $\mathscr{M}$ be a coherent $\mathscr{O}_X$-module. Let $\mathscr{I}_* = \oplus_{n\geq 0}\mathscr{I}^n$ and let $\mathscr{M}_* = \oplus_{n\geq 0}\mathscr{I}^n\mathscr{M}$. The quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{I}_*$ is of finite type and $\mathscr{M}_*$ is a coherent $\mathscr{I}^*$-module. Indeed, this is smooth-local on X, so we may assume that $X = \operatorname{Spec} A$ and $\mathscr{M} = \tilde{M}$, for some finitely generated A-module M; the claim now follows from the affine case [AM69, Lemma 10.8].

Let $I_* = \Gamma(X, \mathscr{I}_*)$ and $M_*^q = \mathrm{H}^q(X, \mathscr{M}_*)$. Let us briefly remark on the graded structure of M_*^q because it will be important. If $x \in I_s = \Gamma(X, \mathscr{I}^s)$, then for all $t \geq 0$ there is an induced homomorphism of $\mathscr{O}_X$-modules that $\mathscr{I}^t\mathscr{M} \to \mathscr{I}^{s+t}\mathscr{M}$ that is multiplication by x. It follows that we obtain an induced morphism:

$$\mu^q_{x,\mathscr{M},t} \colon \mathrm{H}^q(X, \mathscr{I}^t\mathscr{M}) \to \mathrm{H}^q(X, \mathscr{I}^{t+s}\mathscr{M}).$$

This is how M_*^q becomes a graded I_*-module. In particular, $I_s M_t^q \subseteq M_{t+s}^q$ denotes the image of the natural $R = I_0$-module homomorphism $I_s \otimes_R M_t^q \to M_{t+s}^q$.

Further, the canonical inclusions $\mathscr{I}^t\mathscr{M} \subseteq \mathscr{I}^{t'}\mathscr{M}$ for $t \geq t'$ give rise to an inverse system $(\mathrm{H}^q(X, \mathscr{I}^t\mathscr{M}))_{t\geq 0}$ with transition map $\nu^q_{\mathscr{M},t,t'} \colon \mathrm{H}^q(X, \mathscr{I}^t\mathscr{M}) \to \mathrm{H}^q(X, \mathscr{I}^{t'}\mathscr{M})$ when $t \geq t'$. It follows that the composition:

$$\mathrm{H}^q(X, \mathscr{I}^t\mathscr{M}) \xrightarrow{\mu^q_{x,\mathscr{M},t}} \mathrm{H}^q(X, \mathscr{I}^{t+s}\mathscr{M}) \xrightarrow{\nu^q_{\mathscr{M},t+s,t}} \mathrm{H}^q(X, \mathscr{I}^t\mathscr{M})$$

coincides with multiplication by x on $\mathrm{H}^q(X, \mathscr{I}^t\mathscr{M})$ as an R-module. In particular, if $P \subseteq \mathrm{H}^q(X, \mathscr{I}^t\mathscr{M})$ is an R-submodule, then we have the equality of R-submodules of $\mathrm{H}^q(X, \mathscr{I}^t\mathscr{M})$:

$$\Gamma(X, \mathscr{I}^s)P = \nu^q_{\mathscr{M},t+s,t}(I_s P). \tag{B.1.1}$$

By strong cohomological properness, I_* is Noetherian and M_*^q is a finitely generated and graded I_*-module. Since graded R-algebras are of finite type, it follows that for some sufficiently large N that as ideals of R, $\Gamma(X, \mathscr{I}^N)^k = \Gamma(X, \mathscr{I}^{Nk})$ for all $k \geq 0$. By assumption R is $I = I_1 = \Gamma(X, \mathscr{I})$-adically complete, so it is also $I_N = \Gamma(X, \mathscr{I}^N)$-adically complete. Hence, we may replace $\mathscr{I}$ by $\mathscr{I}^N$ and assume henceforth that $\Gamma(X, \mathscr{I}^k) = \Gamma(X, I)^k$ for all $k \geq 0$ and so $I_* = \oplus_{k\geq 0} I^k = R_*$.

We now let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Let $q \geq 0$ and $n \geq -1$ and consider the exact sequence of R-modules:

$$0 \longrightarrow R_n^q \longrightarrow H^q \longrightarrow H_n^q \longrightarrow Q_n^q \longrightarrow 0, \tag{B.1.2}$$

where $H^q = \mathrm{H}^q(X, \mathscr{F})$, $H^q_n = \mathrm{H}^q(X, \mathscr{F}/\mathscr{I}^{n+1}F)$, $L^q_n = \mathrm{H}^q(X, \mathscr{I}^{n+1}\mathscr{F})$,

$$R^q_n = \ker(H^q \to H^q_n) = \mathrm{im}(L^q_n \to H^q), \text{ and}$$
$$Q^q_n = \mathrm{im}(H^q_n \to L^{q+1}_n) = \ker(L^{q+1}_n \to H^{q+1}).$$

The result follows from the following three claims:

1. the filtration $(R^q_n)_{n\geq -1}$ on H^q is I-good;
2. the inverse system (Q^q_n) is Artin–Rees zero (i.e., there exists an s such that $Q^q_{n+s} \to Q^q_n$ is 0 for all n);
3. the inverse system (H^{q-1}_n) satisfies the uniform Artin–Rees condition (i.e., there is an s such that the images of the morphisms $H^{q-1}_{n'} \to H^{q-1}_n$ agree for all $n' \geq n+s$).

Indeed, the exact sequence (B.1.2) induces the following short exact sequence:

$$0 \longrightarrow H^q/R^q_n \longrightarrow H^q_n \longrightarrow Q^q_n \longrightarrow 0. \tag{B.1.3}$$

We now take inverse limits, and obtain the following exact sequence:

$$0 \longrightarrow \varprojlim_n H^q/R^q_n \longrightarrow \varprojlim_n H^q_n \longrightarrow \varprojlim_n Q^q_n.$$

Since the system (Q^q_n) is Artin–Rees zero, it follows immediately that $\varprojlim_n Q^q_n = 0$. Moreover, the filtration (R^q_n) on H^q is I-good and since H^q is a finitely generated R-module, it follows that the natural map $H^q \to \varprojlim_n H^q/R^q_n$ is an isomorphism. What results from all of this is an isomorphism:

$$\mathrm{H}^q(X, \mathscr{F}) \simeq \varprojlim_n \mathrm{H}^q(X, \mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}).$$

Now the Milnor exact sequence furnishes us with an exact sequence:

$$0 \longrightarrow \varprojlim^1_n H^{q-1}_n \longrightarrow \mathrm{H}^q(\widehat{X}_Z, \widehat{\mathscr{F}}_Z) \longrightarrow \varprojlim_n H^q_n \longrightarrow 0.$$

But (H^{q-1}_n) satisfies the uniform Artin–Rees condition, so $\varprojlim^1_n H^{q-1}_n = 0$. It follows that we have isomorphisms:

$$\mathrm{H}^q(X, \mathscr{F}) \simeq \mathrm{H}^q(\widehat{X}_Z, \widehat{\mathscr{F}}_Z) \simeq \varprojlim_n \mathrm{H}^q(X, \mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}).$$

That is, formal functions holds for the pair (X, Z).

We first establish that the filtration $(R^q_n)_{n\geq -1}$ on H^q is I-good. To see this, we first note that $R_{-1} = H^q$. We now apply the previous discussion to the $\mathscr{O}_X$-module $\mathscr{I}\mathscr{F}$. It follows that

$L^q_* = \oplus_{n\geq 0} L^q_n$ is a finitely generated R_*-module. But the graded R_*-module $\oplus_{n\geq 0} R^q_n$ is the image of the graded R_*-module homomorphism

$$\oplus_{n\geq 0} L^q_n \to \oplus_{n\geq 0} H^q,$$

and so $\oplus_{n\geq -1} R^q_n$ is also a finitely generated graded I_*-module. By [AM69, Lemma 10.8], it follows that the filtration $(R^q_n)_{n\geq -1}$ is I-good.

We next prove that the inverse system (Q^q_n) is Artin–Rees zero. First observe that $Q^q_* = \oplus_{n\geq 0} Q^q_n$ is a R_*-submodule of $L^{q+1}_* = \oplus_{n\geq 0} L^{q+1}_n$, which is a finitely generated R_*-module. Hence, Q^q_* is a finitely generated R_*-module. In particular, there exist integers h, $l \geq 0$ such that $I_h Q^q_k = Q^q_{h+k}$ for all $k \geq l$. But Q^q_n is always a quotient of H^q_n and H^q_n is annihilated by I^{n+1} and so if $m \geq l+h$, then write $m = th + r + l$, where $0 \leq r < h$. Then

$$I^{l+h+1} Q^q_m = I^{l+h+1} I_{th} Q_{l+r} \subseteq I^{l+r+1} I_{th} Q_{l+r} = 0.$$

It follows from (B.1.1) and the above that if $s = (h+2)(l+h) \geq l+h+1$, then for $t \geq 0$ we have

$$\nu^{q+1}_{\mathscr{I}\mathscr{F},t+s,t}(Q^q_{t+s}) = \nu^{q+1}_{\mathscr{I}\mathscr{F},t+s,t}(I_{(h+1)(l+h)} Q^q_{t+h+l}) = I^{(h+1)(l+h)} Q^q_{t+l+h} \subseteq I^{l+h+1} Q^q_{t+l+h} = 0.$$

Finally, the exact sequence of (B.1.3) and basic properties of the Artin–Rees condition shows that it suffices to prove that the inverse systems (H^q/R_n) and (Q_n) satisfy the uniform Artin–Rees condition. Since (Q_n) is Artin–Rees zero, it satisfies the uniform Artin–Rees condition. Further, since (H^q/R_n) is a surjective system it trivially satisfies the uniform Artin–Rees condition. The result follows. □

Bibliography

[AHR19] Jarod Alper, Jack Hall, and David Rydh, *The étale local structure of algebraic stacks*, 2019, in preparation.

[AHR20] Jarod Alper, Jack Hall, and David Rydh, *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. (2) **191** (2020), no. 3, 675–738. MR 4088350

[Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.

[Alp14] ———, *Adequate moduli spaces and geometrically reductive group schemes*, Algebraic Geometry **1** (2014), 489–531.

[AM69] Michael Francis Atiyah and I. G. MacDonald, *Introduction to commutative algebra.*, Addison-Wesley-Longman, 1969.

[Art69] Michael Artin, *Algebraization of formal moduli. I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art70] ———, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. (2) **91** (1970), 88–135.

[Art74] ———, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397. MR 3905467

[Bri07] Tom Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345.

[Con99] Brian Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 473–541. MR 1697371

[Con05] ———, *Formal GAGA for Artin stacks*, 2005, Preprint.

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 143–316. Lecture Notes in Math., Vol. 349.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, I.H.E.S. Publ. Math **4, 8, 11, 17, 20, 24, 28, 32** (1960, 1961, 1961, 1963, 1964, 1965, 1966, 1967).

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck's FGA explained. MR 2222646

[GZB15] Anton Geraschenko and David Zureick-Brown, *Formal GAGA for good moduli spaces*, Algebr. Geom. **2** (2015), no. 2, 214–230.

[Hal18] Jack Hall, *GAGA theorems*, arXiv e-prints (2018), arXiv:1804.01976.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[HP14] Daniel Halpern-Leistner and Anatoly Preygel, *Mapping stacks and categorical notions of properness*, arXiv e-prints (2014), arXiv:1402.3204.

[HR17] Jack Hall and David Rydh, *Perfect complexes on algebraic stacks*, Compos. Math. **153** (2017), no. 11, 2318–2367.

[HR19] ———, *Coherent Tannaka duality and algebraicity of Hom-stacks*, Algebra Number Theory **13** (2019), no. 7, 1633–1675.

[Hub96] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. MR 1734903

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin-New York, 1971.

[KP20] Dimitry Kubrak and Artem Prikhodko, *p-adic hodge theory for artin stacks*, 2020, in preparation.

[Lie06] Max Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. **15** (2006), no. 1, 175–206.

[Lim19] David Benjamin Lim, *Grothendieck's existence theorem for relatively perfect complexes on algebraic stacks*, arXiv e-prints (2019), arXiv:1907.05025.

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000.

[LMS14] Max Lieblich, Davesh Maulik, and Andrew Snowden, *Finiteness of K3 surfaces and the Tate conjecture*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 2, 285–308. MR 3215924

[LO08] Yves Laszlo and Martin Olsson, *The six operations for sheaves on Artin stacks. I. Finite coefficients*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 107, 109–168.

[Muk87] S. Mukai, *On the moduli space of bundles on $K3$ surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413. MR 893604

[Nag62] Masayoshi Nagata, *Complete reducibility of rational representations of a matric group*, J. Math. Kyoto Univ. **1** (1961/62), 87–99. MR 142667

[Ols05] Martin C. Olsson, *On proper coverings of Artin stacks*, Adv. Math. **198** (2005), no. 1, 93–106.

[Ols07] Martin Olsson, *Sheaves on Artin stacks*, J. Reine Angew. Math. **603** (2007), 55–112.

[Ryd16] David Rydh, *Functorial flatification of proper morphisms*, 2016, in preparation.

[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278. MR 68874

[Ser03] Christian Serpé, *Resolution of unbounded complexes in Grothendieck categories*, J. Pure Appl. Algebra **177** (2003), no. 1, 103–112.

[SGA6] *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[Spa88] Nicholas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.

[Stacks] The Stacks Project Authors, *Stacks Project*, `https://stacks.math.columbia.edu`, 2020.

[Tot18] Burt Totaro, *Hodge theory of classifying stacks*, Duke Math. J. **167** (2018), no. 8, 1573–1621. MR 3807317