\( \mathcal{N} = 4 \) COMPACTIFICATIONS OF STRING THEORY AND SUPERGRAVITY

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Date: May 20, 2018.
0. Introduction

Type II string theory compactified on a Calabi-Yau threefold $M$ yields a four-dimensional, i.e. $(3+1)$-dimensional, supergravity theory with $\mathcal{N} = 2$, $\mathcal{N} = 4$, or $\mathcal{N} = 8$ supersymmetries. The holonomy of $M$ is one of the following three Lie groups: $\mathbf{1}$, SU(2), or SU(3), yielding supergravity theories with $\mathcal{N} = 8$, $\mathcal{N} = 4$, or $\mathcal{N} = 2$ supersymmetries respectively. This motivates the following question:

“What are the threefolds of $G$ holonomy and what do they look like?”

for $G \in \{\mathbf{1}, \text{SU}(2), \text{SU}(3)\}$. Given such a threefold, the natural follow-up is:

“What does Type II string theory compactified on it look like?”

In this paper, we will consider the case $G = \text{SU}(2)$, corresponding to four-dimensional supergravity theories with $\mathcal{N} = 4$, i.e. half-maximal, supersymmetry. These models are known as (geometric) CHL models after Chaudhuri, Hockney, and Lykken, who first provided their heterotic description [1]. Chaudhuri and Lowe [2], along with Schwarz and Sen [3], subsequently provided their dual Type II description.

The choice between less or more supersymmetry is often one between phenomenological desirability and mathematical tractability respectively, or conversely between Scylla and Charybdis, between mathematical intractability and phenomenological undesirability. CHL models take the latter horn of the dilemma. They are toy models, rather unrealistic, given the apparent lack of $\mathcal{N} = 4$ supersymmetry in nature, but are nevertheless useful examples through which we can better understand the mathematical framework of string theory and the manner in which it resolves the paradoxes of quantum gravity. In this vein, the enhanced supersymmetry present within CHL models is often sufficient to enable calculations which would otherwise be impossible.

Before sketching out the situation with threefolds of SU(2) holonomy, I think it is worth mentioning the situation with threefolds of trivial holonomy and with threefolds of SU(3) holonomy. There is a single diffeomorphism class of threefolds of trivial holonomy, namely the six-dimensional torus $T^6$. In contrast, there are many known diffeomorphism classes of threefolds of SU(3) holonomy, and it is not even known whether or not there are infinitely many. Furthermore, it does not appear as if there is even a consensus among mathematicians as to which possibility holds [4]. So, the classification of threefolds of trivial holonomy is so simple as to be boring, while the classification of threefolds of SU(3) holonomy is a tremendously complex, wide open problem. The classification of threefolds of SU(2)
holonomy is, perhaps expectedly, intermediate. There are a handful of diffeomorphism classes of threefolds of SU(2) holonomy, thirteen to be exact, yielding a handful of geometric CHL models to work with. Enough to have some diversity, while not enough to become unwieldy.

An outline for this thesis is as follows: The first couple sections are comprised entirely of exposition. The first section (following this introduction) is a review of spacetime supersymmetry. Since our eventual concern is with four-dimensional theories with $\mathcal{N} = 4$ spacetime supersymmetries, this is our focus here. The following section is a review of the RNS construction of the Type II superstring, up to the determination of their massless spectra, largely following [5][6][7]. The section after that is a brief overview of classical Type II supergravity, once again with an eye towards four-dimensional theories with $\mathcal{N} = 4$ supersymmetry. In particular, we will dimensionally reduce ten-dimensional supergravity to concretely determine the massless field content of CHL models. The subsequent sections are all a mixture of original research and exposition. In section 4, following the proof sketch given by Aspinwall [8], we will classify all CHL manifolds, i.e. complex threefolds of SU(2) holonomy, up to diffeomorphism, and then use this classification in order to compute the rank of the cohomology lattices of all CHL manifolds. A key component of this computation involves a recursive formula for the orbifold Euler characteristic of $K3/G$, and the physically motivated assumption that the resultant orbifold CFT is that of a K3 surface. I have not seen this particular line of argumentation elsewhere, but the results are all to be found scattered around the literature. We will also give a direct construction of all CHL manifolds as $K3$ fibrations over $T^2$. This direct construction will allow us to apply a fundamental technique in algebraic topology, the Mayer-Vietoris sequence, in order to determine their cohomology lattices. In section 5, we will use the results of the preceding section along with dimensional reduction in order to describe certain aspects of CHL models, including their scalar fields, gauge fields, and charge lattices, all from a Type II perspective. I have only seen this done from a heterotic perspective elsewhere, in [9] for example. In section 6, we will ask how half-BPS black holes look in these models, and in particular investigate their attractor geometries and counts. The second half of this section represents original research, and in particular extends the observation in [10] to some CHL models. The final section, which is largely based on my own paper Rademacher series for $\eta$-quotients [11], uses some techniques borrowed from analytic number theory in order to extract from the relevant partition functions an exact formula for the entropy of half-BPS black holes in “$G = \mathbb{Z}_N$” CHL models. For the prime-$p$ CHL models, this latter result is clarified and
complemented by the analysis of Nally in [12]. We will briefly sketch how to extend Nally’s result to the composite-$N$ CHL models, and in particular the cases $N = 4, 8$, where the inclusion $\Gamma_0(N)^+ \subset N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(N))$ is proper, and therefore non-Atkin-Lehner dualities need to be taken into account.

A couple of clarifications are in order. The first clarification is that we are dealing exclusively with geometric CHL models, those that arise from compactifying Type II string theory on a smooth manifold. The more general class of non-geometric CHL models is well-understood, and there exists a classification due to Persson and Volpato [9], who analyzed the stringy symmetries of a K3 surface. We will not touch upon non-geometric CHL models here, although it should be noted that the analysis in section 7 applies to the $\eta$-quotients which arise (there) as the partition functions of half-BPS black holes given in [13]. The second clarification is that, although CHL models are perhaps most easily understood on the heterotic side, where they are to be understood as compactifications on $T^6$, we will work almost exclusively on the Type II side, hopefully maintaining a geometric perspective.
1. Spacetime Supersymmetry

There are two relevant types of supersymmetry algebras, corresponding to the two relevant types of field theory. The first is spacetime supersymmetry, corresponding to spacetime field theories. The second is worldsheet supersymmetry, corresponding to the worldsheet theory on the worldsheet of a propagating string. Here we will review spacetime supersymmetry, which is relevant to macroscopic physics, focusing on the four-dimensional case. In the next section we will touch on worldsheet supersymmetry, which is relevant to the construction of the Type II string. This section draws on expositions in [7] and [5] regarding the relevant physics and [14], [15], and [16] regarding the relevant details regarding spinors.

1.1. Spacetime Supersymmetry. The Poincaré group in $D$-dimensions is the group of isometries of $\mathbb{R}^{1,D-1}$, which is given by $\text{Iso}(\mathbb{R}^{1,D-1}) \cong \mathbb{R}^{1,D-1} \rtimes \text{O}(1,D-1)$, the semidirect product of the group of translations and the group of Lorentz transformations. The double cover of the Poincaré group in $D$-dimensions is $\mathbb{R}^{1,D-1} \rtimes \text{Spin}(1,D-1)$. Both of these groups are not connected and consist of four connected components, related by the parity and time reversal operators. The component of the Poincaré group connected to the identity is known as the isochronous Poincaré group, and is given by the semidirect product of the group of translations and the group of isochronous Lorentz transformations. Analogously, the double cover of the isochronous Poincaré group is the component of the double cover of the Poincaré group connected to the identity. Since this group is both connected and simply connected, the relevant representations are exactly those of its Lie algebra.

Definition 1.1. The Poincaré Algebra in $D$-dimensions is the Lie algebra of all of these groups. This Lie algebra is generated by the momentum operators $P^\mu$, which generate the translations, and the rotation and boost operators $+M_{\mu\nu} = -M_{\nu\mu}$. The Lie algebra generators satisfy

\begin{align*}
[&P^\mu, P^\nu] = 0 \quad \text{(1)} \\
[&P^\mu, M^{\nu\rho}] = i(\eta^{\mu\rho} P^\nu - \eta^{\mu\nu} P^\rho) \quad \text{(2)} \\
[&M_{\mu\nu}, M_{\rho\tau}] = i(\eta_{\mu\rho} M_{\nu\tau} - \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\tau} + \eta_{\nu\tau} M_{\mu\rho}) \quad \text{(3)}
\end{align*}

The generators of rotations $\text{SO}(D-1) \subseteq \text{O}(1,D-1)$ are given by $J_{ij} = M_{ij}$ and the boosts are given by $K^i = M^{i0}$ for $i, j \in \{1, \ldots, D-1\}$.

The Poincaré group can be extended to the super-Poincaré group with some number of supersymmetries. If $D \equiv 2 \mod 4$ then the supersymmetries in question are Majorana-Weyl,
and we must distinguish between right-handed and left-handed supersymmetries. Otherwise, the supersymmetries in question are Majorana, and we need only specify the number of supersymmetries to specify the group.

We are ultimately interested in the case of $(3 + 1) = 4$ dimensions, so let $D = 4$. Let $\Gamma^\mu$ be a four-dimensional Majorana representation of the Dirac algebra. To be concrete, we can take
\[
\Gamma^0 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & +i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}
\]
where $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices. The chirality matrix and charge conjugation matrix are then given by
\[
\Gamma^5 = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}.
\]

**Definition 1.2.** The **Super-Poincaré algebra** in four dimensions with $N$ supersymmetries ($4N$ supercharges) is generated by the Poincaré algebra and $N$ supersymmetry generators $Q^A_\alpha$ and $\overline{Q}^A_\alpha = Q^A_\alpha \Gamma^0_\alpha$, along with (real) central charges $Z^{AB}$ for $A, B \in \{1, \ldots, N\}$, where $Q^A_\alpha$ is a Majorana spinor and $\alpha$ is a spinor index. The spinor $Q^A$ is known as a **supersymmetry** while the individual components $Q^A_\alpha$ are known as **supercharges**. The yet unspecified Lie brackets are given by [17]
\[
\{Q^A_\alpha, \overline{Q}^B_\beta\} = -2\delta^{AB}P_\mu \Gamma^\mu_{\alpha\beta} - 2iZ^{AB}\delta_{\alpha\beta} \quad \text{and} \quad [M_{\mu\nu}, Q^A_\alpha] = \frac{i}{4}[\Gamma_{\mu}, \Gamma_{\nu}]_{\alpha\beta} Q^A_\beta
\]
\[
[P^\mu, Q^A_\alpha] = [Z^{AB}, Q^C_\alpha] = [Z^{AB}, P^\mu] = 0.
\]

The Super-Poincaré group is then the group generated by exponentiating this algebra and adding the parity and time-reversal operators.

1.1.1. **R-Symmetry.** The Super-Poincaré algebra admits a group of automorphisms $\text{SU}(\mathcal{N}) \times \text{U}(1)$, where $\text{SU}(\mathcal{N})$ acts on the indices $A, B, C$ and $\text{U}(1)$ acts on the supercharges and central charge by multiplication. These symmetries are known as **R-symmetries** and consequently $A, B, C$ are called **R-symmetry indices**. The existence of this symmetry simply reflects the fact that our choice of generators of the odd part of the algebra is rather arbitrary, as is a global phase on the spinors and central charge. These symmetries can be added into the algebra itself in the usual way and therefore extend the relevant group, but this is not necessary for our purposes.
1.1.2. Structure of Irreducible Representations. We want to understand the structure of an irreducible unitary representation $V$ of the Super-Poincaré algebra and the CPT operator. Such a representation is known as a multiplet. Since the mass squared operator $P_\mu P^\mu = M^2$ commutes with the entire group, $M^2$ is proportional to the identity: $M^2 = m^2 \cdot \text{id}_V$. We only consider the case $m^2 > 0$, so that $m$ is real and we can take $m > 0$. By similar reasoning, each central charge $Z^{AB} = Z^{AB} \cdot \text{id}_V$ is proportional to the identity. Of course, $V$ might not be an irreducible representation of the Super-Poincaré algebra. Either $V$ is already an irreducible representation of the Super-Poincaré algebra, in which case we say that $V$ is CPT self-conjugate, or we can decompose

(5) \[ V = V_{\text{iso}} \oplus (CPT)V_{\text{iso}} \]

where $V_{\text{iso}} \subset V$ is an irreducible representation of the Super-Poincaré algebra. Consequently $V_{\text{iso}}$ is an irreducible representation of the component of the double cover of the Super-Poincaré group connected to the identity, on which $\text{sign}(P^0) = +\text{id}_{V_{\text{iso}}}$. We now consider this representation. We can decompose $V_{\text{iso}}$ into a direct sum of eigenspaces of $P^\mu$:

(6) \[ V_{\text{iso}} = \bigoplus_{p^\mu \in \mathbb{R}^{1,3}} V_{p^\mu} = \bigoplus_{\Lambda \in \text{O}(1,3)_{\text{iso}}} V_{\Lambda, p^\mu} = \bigoplus_{\Lambda \in \text{O}(1,3)_{\text{iso}}} \Lambda V_{\text{can}} \]

where

(7) \[ p^\mu_{\text{can}} = \begin{cases} (m, 0, 0, 0) & (m > 0) \\ (1, 1, 0, 0) & (m = 0). \end{cases} \]

Note that $V_{p^\mu}$ is trivial if $p^2 \neq m^2$ or $\text{sign}(p^0) \leq 0$. The little group of Spin$(1,3)$ which preserves $p^\mu_{\text{can}}$ is Spin$(3)$ if $m > 0$ and Iso$(\mathbb{R}^2)$ (1) if $m = 0$. The subspace $V_{p^\mu_{\text{can}}}$ is an irreducible representation of the group generated by the little group and the supersymmetries, and consequently of the corresponding algebra. We can further decompose $V_{p^\mu_{\text{can}}}$ into a direct sum of eigenspaces of the total spin operator $J^2$:

(8) \[ V_{p^\mu_{\text{can}}} = \bigoplus_{2j \in \mathbb{Z}} V_j, \]

where each $V_j$ is a direct sum of irreducible representations of the little group of total spin $j$. Since we understand irreducible representations of the little group, we only need to understand this decomposition. The number of spins appearing in the spin decomposition (Eq. 8) is determined by the number of supersymmetry operators which are zero in this

\[ ^{1}\text{Or its double cover.} \]
representation. This is determined by the number of BPS bounds which our representation saturates.

1.1.3. BPS Bounds and States. Let $V$ be an irreducible (unitary, nontrivial) representation of the Super-Poincaré algebra. The supersymmetry anti-commutation relation implies that the central charge matrix is anti-symmetric:

$$-8iZ^{AB} = + \{Q^A, \overline{Q}_B\} = Q^A_R \Gamma^0_{\beta \alpha} Q^B_{\beta} + Q^B_R \Gamma^0_{\gamma \alpha} Q^A_{\gamma} = - \{Q^B, \overline{Q}_A\} = +8iZ^{BA}.$$

Using $R$-symmetry, we can replace $Z$ with $SZSZ^T$ for any $S \in \text{SO}(N)$. In this manner, we can conjugate $Z$ into a sufficiently nice form. If $N$ is even, we can assume that $Z$ is block diagonal, with the $i$th block given by the $2 \times 2$ matrix

$$(9) \begin{pmatrix} 0 & +Z_i \\ -Z_i & 0 \end{pmatrix} \text{ for } i \in \{1, \ldots, N/2\}$$

where $Z_i$ is real. If $N$ is odd, we can assume that $Z$ is block diagonal, with the first $(N-1)/2$ blocks of the above form, and the last block zero. For simplicity, in the following analysis assume that $N$ is even.

We can multiply the supersymmetry anti-commutation relation by $\Gamma^0$ to find

$$(10) \{Q^A, Q^B_{\dagger}\} = 2\delta^{AB}P_\mu \Gamma^\mu_{\alpha \gamma} \Gamma^0_{\gamma \beta} + 2iZ^{AB} \Gamma^0_{\alpha \beta}.$$ 

We can write $\{Q^A, Q^B_{\dagger}\}$ in matrix form as a tensor product of a $4N \times 4N$ matrix $M^{AB}_{\alpha \beta}$, where $A$ and $\alpha$ together index the rows and $B$ and $\beta$ together index the columns, and the identity matrix $\text{id}_V$:

$$(11) \{Q^A, Q^B_{\dagger}\} = M^{AB}_{\alpha \beta} \otimes \text{id}_V.$$ 

Let $v \in V$ be an arbitrary nonzero vector and $\lambda^A_A \in \mathbb{C}$ be any collection of $4N$ complex numbers. Note that

$$(12) \sum_{A} M^{AB}_{\alpha \beta} \lambda^A_B \cdot \|v\|^2 = \sum_{A} v^\dagger (Q^A, Q^B_{\dagger}) v \lambda^A_B = \|\lambda^\beta_B Q^B_{\dagger} v\|^2 + \|\lambda^A_A Q^A_{\dagger} v\|^2.$$ 

The right hand side is nonnegative. In other words, the matrix $M^{AB}_{\alpha \beta}$ is positive semi-definite.

Now suppose that our representation $V$ has mass $M^2 > 0$. We examine $V_{\text{pcan}}$, where $p^\mu_{\text{can}} = (M, 0, 0, 0)$. The supercharge anti-commutation relation (Eq. 10) becomes

$$(13) \{Q^A, Q^B_{\dagger}\} = 2M\delta^{AB} \delta_{\alpha \beta} + 2iZ^{AB} \Gamma^0_{\alpha \beta}.$$
The positive semi-definite matrix $M_{\alpha\beta}^{AB}$ is given by

\begin{equation}
M_{\alpha\beta}^{AB} = 2 \begin{pmatrix}
M & 0 & 0 & -iZ \\
0 & M & +iZ & 0 \\
0 & -iZ & M & 0 \\
+iZ & 0 & 0 & M
\end{pmatrix}
\end{equation}

where the $A, B$ indices are implicit. This matrix is a direct sum of four copies, up to
transposes and a factor of two, of each of the matrices

\begin{equation}
\begin{pmatrix}
M & -iZ_i \\
+iZ_i & M
\end{pmatrix}
\end{equation}

for $i \in \{1, \ldots, N/2\}$.

The whole matrix is positive semi-definite if and only if each of these smaller matrices is,
which is true if and only if $M^2 \geq |Z_i|^2$ since $M > 0$.

Now suppose our representation $V$ has mass $M^2 = 0$, that is, is massless. We examine
$V_{\text{can}}$, where $p_{\text{can}}^\mu = (1, 1, 0, 0)$. The supercharge anti-commutation relation (Eq. 10) becomes

\begin{equation}
\{Q^A_\alpha, Q^B_\beta^\dagger\} = -2\delta^{AB} \left( \Gamma^0_{\alpha\gamma} - \Gamma^1_{\alpha\gamma} \right) \Gamma^0_{\gamma\beta} + 2iZ^{AB}\Gamma^0_{\alpha\beta}.
\end{equation}

The positive semi-definite matrix $M_{\alpha\beta}^{AB}$ is given by

\begin{equation}
M_{\alpha\beta}^{AB} = 2 \begin{pmatrix}
-2 & 0 & 0 & -iZ \\
0 & -2 & +iZ & 0 \\
0 & -iZ & 0 & 0 \\
+iZ & 0 & 0 & 0
\end{pmatrix}
\end{equation}

where the $A, B$ indices are implicit. This matrix is a direct sum of four copies, up to
transposes and a factor of two, of each of the matrices

\begin{equation}
\begin{pmatrix}
-2 & -iZ_i \\
+iZ_i & 0
\end{pmatrix}
\end{equation}

for $i \in \{1, \ldots, N/2\}$.

The whole matrix is positive semi-definite if and only if each of these smaller matrices is,
which is true if and only if $Z = 0$. So, once again, $M^2 \geq |Z_i|^2$ for each $i$.

Combining these two cases, we have proven the Bogomol’nyi-Prasad-Sommerfield (BPS)
bound:

**Proposition 1.3** (BPS). Assume that $\mathcal{N}$ is even. Given an irreducible unitary representa-
tion of the super-Poincaré algebra of mass $M \geq 0$, then

\[ M \geq \max\{|Z_1|, \ldots, |Z_{N/2}|\} \]
where $+iZ_1, -iZ_1, \ldots, +iZ_{N/2}, -iZ_{N/2}$ are the eigenvalues of the central charge matrix counted in pairs. This constitutes $\mathcal{N}/2$ bounds $M \geq |Z_i|$ for $i \in \{1, \ldots, \mathcal{N}/2\}$. We say that this bound is saturated if equality holds.

From the preceding analysis, in a massless representation all BPS bounds are automatically saturated. In contrast, in a massive representation, any number of the $\mathcal{N}/2$ bounds can be saturated. The number of saturated bounds determines the number of distinct spins in our representation. A representation in which $4M$ of the supercharges are zero is known as a $(M/\mathcal{N})$-BPS state. A $(M/\mathcal{N})$-state is therefore a state possessing $(\mathcal{M}/\mathcal{N})$ of the possible supersymmetries. For each $A \in \{1, \ldots, \mathcal{N}\}$, the supercharge anti-commutation relation implies that

- if the corresponding BPS bound is not saturated, the supersymmetry $Q^A$ provides a two dimensional subspace of creation operators and a two dimensional subspace of annihilation operators,
- if the corresponding BPS bound is saturated, the supersymmetry $Q^A$ provides instead a one dimensional subspace of creation operators and a one dimensional subspace of annihilation operators, with a complementary two dimensional subspace of unbroken supersymmetries.

1.1.4. $D = 4$, $\mathcal{N} = 2$ Massless Representations. The massless irreducible representations of the four-dimensional Super-Poincaré algebra, including CPT, with $\mathcal{N} = 2$ supersymmetries are characterized only by the lowest spin state $j_0$. The particle content is given by $((j_0)^1, (j_0 + 1/2)^2, (j_0 + 1)^1)$ plus the CPT conjugate if necessary. There are three important multiplets which show up in $\mathcal{N} = 2$ Type II supergravity, plus another multiplet which shows up upon truncating $\mathcal{N} = 4$ Type II supergravity to $\mathcal{N} = 2$.

- $j_0 = -1/2$: This is the hypermultiplet $\mathcal{M}^{\text{hyper}}_{D=4,\mathcal{N}=2}$, whose particle content is given by two copies of $((-1/2)^1, (0)^2, (+1/2)^1)$. Each half alone is not CPT self-conjugate, hence the factor of two. In terms of fields, the particle content is given by two copies of $(\bar{\chi}_A, z^A, \lambda_\alpha)$, where $A \in \{1, 2\}$. The spinors $\bar{\chi}$ and $\lambda$ denote two dilatinos, and the remaining two fields $z^1$ and $z^2$ are real scalars.

- $j_0 = 0$: This is the vector multiplet, $\mathcal{M}^{\text{vector}}_{D=4,\mathcal{N}=2}$, whose particle content is given by $((0)^1, (1/2)^2, (1)^1)$ plus its CPT conjugate. In terms of fields, the particle content is given by $(A^*_\mu, \bar{X}^A, z, \lambda^A, A_\mu)$. As before, $A \in \{1, 2\}$. The field $z$ is a scalar, $\lambda^A$ is a pair of spinors, and $A_\mu$ is a gauge field.
• \( j_0 = 1/2 \): This is the gravitino multiplet \( \mathcal{M}_{D=4,N=2}^{\text{gravitino}} \), whose particle content is given by \(((1/2)^1, (3/2)^1)\) plus its CPT conjugate. In terms of fields, the particle content is given by \((\lambda_\alpha, A_\mu^A, \chi_{\alpha\mu})\). The spinor \( \lambda_\alpha \) is a dilatino, \( A_\mu^A \) is a pair of gauge fields, and \( \chi_{\alpha\mu} \) is a gravitino. This multiplet arises when truncating \( N = 4 \) Type II supergravity to \( N = 2 \).

• \( j_0 = 1 \): This is the supergravity multiplet, \( \mathcal{M}_{D=4,N=2}^{\text{supergravity}} \), whose particle content is given by \(((1)^1, (3/2)^2, (2)^1)\) plus its CPT conjugate. In terms of fields, the particle content is given by \((A_\mu, \chi_{\alpha\mu}, g_{\mu\nu})\). The field \( A_\mu \) is a gauge field called the graviphoton, \( \chi_{\alpha\mu} \) is a pair of gravitinos, and \( g_{\mu\nu} \) is the graviton.

1.1.5. \( D = 4, N = 4 \) Representations. The massless irreducible representations of the four-dimensional Super-Poincaré algebra with \( N = 4 \) supersymmetries are also characterized only by the lowest spin state \( j_0 \). The particle content is given by \(((j_0)^1, (j_0 + 1/2)^4, (j_0 + 1)^6, (j_0 + 3/2)^4, (j_0 + 2)^1)\), plus the CPT conjugate if necessary. There are only two relevant multiplets.

• \( j_0 = 0 \): This is the supergravity multiplet \( \mathcal{M}_{D=4,N=4}^{\text{supergravity}} \), whose particle content is given by \(((0)^1, (1/2)^4, (1)^6, (3/2)^4, (2)^1)\) plus its CPT conjugate. In terms of fields, the particle content is given by

\[
(\tau, \lambda^A_\alpha, A^{AB}_\mu, \chi^{ABC}_{\mu\nu}, G_{\mu\nu}).
\]

Here \( \tau \) is a complex scalar formed by combining the scalar field in the multiplet definition with the trace of the spin two field, \( \lambda^A_\alpha \) is a quadruple of dilatinos, \( A^{AB}_\mu \) is a collection of 6 graviphotons arranged in an anti-symmetric matrix, \( \chi^{ABC}_{\mu\nu} \) is a collection of 4 gravitinos arranged in a three-form, and \( G_{\mu\nu} \) is the graviton. Here \( A, B, C \in \{1, 2, 3, 4\} \) are \( R \)-symmetry indices. To index the fields without repetitions, we may take \( A < B < C \).

• \( j_0 = -1 \): This is the matter multiplet \( \mathcal{M}_{D=4,N=2}^{\text{matter}} \), whose particle content is given by \(((−1)^1, (−1/2)^4, (0)^6, (1/2)^4, (1)^1)\) and is CPT self-conjugate. In terms of fields, the particle content is given by

\[
(A_\mu, \lambda^A_\alpha, z^{AB}, \chi^A_{\alpha\mu}, A^*_\mu).
\]

The indices are as before. Here \( z^{AB} \) is a collection of six scalars arranged in an antisymmetric tensor under \( R \)-symmetry.
1.1.6. Truncation. The $\mathcal{N} = 4$ supersymmetry algebra naturally contains two copies of the $\mathcal{N} = 2$ supersymmetry algebra. We can consider how irreducible representations of the original algebra decompose as direct sums of irreducible representations of the subalgebra. In particular, we need to know how the $\mathcal{N} = 4$ multiplets break up as a sum of $\mathcal{N} = 2$ multiplets. In order to apply arguments from settings with $\mathcal{N} = 2$ supersymmetry to settings with $\mathcal{N} = 4$ supersymmetry, it is often useful to understand this decomposition.

We begin with the $\mathcal{N} = 4$ supergravity multiplet. This breaks up as the direct sum of a $\mathcal{N} = 2$ supergravity multiplet, a vector multiplet, and two gravitino multiplets:

\begin{equation}
\mathcal{M}_{D=4,\mathcal{N}=4}^{\text{supergravity}} = \mathcal{M}_{D=4,\mathcal{N}=2}^{\text{supergravity}} \oplus 2 \mathcal{M}_{D=4,\mathcal{N}=2}^{\text{gravitino}} \oplus \mathcal{M}_{D=4,\mathcal{N}=2}^{\text{vector}}.
\end{equation}

The $\mathcal{N} = 4$ matter multiplet, on the other hand, breaks up into a sum of one vector multiplet and one hypermultiplet:

\begin{equation}
\mathcal{M}_{D=4,\mathcal{N}=4}^{\text{matter}} = \mathcal{M}_{D=4,\mathcal{N}=2}^{\text{vector}} \oplus \mathcal{M}_{D=4,\mathcal{N}=2}^{\text{hyper}}.
\end{equation}

These decompositions can be confirmed by counting the number of particles of each spin in the given multiplets.
2. The Type II Superstring

In this section we will review the RNS (Ramond-Neveu-Schwarz) construction of the closed Type II superstring in \( D \) dimensions. Our purpose here is purely expository, and so we will not make any attempt to be comprehensive or particularly careful. The key result which we will use later is the massless spectrum of the superstring in Type IIA and Type IIB, as this constitutes the fields of the resultant supergravity theory.

2.1. The Classical Superstring. The classical superstring is constructed by taking a string worldsheet \( \Sigma \), which is topologically a cylinder without boundary at the tree level, and considering a classical field theory on the worldsheet.

2.1.1. Worldsheet Fields. We adorn the string worldsheet \( \Sigma \) with two types of fields. The first type is the worldsheet boson \( X \), which represents a sufficiently smooth embedding \( X : \Sigma \to M \) of the string worldsheet into spacetime. If we fix a point \( \sigma_0 \in \Sigma \) and a local coordinate system \((x^0, \ldots, x^{D-1})\) around \( X(\sigma_0) \), we can identify \( X(\sigma) \) near \( \sigma_0 \) with its coordinate representation \( X^\mu = X^\mu(\sigma) \), where \( \mu \in \{0, \ldots, D-1\} \) is a spacetime vector index. Since \( X^\mu \) is merely a collection of real numbers,
\[
[X^\mu, X^\nu] = 0.
\]
The field \( X^\mu \) transforms as a worldsheet scalar and as a spacetime vector. The second type is the worldsheet fermion \( \Psi = (\psi^-, \psi^+)^T \), whose components \( \psi_+ \) and \( \psi_- \) each represent a sufficiently smooth section of bundles \( S_\pm \otimes X^* (TM) \), where \( S_\pm \) are two Weyl spin bundles over \( \Sigma \), corresponding to two spin structures on \( \Sigma \). Using the same local coordinate system as before, we can identify \( \Psi(\sigma) \) near \( \sigma_0 \) with its coordinate representation \( \Psi^\mu(\sigma) = \psi_\mu(\sigma) \), where \( \mu \in \{0, \ldots, D-1\} \) is a spacetime vector index. For each \( \mu \in \{0, \ldots, D-1\} \), \( \psi_\pm^\mu \) is a worldsheet Weyl spinor and a smooth real Grassmann valued function, so that
\[
\{\psi_\mu^+, \psi_\nu^-\} = \{\psi_\mu^+, \psi_\nu^+\} = \{\psi_\mu^-, \psi_\nu^-\} = 0.
\]
The two fields \( \psi_\pm^\mu \) are combined into a two-component worldsheet spinor \( \Psi^\mu = \psi_\mu = (\psi_\mu^-, \psi_\mu^+)^T \), where \( m \in \{-, +\} \) is a worldsheet spinor index. In other words, \( \Psi^\mu \) transforms as a worldsheet spinor and a spacetime vector. Regarding our choice of spin structure, there are \( |H^1(\Sigma, \mathbb{Z}_2)| \) possible spin structures on \( \Sigma \) up to isomorphism. Each spin structure is associated with an element of \( H^1(\Sigma, \mathbb{Z}_2) \), which describes how the corresponding bundle twists around the various non-contractible curves of \( \Sigma \). Consequently, each distinct spin structure gives rise to a distinct fermionic sector, and the fermions in two distinct sectors differ in the “signs” accrued when traversing some cycle.
Consider temporarily the special case when the worldsheet $\Sigma$ is an infinite cylinder: $\Sigma \cong S^1 \times \mathbb{R}$. We parameterize the $S^1$ factor by $\sigma = \sigma^1 \in [0, 1)$ and the $\mathbb{R}$ factor by $\tau = \sigma^0 \in \mathbb{R}$. This parameterization is not canonical, as the factorization of a cylinder as $S^1 \times \mathbb{R}$ is not canonical topologically speaking. The pair $(\tau, \sigma) = (\sigma^0, \sigma^1)$ is a worldsheet vector $\sigma^\alpha$, where $\alpha \in \{0, 1\}$ is a worldsheet vector index. This surface admits two distinct spin structures, the Ramond structure, yielding a spin bundle, the Ramond bundle, analogous to the trivial bundle over $S^1$, and the Neveu-Schwarz structure, yielding a spin bundle, the Neveu-Schwarz bundle, analogous to the Möbius bundle over $S^1$, and this gives rise to two distinct sectors of worldsheet fermions. It is convenient to consider the worldsheet bosons $X^\mu = X^\mu(\sigma, \tau)$ as a smooth function on some domain of $\mathbb{R}^2$, subject to the periodicity condition

(23) \[ X^\mu(\sigma, \tau) = X^\mu(\sigma + 1, \tau). \]

Likewise, it is convenient to consider $\psi^\mu_m = \psi^\mu_m(\sigma, \tau)$ as a smooth Grassmann function on some domain of $\mathbb{R}^2$, subject to the periodicity condition

(24) \[ \psi^\mu_m(\sigma, \tau) = \pm \psi^\mu_m(\sigma + 1, \tau), \]
with the precise sign depending on the fermion sector. Specifically, in the Ramond (R) sector,
\begin{equation}
\psi^\mu_m(\sigma, \tau) = +\psi^\mu_m(\sigma + 1, \tau),
\end{equation}
and in the Neveu-Schwarz (NS) sector
\begin{equation}
\psi^\mu_m(\sigma, \tau) = -\psi^\mu_m(\sigma + 1, \tau).
\end{equation}
It should be noted that this anti-periodicity is an artifact of attempting to describe the Neveu-Schwarz bundle using a single coordinate chart. The actual spinor, as a section of the given bundle, is single valued, and so this sign has nothing to do with projectivization.

2.1.2. Background and Auxiliary Fields. In addition to the worldsheet fields $X$ and $\Psi$, we consider background spacetime fields. These include a spacetime metric $G(x) \in \text{Sym}^2(T_x^* M)$, given in coordinates by $G_{\mu\nu} \, dx^\mu \, dx^\nu$, a scalar $\Phi \in C^\infty(M)$, a whole host of gauge fields, and fermions. The particular fields considered are those arising from string states relevant to the low-energy effective action, which are precisely the massless states. Each spacetime field configuration is a section over some spacetime bundle. Given an embedding $X : \Sigma \to M$, we can pull back these spacetime bundles over $M$ to worldsheet bundles over $\Sigma$. The spacetime sections are then pulled back to worldsheet sections, the spacetime fields are pulled back to worldsheet fields. For example, the spacetime metric induces a worldsheet metric $g = X^*(G)$, given explicitly in coordinates by
\begin{equation}
g_{\alpha\beta} = G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta},
\end{equation}
and similarly for the other fields. It is convenient to forget this relation between $g$ and $G$, and allow $g$ to be an auxiliary field in the action. We denote the inverse of $g_{\alpha\beta}$ as $g^{\alpha\beta}$, and use these to raise and lower worldsheet indices. We associate with $g_{\alpha\beta}$ the frame $e^\alpha_a$, where $e_a \in T \Sigma$ and $a \in \{0, 1\}$ is a worldsheet frame index. This satisfies $g^{\alpha\beta} = e^\alpha_a e^\beta_b g_{\alpha\beta}$, and $\eta_{ab} = e^\alpha_a e^\beta_b g_{\alpha\beta}$, where $\eta_{ab}$ is the canonical Minkowski metric
\begin{equation}
\eta = \begin{pmatrix}
\eta_{00} & \eta_{01} \\
\eta_{10} & \eta_{11}
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & +1
\end{pmatrix}.
\end{equation}
The frame is not canonically defined, since we can modify a frame by a local Lorentz transformation to get another frame. Consequently, we will consider the frame to be an auxiliary field subject to the constraint
\begin{equation}
\eta_{ab} = e^\alpha_a e^\beta_b g_{\alpha\beta},
\end{equation}
whose gauge symmetries can be used to eliminate its degrees of freedom. Additionally, we consider an auxiliary Rarita-Schwinger field $\chi$, that is a gravitino, $\chi_a = \chi_m^a = (\chi_-^a, \chi_+^a)^T$, where $m \in \{-, +\}$ is a worldsheet spinor index and $a \in \{0, 1\}$ is a worldsheet frame index. We require that $\chi_m^a$ and $\psi^m$ derive from the same spin structure.

Our action is a functional $S[X, \Psi; g, \chi, e] = S$ of $X$ and $\Psi$ and the auxiliary fields $g$, $\chi$, and $e$, which depends implicitly on background fields, and we have imposed additional constraints. In general, the background fields are rather complicated, as are their interactions with the propagating string.

2.1.3. Without Background Fields. We begin when $M = \mathbb{R}^{1,D-1}$ is Minkowski space, with $G_{\mu\nu} = \eta_{\mu\nu}$, the usual flat metric, and with no other background fields present. Then, the action $S = S[X, \Psi; g, \chi, e]$ is defined by

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha X \partial_\beta X + \Psi \rho^a e_a^\alpha \nabla_\alpha \Psi - \Psi \rho^a \chi_a e_a^\alpha \partial_\alpha X - \frac{1}{4} \Psi \rho^a \rho^b \chi_a \chi_b \Psi \right].$$

There are four types of index which appear explicitly or implicitly in this action: (i) worldsheet vector indices $\alpha, \beta, \ldots \in \{0, 1\}$, (ii) worldsheet frame indices $a, b, \ldots \in \{0, 1\}$, (iii) worldsheet spinor indices $m, n, \ldots \in \{-, +\}$, (iv) spacetime vector indices $\mu, \nu, \ldots \in \{0, \ldots, D - 1\}$. The sums over spacetime vector indices are left implicit, and the sums over worldsheet spinor indices are written using matrix multiplication. The matrices $\rho^a$ are a real representation of the worldsheet frame Clifford algebra

$$\{\rho^a, \rho^b\} = 2\eta^{ab}.$$

To be explicit, we may define

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}.$$

We have defined conjugate spinors $\Psi^\mu = \Psi^T \rho^0$ and $\chi_a = \chi_a^T \rho^0$ and the covariant derivative $\nabla_\alpha$. Finally, the dimensional parameter $\alpha' = \ell_S^2$ is the Regge slope parameter.

From the action, we can compute the stress-energy tensor and supercurrent

$$T_{\gamma\delta}(\sigma, \tau) = \left[ \frac{1}{2} \partial_\alpha X \partial_\beta X + \frac{1}{4} \Psi \rho^a \partial_\beta \Psi + \frac{1}{4} \Psi \rho^a \partial_\alpha \Psi \right] \left[ \eta^\alpha_\gamma \eta^\beta_\delta - \frac{1}{2} \eta^\alpha_\delta \eta^\beta_\gamma \right]$$

and

$$J_\alpha^\pm = -\frac{1}{2} (\rho^a \rho^\alpha \Psi)_\pm \partial_\beta X.$$

**Proposition 2.1.** The action $S[X, \Psi; g, \chi, e]$ is invariant under the groups $\text{Iso}(M)$, $\text{Diff}(\Sigma)$, $\text{Weyl}(\Sigma)$, $\text{Lorentz}(\Sigma)$, $\text{SuperWeyl}(\Sigma)$, $\text{SuperSym}(\Sigma)$. 
Here

(a) $\text{Iso}(M)$ is the group of spacetime isometries (acting on spacetime vector indices). Their action on the fields and coordinates is of the form

$$x \rightarrow f(x), \quad X \rightarrow f(X), \quad \Psi^\mu \rightarrow \partial_\nu f^\mu(X)\Psi^\nu$$

for isometries $f : M \rightarrow M$.

(b) $\text{Diff}(\Sigma)$ is the group of worldsheet diffeomorphisms (acting on worldsheet vector indices). Their action on the fields and coordinates is of the form

$$\sigma \rightarrow f(\sigma), \quad g_{\alpha\beta} \rightarrow (\partial_{\alpha} f^\gamma)(\partial_{\beta} f^\delta)g_{\gamma\delta}, \quad e^a_\alpha \rightarrow (\partial_{\beta} f^\alpha)e^\beta_\alpha$$

for diffeomorphisms $f : \Sigma \rightarrow \Sigma$, where $\Psi$ and $\chi$ remain unchanged.

(c) $\text{Weyl}(\Sigma)$ is the group of worldsheet Weyl transformations. Their action on fields is of the form

$$g_{\alpha\beta} \rightarrow e^{2f}g_{\alpha\beta}, \quad \Psi \rightarrow e^{-f/2}\Psi, \quad \chi_\alpha \rightarrow e^{f/2}\chi_\alpha, \quad e^a_\alpha \rightarrow e^{-f}e^a_\alpha$$

for smooth functions $f : \Sigma \rightarrow \mathbb{R}$.

(d) $\text{Lorentz}(\Sigma)$ is the group of local worldsheet frame transformations (acting on worldsheet frame indices and worldsheet spinor indices). Their action on fields is of the form

$$e^a_\alpha \rightarrow \Lambda^b_\alpha e^a_\beta, \quad \Psi^\mu \rightarrow \Lambda_S^b \Psi^\mu, \quad \chi_a \rightarrow \Lambda^b_a \Lambda S \chi_b,$$

where

$$\Lambda = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \quad \text{and} \quad \Lambda_S = \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}$$

for rapidity $\theta(\sigma) : \Sigma \rightarrow \mathbb{R}$.

(e) $\text{SuperWeyl}(\Sigma)$ is the group of worldsheet super-Weyl transformations ($^2$). These only alter the worldsheet gravitino, and their action is of the form

$$\chi^m_a \rightarrow \chi_a + \eta_{ab}\rho^b \chi^m_a,$$

for an arbitrary spinor $\lambda^m$ with the same spin structure as $\psi^\mu_m$ and $\chi^m_a$.

(f) $\text{SuperSym}(\Sigma)$ is the ($\mathcal{N} = 1$) group of supersymmetries.

In light of these symmetries, the action $S$ defines a super-conformal theory. We can use the $\text{Diff}(\Sigma) \times \text{Weyl}(\Sigma)$ symmetry to choose a gauge for the worldsheet metric $g$. Ideally, we could choose the flat worldsheet gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$. This is easily done locally, but there is a global topological obstruction, captured by the Gauss-Bonnet theorem. Analogously, we can

$^2$This tad strange symmetry is a consequence of the identity $\eta_{\alpha\rho} \rho^a \rho^b \rho^c = 0$. 
use the SuperWeyl(Σ) symmetry and supersymmetry to choose a gauge for the worldsheet gravitino \( \chi \). Ideally, we could choose \( \chi = 0 \) identically. This is easily done locally, but there is a global topological obstruction. When \( \Sigma \) is a cylinder, these topological obstructions vanish, and we may choose the flat worldsheet (conformal) gauge

\[
g_{\alpha\beta} = \eta_{\alpha\beta} \quad \text{and} \quad \chi = 0.
\]

After we choose this gauge, the equations of motion for \( g \) and \( \chi \) become constraints on the other fields, specifically the vanishing of the worldsheet stress-energy tensor and supercurrents:

\[
T_{\gamma\delta}(\sigma, \tau) = \left[ \frac{1}{2} \partial_\alpha X \partial_\beta X + \frac{1}{4} \overline{\Psi} \rho_\alpha \partial_\beta \Psi + \frac{1}{4} \overline{\Psi} \rho_\beta \partial_\alpha \Psi \right] \left[ \eta^\gamma_\alpha \eta^\delta_\beta - \frac{1}{2} \eta^\gamma_\beta \eta^\alpha_\delta \right] = 0
\]

and

\[
J^\alpha_\pm = -\frac{1}{2} (\rho^\beta \rho^\alpha \Psi) \pm \partial_\beta X = 0.
\]

The vanishing of the supercurrents is equivalent to

\[
\psi_+ \partial_- X = 0 = \psi_- \partial_+ X
\]

while the vanishing of the stress-energy tensor is equivalent to

\[
T_{++} = \frac{1}{2} \partial_+ X \partial_+ X + \frac{1}{2} \psi_+ \partial_+ \psi_+ = 0 = \frac{1}{2} \partial_- X \partial_- X + \frac{1}{2} \psi_- \partial_- \psi_- = T_{--}.
\]

We have used worldsheet lightcone coordinates \( \sigma^\pm = (\sigma^0 \pm \sigma^1)/2 \). The gauge-fixed action \( S[X, \Psi] = S \) is given by \( S = S_{\text{bos}} + S_{\text{fer}} \) where \( S_{\text{bos}} \) is the bosonic action and \( S_{\text{fer}} \) is the fermionic action. The bosonic part of the action is given by

\[
S_{\text{bos}} = -\frac{1}{8 \pi \alpha'} \int d^2\sigma (\partial_\alpha X \partial^\alpha X) = + \frac{1}{2 \pi \alpha'} \int d^2\sigma (\partial_- X \partial_+ X)
\]

while the fermionic part of the action is given by

\[
S_{\text{fer}} = -\frac{1}{4 \pi \alpha'} \int d^2\sigma \left( \overline{\Psi} \rho^\alpha \partial_\alpha \Psi \right) = + \frac{1}{2 \pi \alpha'} \int d^2\sigma \left( \psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ \right).
\]

Varying the action, we find the equations of motion

\[
\partial_- \partial_+ X^\mu = 0,
\]

\[
\partial_- \psi^\mu_+ = 0,
\]

\[
\partial_+ \psi^\mu_- = 0.
\]

By the equations of motion for \( X^\mu \), we may write \( X^\mu \) as \( X^\mu(\tau, \sigma) = X^\mu_+(\tau + \sigma) + X^\mu_-(\tau - \sigma) \), where \( X^\mu_+(\sigma_\pm) \) are real (and smooth) but do not necessarily satisfy any periodicity
requirement regarding $\sigma$. However, $X'_\pm (\sigma) = \pm \partial_\pm X (\tau, \sigma)$ is periodic in $\sigma$, and consequently in $\sigma_\pm$. We can then expand $X'_\pm (\sigma)\pm$ in Fourier series. Integrating, our original fields can be written as

\begin{align}
X^\mu_+ (\tau, \sigma) &= \frac{x^\mu}{2} + \ell^2 S^\mu (\tau + \sigma) + i \frac{\ell S}{\sqrt{2}} \sum_{n \neq 0} \frac{x^\mu_n e^{-2\pi in(\tau + \sigma)}}{n} \\
X^\mu_- (\tau, \sigma) &= \frac{x^\mu}{2} + \ell^2 S^\mu (\tau - \sigma) + i \frac{\ell S}{\sqrt{2}} \sum_{n \neq 0} \frac{\tilde{x}^\mu_n e^{-2\pi in(\tau - \sigma)}}{n}
\end{align}

where $x^\mu, p^\mu \in \mathbb{R}^{1, D-1}$ and $x^\mu_n, \tilde{x}^\mu_n$ are arbitrary complex numbers, subject to the reality constraints $x^\mu_n = (x^\mu_{-n})^*$ and $\tilde{x}^\mu_n = (\tilde{x}^\mu_{-n})^*$. The particular normalizations chosen above are chosen for convenience. We'll take $\ell S = 1$. By the equations of motion for $\Psi^\mu$, we may write

\begin{align}
\psi^\mu_+ (\tau, \sigma) &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \psi^\mu_n e^{-2\pi in(\tau + \sigma)} \\
\psi^\mu_- (\tau, \sigma) &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{\psi}^\mu_n e^{-2\pi in(\tau - \sigma)}
\end{align}

while in the Neveu-Schwarz sector

\begin{align}
\psi^\mu_+ (\tau, \sigma) &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \psi^\mu_n e^{-2\pi in(n+1/2)(\tau + \sigma)} \\
\psi^\mu_- (\tau, \sigma) &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{\psi}^\mu_n e^{-2\pi in(n+1/2)(\tau - \sigma)}
\end{align}

where $\psi^\mu_n$ and $\tilde{\psi}^\mu_n$ are arbitrary complex Grassmann numbers, subject to the reality (Majorana) constraints $\psi^\mu_n = (\psi^\mu_{-n})^*$ and $\tilde{\psi}^\mu_n = (\tilde{\psi}^\mu_{-n})^*$.

In worldsheet lightcone coordinates, the diagonal entries of the stress energy tensor are given by

\begin{align}
T_{++}(\tau + \sigma) &= \frac{1}{2} \partial_+ X_+ \partial_+ X_+ + \frac{1}{2} \psi_+ \partial_+ \psi_+ \\
T_{--}(\tau - \sigma) &= \frac{1}{2} \partial_- X_- \partial_- X_- + \frac{1}{2} \psi_- \partial_- \psi_-
\end{align}
While these are zero classically, these will be nonzero quantume mechanically. Even without taking this into account, we can take a Fourier expansion
\[ T_{++}(\tau + \sigma) = \sum_{n \in \mathbb{Z}} L_n e^{-2\pi in(\tau + \sigma)} \]
\[ T_{--}(\tau - \sigma) = \sum_{n \in \mathbb{Z}} \tilde{L}_n e^{-2\pi in(\tau - \sigma)} \]

2.1.4. With Background Fields. Let’s discuss what changes in the preceding analysis if we turn on background fields and allow the spacetime manifold to be arbitrary. We can assume that the background fields are solutions to their respective equations of motion, which are part of the relevant supergravity theory. Whatever action we choose, we require that it possesses the analogous symmetries to the case without background fields, those listed in Proposition 2.1. Then, we can use $\text{Diff}(\Sigma) \ltimes Weyl(\Sigma)$ symmetry and the analogous supersymmetries to choose the flat worldsheet gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $\chi_\alpha = 0$.

The bosonic part of the action $S_{\text{bos}}[X]$, which we write as a sum of terms $S_{\text{bos}} = S_G + S_B + S_\Phi$ representing the coupling of the string to the gravitational, two-form, and scalar fields respectively. We might additionally attempt to add a cosmological constant term
\[ S_\Lambda[X] \propto \int d^2 \sigma \sqrt{-g}, \]
but this is not allowed by the equations of motion.

The natural choice for $S_\Phi$ is
\[ S_\Phi[X] \propto \int d^2 \sigma \sqrt{-g} R(2) \Phi \]
where $R(2)$ is the Ricci scalar curvature constructed from the worldsheet metric $g_{\alpha\beta}$. In order to enforce conformal symmetry, we assume that $\Phi$ is constant. Then, by the Gauss-Bonnet theorem, $S_\Phi[X]$ is purely topological: $S_\Phi[X] \propto \chi(\Sigma)$. Since we are only considering a single worldsheet topology, we may neglect this term, and therefore assume $S_\Phi[X] = 0$. Similarly, the natural choice for $S_B[X]$ is
\[ S_B[X] \propto \int d^2 \sigma \sqrt{-g} B_{\mu\nu}(X) e^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} = \int_{\Sigma} X^*(B). \]
In order to enforce conformal symmetry, we assume that $B$, and therefore $X^*(B)$, is closed. Then, by Stokes’s theorem, $S_B[X]$ is purely topological and can be neglected. Finally, the natural choice for $S_G[X]$ is
\[ S_G[X] \propto \int d^2 \sigma \sqrt{-g} g^{\alpha\beta} G_{\mu\nu}(X) \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}, \]
which is simply the Polyakov action in nontrivial background. A similar analysis allows us to restrict the form of the fermionic action $S_{\text{fer}}[\Psi]$.

2.2. Quantization.

2.2.1. Outline. In this section we will quantize the superstring using the fairly direct method of canonical quantization. In order to quantize the superstring, we need to construct an appropriate Hilbert space of states, $\mathcal{H}_{\text{closed}}^{RNS} = \mathcal{H}_{\text{closed}}^{RNS}(M)$. For simplicity, we assume that $M = \mathbb{R}^{1,D-1}$ is $D$-dimensional Minkowski space. The general case, when $M$ is an arbitrary manifold, can be handled using BRST quantization. The Fourier modes $x^\mu_n, \tilde{x}^\mu_n, \psi^\mu_n, \tilde{\psi}^\mu_n, \psi^\mu_r, \tilde{\psi}^\mu_r$ in the classical theory are promoted to operators in the quantum theory, acting on $\mathcal{H}_{\text{closed}}^{RNS}$ and the Fock spaces involved in its construction.

For each momentum $p \in \mathbb{R}^{1,D-1}$, we will construct a Fock space $F_p$ capturing the left-moving bosonic excitations of the string and Fock spaces $F^R$ and $F^{NS}$ capturing the left-moving fermionic excitations of the string in the Ramond and Neveu-Schwarz sectors respectively. Analogously, we will construct Fock spaces $\tilde{F}_p, \tilde{F}^R, \tilde{F}^{NS}$ capturing the right-moving excitations of the string. These spaces do not immediately yield a physical theory, as they contain states of negative and zero norm. We will restrict ourselves to smaller Fock spaces $F_{p,\text{phys}} \subset F_p \otimes F^R, F^{NS}_{p,\text{phys}} \subset F_p \otimes F^{NS}$ in order to eliminate the negative norm states. We will then quotient out by the subspaces $F_{p,\text{null}} \subset F_{p,\text{phys}}, F^{NS}_{p,\text{null}} \subset F^{NS}_{p,\text{phys}}, \tilde{F}_{p,\text{null}} \subset \tilde{F}_{p,\text{phys}}$ of zero norm vectors to get the quotient spaces

$$F^R_{p,+,\text{phys}} = F^R_{p,\text{phys}}/F^R_{p,\text{null}}, \quad F^{NS}_{p,+,\text{phys}} = F^{NS}_{p,\text{phys}}/F^{NS}_{p,\text{null}},$$

These spaces will have positive norm and can be considered as Hilbert spaces. We can then combine these spaces

$$F^{RNS}_{p,+,\text{phys}} = F^R_{p,+,\text{phys}} \oplus F^{NS}_{p,+,\text{phys}}$$

We can then combine these spaces to get a Hilbert space

$$(59) \quad \mathcal{H}_{\text{closed}}^{RNS} = \bigoplus_{p \in \mathbb{R}^D} F^{RNS}_{p,+,\text{phys}} \otimes \tilde{F}^{RNS}_{p,+,\text{phys}}.$$
The success of this procedure depends critically on the details of the quantization, the selection of the physical subspaces, and the overall spacetime dimension. In fact, this will only be possible if \( D = 10 \).

2.2.2. Operator Definitions. We upgrade the Fourier coefficients of our worldsheet fields to operators with the commutation and anti-commutation relations

\[
[x_\mu^n, x_\nu^n] = m\eta^{\mu\nu}\delta_{m+n,0},
\]

\[
\{\psi_\mu^m, \psi_\nu^n\} = \eta^{\mu\nu}\delta_{m+n,0},
\]

To quantize the superstring, the modes of the stress-energy tensor and supercurrents are given in the Ramond sector by \( L_m - a_R\delta_{m,0} \) and \( G_m \) respectively, where

\[
L_0 = \frac{x_0^2}{2} + \sum_{n \in \mathbb{Z}^+} (x_{-n} \cdot x_n + n\psi_{-n} \cdot \psi_n)
\]

\[
L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} (x_{m-n} \cdot x_n + n\psi_{m-n} \cdot \psi_n)
\]

\[
G_m = \sum_{n \in \mathbb{Z}} x_{-n} \cdot \psi_{n+m}.
\]

Here \( \mathbb{Z}^+ \) is the set of positive integers. The constant \( a_R \) is currently undetermined, and reflects the normal ordering ambiguity that arises upon quantizing the theory. The elimination of negative norm states from the spectrum requires a specific value of \( a_R \). The modes of the stress-energy tensor and supercurrents are given in the Neveu-Schwarz sector by \( L_m - a_{NS}\delta_{m,0} \) and \( G_m \) respectively, where

\[
L_0 = \frac{p_0^2}{2} + \sum_{n \in \mathbb{Z}^+} (x_{-n} \cdot x_n + (n + 1/2)\psi_{-n-1/2} \cdot \psi_{n+1/2})
\]

\[
L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} (x_{m-n} \cdot x_n + (n + 1/2)\psi_{m-n-1/2} \cdot \psi_{n+1/2})
\]

\[
G_m = \sum_{n \in \mathbb{Z}} x_{-n} \cdot \psi_{m+n+1/2}
\]

where \( a_{NS} \) is another currently undetermined constant. In these expressions the sums over spacetime vector indices are implicit. These satisfy the \( (\mathcal{N} = 1) \) super-Virasoro algebra with central charge \( c = D \).
2.2.3. Fock Space Construction. We now operationally define the Fock spaces involved in the construction of $\mathcal{H}_{\text{RNS}}^{\text{closed}}$. We give the construction explicitly for the left-moving Fock spaces only, as the constructions for the right-moving Fock spaces are analogous.

The Fock spaces $\mathcal{F}_R$ and $\mathcal{F}_{NS}$ contain fermionic excitations of the superstring with arbitrary momentum, while $\mathcal{F}_p$ contains bosonic excitations of the superstring with momentum $p$. The space $\mathcal{F}_p$ is filled out by acting on the (unique) ground state $|p\rangle \in \mathcal{F}_p$ with bosonic creation operators. The (unique) ground state $|0\rangle_{\text{NS}} \in \mathcal{F}_{\text{NS}}$ is characterized by

$$\psi_{n+1/2}^\mu |0\rangle_{\text{NS}} \text{ for all } n \in \{0, 1, 2, \ldots\}$$

This is a spacetime scalar. The Neveu-Schwarz Fock space $\mathcal{F}_{\text{NS}}$ is obtained by applying the creation operators $\psi_n^\mu$ for $n \in \{-1, -2, \ldots\}$ to $|0\rangle_{\text{NS}}$. The states created in this manner are spacetime bosons. The ground state of $\mathcal{F}_R$ is degenerate. The source of the degeneracy is the fact that $\psi_0^\mu$ satisfies the spacetime Clifford algebra

$$(\psi_0^\mu, \psi_0^\nu) = \eta^{\mu\nu}.$$ 

It follows that the subspace of Ramond ground states furnishes a representation of $\text{Spin}(1, D - 1)$, and specifically constitutes a Dirac spinor. We can label a basis of ground states $|0, A\rangle_R \in \mathcal{F}_p \otimes \mathcal{F}_R$ using a spacetime spinor index $A \in \{1, \ldots, 2^{\lfloor D/2 \rfloor}\}$. The subspace of ground states is characterized by

$$\psi_n^\mu |0, A\rangle_R = 0 \text{ for all } n \in \{1, 2, \ldots\}.$$ 

The Ramond Fock space $\mathcal{F}_R$ is obtained by applying the creation operators $\psi_n^\mu$ for $n \in \{-1, -2, \ldots\}$ to $|0, A\rangle_R$. The states created in this manner are all spacetime fermions. We label the ground states of $\mathcal{F}_p \otimes \mathcal{F}_{\text{NS}}$ and $\mathcal{F}_p \otimes \mathcal{F}_R$ as $|0, p\rangle_{\text{NS}}$ and $|0, p\rangle_R$ respectively.

We define the physical subspace $\mathcal{F}_p^{\text{phys}}$ in the Ramond sector to contain exactly those states $|\psi, p\rangle_R \in \mathcal{F}_p \otimes \mathcal{F}_R$ which satisfy

$$(L_0 - a_R) |\psi, p\rangle_R = L_n |\psi, p\rangle_R = G_m |\psi, p\rangle_R = 0 \text{ for all } m \in \mathbb{N} \text{ and } n \in \mathbb{Z}^+.$$ 

The relation $G_0^2 = L_0$ implies that, in order for $\mathcal{F}_p^{\text{phys}}$ to be nontrivial,

$$a_R = 0.$$ 

Similarly, we define the physical subspace $\mathcal{F}_p^{\text{phys}}$ in the Neveu-Schwarz sector to contain exactly those states $|\psi, p\rangle_{\text{NS}} \in \mathcal{F}_p \otimes \mathcal{F}_{\text{NS}}$ which satisfy

$$(L_0 - a_{\text{NS}}) |\psi, p\rangle_{\text{NS}} = L_n |\psi, p\rangle_{\text{NS}} = G_{m+1/2} |\psi, p\rangle_{\text{NS}} = 0 \text{ for all } m \in \mathbb{N} \text{ and } n \in \mathbb{Z}^+.$$
This requires
\begin{equation}
 a_{NS} = 1/2.
\end{equation}

If \(|\phi, p\rangle\) and \(|\psi, p\rangle\) are two physical states, both elements of either \(F_{p,phys}^R\) or \(F_{p,phys}^{NS}\), then
\begin{equation}
 \langle \phi, p| T| \psi, p \rangle = 0 = \langle \phi, p| G| \psi, p \rangle.
\end{equation}

This is the quantum version of the classical vanishing of the stress-energy tensor and supercurrent. Classically, this is a consequence of the equations of motion of the worldsheet metric and gravitino. Quantum mechanically, this is a constraint built into the definition of our Hilbert space.

The fact that this rigmarole works is the No Ghost Theorem.

**Theorem 2.2 (No Ghost Theorem).** The spaces \(F_{p,phys}^{NS}\) and \(F_{p,phys}^R\) have non-negative norm if \(a_R = 0, a_{NS} = 1/2,\) and \(D = 10\). The elements of the quotient spaces \(F_{p,+}^{NS}\) and \(F_{p,+}^R\) are in one-to-one correspondence with the states in \(F_p \otimes F^R\) and \(F_p \otimes F^{NS}\) formed by acting on the Ramond sector and Neveu-Schwarz sector ground states with the transverse operators \(x_{-n}^i\) and \(\psi^i_{-r}\) for \(n, r > 0\) and \(i \in \{2, \cdots, D\}\). After passing to the quotient spaces \(F_{p,+}^R\) and \(F_{p,+}^{NS}\), these states lie in distinct nontrivial equivalence classes, and therefore constitute canonical representatives of their respective classes.

So, retroactively, set
\begin{equation}
 a_R = 0, \quad a_{NS} = 1/2, \quad D = 10
\end{equation}

and then things make sense.

2.3. **GSO Projection.** The Hilbert space \(H_{\text{closed}}^{RNS}\) is still problematic, as it contains tachyons. The solution, following Green, Schwarz, and Olive, is to project out half of our degrees of freedom based on fermion number. This is known as GSO projection, after its creators, and results in a spectrum with spacetime supersymmetry.

On the space \(F_{p,+}^{NS}\) we define the fermion number operator \(F\) by
\begin{equation}
 F = \sum_{n=1}^{\infty} \psi^{-(n+1/2)} \cdot \psi^{+(n+1/2)}
\end{equation}

and the GSO projection operator \(P_{NS}^{GSO}\) by
\begin{equation}
 P_{NS}^{GSO} = \frac{1}{2} (1 - (-1)^F).
\end{equation}
On the space $F_{p^+}$ we define the fermion number operator $F$ by

$$F = \sum_{n=1}^{\infty} \psi_{-n} \cdot \psi_{+n}$$

and the GSO projection operator $P_R^{GSO}(\pm)$ by

$$P_R^{GSO}(\pm) = \frac{1}{2} \left( 1 \pm \frac{1}{32} \cdot \psi_0^0 \cdot \psi_0^0 \cdot (-1)^F \right),$$

where the $\pm$ denotes a choice in sign. The operator $\psi_0^0 \cdots \psi_0^0$ is proportional to the chirality operator in the spacetime Dirac algebra. Then, the projected Fock space is

$$F^{GSO}_{p^+}(\pm) = P_{NS}^{GSO} F^{NS}_{p^+} \oplus P_R^{GSO} F^{R}_{p^+}.$$ 

The right-moving Fock spaces are combined analogously. The GSO Hilbert spaces for the closed and open theories, $\mathcal{H}^{GSO}_{closed}(\pm_1, \pm_2)$ and $\mathcal{H}^{GSO}_{open}(\pm_1, \pm_2)$, are then defined by

$$\mathcal{H}^{GSO}_{closed}(\pm_1, \pm_2) = \bigoplus_{p \in \mathbb{R}^D} F^{GSO}_{p}(\pm_1) \otimes \bar{F}^{GSO}_{p}(\pm_2)$$

$$\mathcal{H}^{GSO}_{open}(\pm) = \bigoplus_{p \in \mathbb{R}^D} F^{GSO}_{p}(\pm_1)$$

This is the Hilbert space of the Type II superstring. The choices of signs $\pm_1$ and $\pm_2$ yields two distinct theories. The theory where they disagree is known as Type IIA string theory:

$$\mathcal{H}^{Type\ IIA} = \mathcal{H}^{GSO}_{closed}(+, -).$$

The theory where they agree is known as Type IIB string theory:

$$\mathcal{H}^{Type\ IIB} = \mathcal{H}^{GSO}_{closed}(+, +).$$

2.3.1. Massless Content. As usual, we break up $F^{GSO}_{p}(\pm_1) \otimes \bar{F}^{GSO}_{p}(\pm_2)$ in the NS-NS, R-R, NS-R, and R-NS sectors. Let’s analyze the ground states in the NS-NS and R-R sectors, which are bosons. The GSO projection eliminates tachyons from the spectrum, and the ground states are all massless. Let $\mathbf{8}_v$ be the fundamental representation of $SO(8)$, and $\mathbf{8}_\pm$ the two eight-dimensional irreducible representations.

- **NS-NS:** A basis of ground states is given by

$$\psi^{-1/2}_i \bar{\psi}^{-1/2}_j |0, p\rangle_{NS-NS}.$$ 

The space of ground states therefore consists of states of the form

$$a_{ij} \psi^{-1/2}_i \bar{\psi}^{-1/2}_j |0, p\rangle_{NS-NS}$$
for arbitrary $a_{ij}$. This space therefore constitutes a tensor representation of $\text{SO}(8)$, specifically $8_v \otimes 8_v$, which decomposes as a direct sum of symmetric traceless, antisymmetric, and trivial representations:

$$8_v \otimes 8_v = S \oplus A \oplus 1.$$

This decomposition corresponds to the decomposition of the arbitrary $8 \times 8$ matrix $a = (a_{ij})$ as

$$a_{ij} = \left[ \frac{a_{ij} + a_{ji}}{2} - \frac{\text{Tr}(a)}{8} \right] + \left[ \frac{a_{ij} - a_{ji}}{2} \right] + \left[ \frac{\text{Tr}(a)}{8} \right].$$

Denoting these tensors as $g_{ij}, B_{ij}, \Phi$, respectively,

$$a = [g] + [B] + [\Phi].$$

The tensor $g_{ij}$ corresponds to the graviton, and the scalar $\Phi$ corresponds to the dilaton.

- R-R: The space of R-R ground states after GSO projection is a tensor product of spinor representations. In Type IIA, these spinors have opposite chirality, whereas in Type IIB they have the same chirality. In Type IIA, this corresponds to the decomposition

$$(88) \quad 8_+ \otimes 8_- = [1] \oplus [3],$$

where $[1]$ and $[3]$ are the irreducible representations of $\text{SO}(8)$ of one-forms and three-forms. In Type IIB, this corresponds to the decomposition

$$(89) \quad 8_+ \otimes 8_+ = [0] + [2] + [4]^+,$$

where $[0], [2]$ are the irreducible representations of $\text{SO}(8)$ consisting of functions and two-forms, while $[4]^+$ is the irreducible representation of $\text{SO}(8)$ consisting of self-dual four-forms.

The massless fermionic spectrum, corresponding to the R-NS and NS-R sectors, can be derived in a similar way, but for our purposes the bosonic spectrum suffices.
3. A Bit of Supergravity

Type II string theory yields a spacetime field theory whose low-energy effective action describes classical supergravity. The classical fields in this supergravity theory correspond to the massless modes of the underlying strings. Since there are two types of Type II string theory to consider, there are two types of supergravity theory to consider, Type IIA supergravity and Type IIB supergravity, the former of which is the low-energy effective field theory of Type IIA string theory and the latter of which is the low-energy effective field theory of Type IIB string theory.

3.1. 10D Supergravity. Let $S^{(IIA)}$ be the full action of Type IIA supergravity and $S^{(IIB)}$ be the full action of Type IIB supergravity in ten dimensions. By supersymmetry, the actions $S^{(IIA)}$ and $S^{(IIB)}$ are determined by their bosonic components. Consequently, the full actions are often identified with their bosonic components, and we will do so here. It is also necessary to distinguish the Einstein frame from the string frame, which differ with respect to volume factors which pop up. although we will be sloppy at doing so.

Definition 3.1. The fields in Type IIA supergravity in ten dimensions consist of the metric $G$, the dilaton $\Phi$, the NS two-form $B$, the RR one-form $C^{(1)}$ and three form $C^{(3)}$. The bosonic action of Type IIA supergravity $S^{(IIA)}$ is given in the string frame by

$$S^{(IIA)} = S_{NS} + S^{(IIA)}_R + S^{(IIA)}_{CS}$$

where

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left( R + 4(\partial_m \Phi)(\partial^n \Phi) - \frac{1}{2} |H^{(3)}|^2 \right)$$

$$S^{(IIA)}_R = -\frac{1}{4\kappa^2} \int d^{10}x \left( |F^{(2)}|^2 + |\tilde{F}^{(4)}|^2 \right)$$

$$S^{(IIA)}_{CS} = -\frac{1}{4\kappa^2} \int B^{(2)} \wedge F^{(4)} \wedge F^{(4)}.$$ 

Here $H^{(3)} = dB$ and $\tilde{F}^{(4)} = dC^{(3)} + C^{(1)} \wedge H^{(3)}$ is the field strength of the three-form modified by a Chern-Simons term and $\kappa$ is a constant of proportionality. These formulas are (Eq. 1.2.3) in [17].

This action can be determined by dimensionally reducing the unique action for 11D supergravity on a circle. The RR one-form is then seen to be a Kaluza-Klein gauge field, while the $B$-field is the three-form reduced on the circle. Type IIB supergravity, on the other hand, cannot be obtained by dimensional reduction. This is due to the self-duality of
the five-form field strength, which must be imposed as an additional constraint beyond the equations of motion derived from the action.

**Definition 3.2.** The fields in Type IIB supergravity consist of the metric $G$, the dilaton $\Phi$, the NS two-form $B$, the RR zero-form $C^{(0)}$, the RR two-form $C^{(2)}$, and the self-dual RR four-form $C^{(4+)}$. The bosonic action of type IIB supergravity $S^{(IIB)}$ is given in the string frame by

$$S^{(IIB)} = S_{NS} + S^{(IIB)}_R + S^{(IIB)}_{CS}$$

where $S_{NS}$ is as before and

$$S^{(IIB)}_R = -\frac{1}{4\kappa^2} \int d^{10}x \left( |F^{(1)}|^2 + |\tilde{F}^{(3)}|^2 + \frac{1}{2} |\tilde{F}^{(5)}|^2 \right)$$

$$S^{(IIB)}_{CS} = -\frac{1}{4\kappa^2} \int A^{(4+)} \wedge H^{(3)} \wedge F^{(3)}.$$ 

Here $\tilde{F}^{(3)} = dC^{(2)} - C^{(0)} \wedge H^{(3)}$ and $\tilde{F}^{(5)} = dC^{(4+)} - (1/2)C^{(2)} \wedge H^{(3)} + (1/2)B^{(2)} \wedge dC^{(2)}$ are modified field strengths. The self-duality condition $\tilde{F}^{(5)} = \star \tilde{F}^{(5)}$ is imposed by hand. These formulas are (Eq. 1.2.5) in [17].

As stated before, these classical field theories are the leading order classical field theories associated with the full string theories on a ten-dimensional Lorentzian manifold $M_{10}$.

### 3.2. 4D Supergravity.

To construct four-dimensional theories, we consider the case where the full ten-dimensional (Ricci flat) spacetime $M_{10}$ is diffeomorphic to a product $M_{10} \cong M_4 \times M_6$, where $M_4$ is a four-dimensional manifold, usually Minkowski space or a black hole spacetime, and $M_6$ is a (compact, connected) Calabi-Yau threefold. When the volume of $M_6$ is small, we can ignore all but the lowest energy excitations of the fields along it. We can then perform **dimensional reduction** to obtain a four-dimensional field theory. A ten-dimensional field can be thought of as some tensorfield over $M_{10}$. We can naturally consider this as a section of the bundle constructed by纤维ing the space of sections of this bundle over $M_6$ over $M_4$. If we are only interested in the lowest excitations along $M_6$, this reduces to a finite dimensional vector bundle over $M_4$. We can naturally write this vector bundle as a direct sum of tensor bundles over $M_4$, each of which corresponds to a field of the four-dimensional theory and vice versa. The resultant four-dimensional field theory will also be a supergravity theory, where the number of supersymmetries depends on the holonomy of $M = M_6$. 


Figure 2. A decomposition of ten-dimensional spacetime $M = M_{(10)}$ as Minkowski space, $\mathbb{R}^{1,3}$, and a compact six-dimensional manifold $M_{(6)}$.

3.2.1. Relation between Supersymmetry and Holonomy. The number of spacetime supersymmetries possessed by the compactified theory is determined by the holonomy $\text{Hol}(M)$ of the compactification manifold $M$. Let $\pi : S \to M$ be a Weyl spin bundle on $M$. The number of linearly independent supersymmetries is determined by the number of components of linearly independent Weyl spinors $\eta \in \Gamma(\pi)$ on $M$ that are covariantly constant: $\nabla \eta = 0$. Given a covariantly constant spinor $\eta$, $\eta$ is determined uniquely by its value $\eta(p)$ at any point $p \in M$, since for any point $q \in M$, $\eta(q)$ is the parallel transport of $\eta(p)$ along any path connecting $p$ and $q$.

Conversely, given a (local) spinor $\eta_p \in \pi^{-1}(p)$ we can attempt to define a global spinor $\eta \in \Gamma(\pi)$ such that $\eta(p) = \eta_p$ and such that $\eta$ is covariantly constant. For each point $q \in M$, we define $\eta(q)$ by parallel transporting $\eta_p$ along some path connecting $p$ and $q$. This construction might be path dependent. In fact, the section constructed in this manner is smooth (and continuous) if and only if this construction is path independent. This is true if and only if given any loop $\gamma$ with basepoint $p$ the parallel transport of $\eta_p$ around $p$ is just $\eta_p$ itself. This is true if and only if the holonomy acts trivially on $\eta_p$. Consequently,
Proposition 3.3. Pick arbitrary $p \in M$. The vector space of covariantly constant spinors $\eta \in \Gamma(\pi)$ is isomorphic to the subspace of $\pi^{-1}(p)$ that is acted upon trivially by $\text{Hol}(M)$. An explicit isomorphism is given by $\eta \mapsto \eta(p)$. ■

If the dimension of this subspace is $N/2$, then the four-dimensional theory possess a $N$ dimensional space of supersymmetries. We just need to determine $N$. The fiber $\pi^{-1}(p)$ is four-dimensional, and is acted upon naturally by $\text{SU}(4) \cong \text{Spin}(6)$. That is, $\pi^{-1}(p) \cong 4$ as representations of $\text{SU}(4)$, where $4$ is the defining representation of $\text{SU}(4)$. The double cover of $\text{Hol}(M)$ is a subgroup of $\text{Spin}(6)$, and we identify it with the corresponding subgroup of $\text{SU}(4)$. Then, $4$ splits as a direct sum of irreducible representations of $\text{Hol}(M) \subset \text{SU}(4)$. We can therefore determine $N$ simply by examining this decomposition. There are three cases we are interested in: (i) $\text{Hol}(M) \cong \text{SU}(3)$, (ii) $\text{Hol}(M) \cong \text{SU}(2)$, (iii) $\text{Hol}(M) \cong 1$. In case (i), $4 \cong 3 \oplus 1$, where $3$ is the defining representation of $\text{SU}(3)$ and $1$ is the trivial representation. In case (ii), $4 \cong 2 \oplus 1 \oplus 1$, where $2$ is the defining representation of $\text{SU}(2)$ and $1$ is the trivial representation. In case (iii), $4 \cong 1 \oplus 1 \oplus 1 \oplus 1$, where $1$ is the trivial representation. So:

- If $\text{Hol}(M) \cong \text{SU}(3)$, then the theory possesses $N = 2$ supersymmetry.
- If $\text{Hol}(M) \cong \text{SU}(2)$, then the theory possesses half-maximal $N = 4$ supersymmetry.
- If $\text{Hol}(M) \cong 1$, then the theory possesses the maximal $N = 8$ supersymmetry.

We now briefly describe the resultant four-dimensional bosonic fields, besides the graviton, in terms of their ten-dimensional origin and the geometry or topology of $M$.

3.2.2. Scalar Fields. Four-dimensional Type II supergravity compactified on $M$ contains

$$n_s = 2b^0(M) + 2b^1(M) + b^2(M) + b^3(M) + N$$

independent scalars, where $N$ is the dimension of the moduli space of Einstein metrics on $M$. The ten-dimensional origin in type IIA supergravity of these scalars, in terms of the bosonic fields $\Phi, G, B, C^{(1)}, C^{(3)}$, is given in the following table:

<table>
<thead>
<tr>
<th>number</th>
<th>field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2b^0(M)$</td>
<td>$\Phi, (d_4 \star 4 d_4) B_{\mu \nu}$</td>
</tr>
<tr>
<td>$2b^1(M)$</td>
<td>$C_i, (d_4 \star 4 d_4) C_{\mu \nu j}$</td>
</tr>
<tr>
<td>$b^2(M)$</td>
<td>$B_{ij}$</td>
</tr>
<tr>
<td>$b^3(M)$</td>
<td>$C_{ijk}$</td>
</tr>
<tr>
<td>$N$</td>
<td>$G$</td>
</tr>
</tbody>
</table>
The ten-dimensional origin in type IIB supergravity of these scalars, in terms of the bosonic fields $\Phi, G, B, C^{(0)}, C^{(2)}, C^{(4)}$, is given in the following table:

<table>
<thead>
<tr>
<th>number field</th>
<th>field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4b^0(M)$</td>
<td>$\Phi, (d_4^1 \star_4 d_4) B_{\mu\nu}, C, (d_4^1 \star_4 d_4) C_{\mu\nu}$</td>
</tr>
<tr>
<td>$2b^2(M)$</td>
<td>$B_{ij}, C_{ij}$</td>
</tr>
<tr>
<td>$b^2(M)$</td>
<td>$(d_4^1 \star_4 d_4) C_{\mu\nu ij}, C_{ijkl}^+$</td>
</tr>
</tbody>
</table>

### 3.2.3. Gauge Fields.

Four-dimensional Type II supergravity compactified on $M$ contains

$$r_g = b^0(M) + 2b^1(M) + b^2(M)$$

independent four-dimensional gauge fields. The ten-dimensional origin in type IIA supergravity of these gauge fields is given in the following table:

<table>
<thead>
<tr>
<th>number field</th>
<th>field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^0(M)$</td>
<td>$C_\mu$</td>
</tr>
<tr>
<td>$2b^1(M)$</td>
<td>$G_{\mu i}, B_{\mu i}$</td>
</tr>
<tr>
<td>$b^2(M)$</td>
<td>$C_{\mu ij}$</td>
</tr>
</tbody>
</table>

The ten-dimensional origin in type IIB supergravity of these gauge fields is given in the following table:

<table>
<thead>
<tr>
<th>number field</th>
<th>field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3b^1(M)$</td>
<td>$G_{\mu i}, B_{\mu i}, C_{\mu i}$</td>
</tr>
<tr>
<td>$b^3(M)/2$</td>
<td>$C_{\mu ijk}^+$</td>
</tr>
</tbody>
</table>

### 3.2.4. $\mathcal{N} = 2$ Compactification.

When $M$ has full holonomy, $\text{Hol}(M) \cong \text{SU}(3)$, then most supersymmetry is broken, and the resultant four-dimensional theory possesses $\mathcal{N} = 2$ supersymmetry. This is the generic and arguably most interesting case, and is covered in every introductory textbook. See [5][6][18][7].

### 3.2.5. $\mathcal{N} = 4$ Compactification.

When $M$ has reduced holonomy, $\text{Hol}(M) \cong \text{SU}(2)$, then only half of the supersymmetry is broken, and the resultant four-dimensional theory possesses $\mathcal{N} = 4$ supersymmetry. We will be able to describe these models in more detail later, once we have a classification of the threefolds of $\text{SU}(2)$ holonomy at hand. Here we simply list a few properties which are general consequences of $\mathcal{N} = 4$ supersymmetry and do not require specific knowledge of what $M$ is, beyond its holonomy. One consequence of this enhanced supersymmetry is the following extremely useful result, which allows us to give an exact description of the relevant moduli spaces:
Theorem 3.4. Let $M$ be a complex threefold with holonomy $\text{Hol}(M) \cong \text{SU}(2)$. Then, a Ricci flat metric is an exact solution to the Type II supergravity action. ■

The low-energy effective theory of Type II string theory compactified on $M$ is four dimensional $\mathcal{N} = 4$ supergravity. Recall from the first section that there are two relevant supermultiplets:

- the **supergravity multiplet**, whose bosonic content is given by $(G_{\mu\nu}, A_{\mu}^{AB}, \tau)$, where $G_{\mu\nu}$ is the four-dimensional metric, $A_{\mu}^{AB}$ is a collection of 6 graviphotons, and $\tau$ is a complex scalar,

- the **matter multiplet**, whose bosonic content is given by $(A^\mu, \phi^A)$, where $A^\mu$ is a gauge field and $\phi^A$ is a collection of 6 scalars.

Here $A$ and $B$ are $R$-symmetry indices, as discussed in the first section.

Now suppose that our theory contains $n$ matter multiplets. The complex scalar $\tau$ takes values in the complex upper half-plane $\mathbb{H}$ and the $6n$ real scalars $\phi$ take values in the Grassmannian $\text{SO}(6,n)/(\text{SO}(6) \times \text{SO}(n))$. Together, the scalars parametrize the scalar manifold

\begin{equation}
\mathcal{M} = \mathbb{H} \times \frac{\text{SO}(6,n)}{\text{SO}(6) \times \text{SO}(n)}.
\end{equation}

As a real manifold,

\begin{equation}
\dim(\mathcal{M}) = 2 + 6n.
\end{equation}

The factor of 2 corresponds to the complex scalar in the supergravity multiplet, and the factor of $6n$ corresponds to the six real scalars in each of the $n$ matter multiplets.
In the previous section we noted that compactifying Type II string theory on a (compact, connected) threefold $M$ with SU(2) holonomy, i.e. a CHL manifold, results in a four-dimensional theory with half-maximal supersymmetry, a geometric CHL model. In this section we will classify all the threefolds $M$ with SU(2) holonomy, and investigate a couple of physically relevant issues related to their topology. The ur-example of a threefold of SU(2) holonomy is $M \cong K3 \times T^2$, where K3 is the unique Calabi-Yau twofold of SU(2) holonomy, up to diffeomorphism. In fact, every other example descends from this example, in some sense. Specifically,

**Theorem 4.1.** Suppose that $M$ is a (compact, connected) Calabi-Yau threefold of SU(2) holonomy. Then $M$ is diffeomorphic to a quotient of $K3 \times T^2$ by a finite group $G$ of symplectic automorphisms: $M \cong (K3 \times T^2)/G$, where $G$ acts faithfully. Furthermore, this quotient depends only on the isomorphism class of $G$. That is, if $H_1$ and $H_2$ are two finite groups both acting faithfully on $K3 \times T^2$ by symplectic automorphisms, then the resultant quotients are diffeomorphic: $(K3 \times T^2)/H_1 \cong (K3 \times T^2)/H_2$. ■

This result is more or less well-known, and a proof sketch of the first half of the statement can be found in the lecture notes by Aspinwall [8]. We will fill in the details a bit, and present a sketch of a proof (mostly complete) of the second half of the statement. Note, all group actions in this section will be assumed to be faithful. We will often identify a group of symplectic automorphisms with other representatives of the corresponding isomorphism class and write the orbifolds accordingly. By the second part of this theorem, this notation is well-defined up to diffeomorphism.

Given that symplectic automorphisms feature prominently in the preceding result, we had best define them. A symplectic automorphism is a biholomorphism that respects symplectic structure. This is necessary so that the resultant group action commutes with supersymmetries. Since $M$ is a smooth manifold, the automorphisms in $G$ must act without fixed points. Furthermore, as we will show below, they must be of the form

\begin{align}
(94) &\quad (g, \delta) : K3 \times T^2 \to K3 \times T^2 \\
(95) &\quad (g, \delta) : (x, t) \mapsto (g(x), \delta(t)),
\end{align}

where $g$ is a symplectic automorphism of K3 and $\delta$ is a shift on $T^2$. Note that the commutator subgroup of $G$ acts on $K3 \times T^2$ without fixed points, but acts trivially on the $T^2$ factor. We can take $\delta$ to have the same order as $g$. Since any symplectic automorphism of K3 has fixed
points, this implies that the commutator subgroup of \( G \) is trivial. That is, \( G \) is abelian. We can therefore restrict attention to abelian groups.

The group \( \{ g : (g, \delta) \in G \} \) is isomorphic to \( G \) itself, and is a finite abelian group of symplectic automorphisms acting on \( K3 \), and so we identify it with \( G \). The finite abelian groups acting on \( K3 \) by symplectic automorphisms have been classified by Nikulin, who studied the action of symplectic automorphisms on the second cohomology lattice

\[
\Lambda_{K3} = H^2(K3, \mathbb{Z}) = H \oplus H \oplus H \oplus E_8(-1) \oplus E_8(-1).
\]

Here, as elsewhere, \( H \) is the hyperbolic lattice defined as \( \mathbb{Z}^2 \) with inner product

\[
L_H = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

and \( E_8 \) is the \( E_8 \)-root lattice, which can be taken as \( \mathbb{Z}^8 \) with inner product

\[
L_{E_8} = \begin{pmatrix}
+2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & +2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & +2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & +2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & +2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & +2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & +2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & +2
\end{pmatrix}.
\]

The classification in question is:

**Theorem 4.2** (Nikulin [19]). If \( G \) is a nontrivial, finite abelian group of symplectic automorphisms of a given \( K3 \) surface, then \( G \) is isomorphic to one of the following:

\[
(Z/n\mathbb{Z}) \quad \text{for } n \in \{2, \ldots, 8\}, \quad (Z/n\mathbb{Z}) \times (Z/n\mathbb{Z}) \quad \text{for } n \in \{2, 3, 4\},
\]

\[
(Z/2\mathbb{Z}) \times (Z/2\mathbb{Z}) \times (Z/2\mathbb{Z}), \quad (Z/2\mathbb{Z}) \times (Z/2\mathbb{Z}) \times (Z/2\mathbb{Z}) \times (Z/2\mathbb{Z}),
\]

\[
(Z/2\mathbb{Z}) \times (Z/4\mathbb{Z}), \quad (Z/2\mathbb{Z}) \times (Z/6\mathbb{Z}).
\]

For each of these fourteen groups, there exists a \( K3 \) surface and an isomorphic group of symplectic automorphisms acting on that surface.

Of these fourteen groups, all but two can act by translations without fixed points on \( T^2 \). The exceptions are \((Z/2\mathbb{Z})^3\) and \((Z/2\mathbb{Z})^4\), which cannot act without fixed points on \( T^2 \). Consequently, we can completely classify the compact threefolds of \( \text{SU}(2) \) holonomy:

**Corollary 4.3.** Suppose that \( M \) is a compact threefold of \( \text{SU}(2) \) holonomy. Then \( M \) is diffeomorphic to one of the following: (1) \( K3 \times T^2 \), (2) \( K3 \times T^2/Z_2 \), (3) \( K3 \times T^2/Z_3 \), (4) \( K3 \times T^2/Z_4 \), (5) \( K3 \times T^2/Z_5 \), (6) \( K3 \times T^2/Z_6 \), (7) \( K3 \times T^2/Z_7 \), (8) \( K3 \times T^2/Z_8 \), (9)

Of course, we do not yet know that this notation is well defined, but this will follows from the subsequent analysis. No two of these thirteen manifolds are diffeomorphic, and we can find examples of each [20]. There are therefore exactly thirteen distinct threefolds of SU(2) holonomy up to diffeomorphism and therefore exactly thirteen distinct four dimensional compactifications of Type II string theory with half-maximal (N = 4) supersymmetry in the sense that the moduli space of Type II string theories in four-dimensions with half-maximal supersymmetry consists of thirteen components.

4.1. Proof of Theorem 4.1. We elaborate on the sketch in [8]. Let (M, g) be a compact threefold with SU(2) holonomy. Note that h²,₀ = 1. By the Dolbeault index theorem, h¹,₀ = 1 [8]. Let ˜M be the universal cover of M, and ˜g be the lift of g. Since (M, g) is complete – as a consequence of compactness – (˜M, ˜g) is complete. Since g automatically has vanishing Ricci curvature, ˜g has vanishing Ricci scalar curvature. Therefore ˜g has vanishing Ricci scalar curvature. Finally note that, since b¹(M) = 2, ˜M is noncompact:

\[ \text{vol}(\tilde{M}) = \int_{\tilde{M}} (\tilde{g})^{1/2} \, dx = \sum_{H₁(M,Z)} \int_{M} (g)^{1/2} \, dx \geq \sum_{n=1}^{\infty} \text{vol}(M) = \infty. \]

This implies that ˜M contains a straight line, a geodesic which is length minimizing between any two of its points. We can therefore apply the celebrated Cheeger-Gromoll theorem [21]:

**Theorem 4.4 (Cheeger-Gromoll).** A complete Riemannian manifold with nonnegative Ricci scalar curvature containing a straight-line is isometric to a product of ℝ and a complete Riemannian manifold with nonnegative Ricci scalar curvature.

By an application of the Cheeger-Gromoll theorem, ˜M is isometric to X × ℝⁿ for some compact simply-connected manifold X and some positive integer n ∈ ℤ⁺. Since X × ℝⁿ ≅ ˜M has SU(2) holonomy, X has SU(2) holonomy. By Berger’s classification of the holonomy of simply connected manifolds and the observation that dim(X) ≤ 5, X must have dimension four. Then, by the scarcity of fourfolds of SU(2) holonomy, X is diffeomorphic to K₃. So, ˜M is isometric to K₃ × ℝ². We can quotient this space out by the group of deck transformations corresponding to the free part of H₁(M, Z), which has rank two. The resultant quotient, of which M is itself a quotient, is isometric to K₃ × T². So, M is a quotient of K₃ × T² by an isometry.
I claim that the only isometries $\Phi$ of $K3 \times T^2$ are of the form $\Phi = (g, \delta)$, where $g$ is an isometry of $K3$ and $\delta$ is an isometry of $T^2$, which implies that $\delta$ is a translation. This is a consequence of the fact that $K3$ possesses full SU(2) holonomy. The derivative $D\Phi : T(K3 \times T^2) \rightarrow T(K3 \times T^2)$ must fiberwise preserve the subspaces of the fibers acted upon nontrivially by holonomy and trivially by holonomy. The former are precisely the subspaces tangent to the submanifolds $K3 \times \{t_0\}$ for $t_0 \in T$ and the latter are precisely the subspaces tangent to the submanifolds $\{y_0\} \times T^2$ for $y_0 \in K3$. It follows that $g(D\Phi(v_1), v_2)$ vanishes if one of $v_1, v_2$ is in the former subspace and the other is in the latter subspace. Given any two points on $K3 \times T^2$ with identical $T^2$ coordinates, we can integrate $D(\Phi)(\gamma)\dot{\gamma}$ for a path $\gamma$ connecting them, and conclude that $\Phi$ maps these points to points with the same $T^2$ coordinate. Similarly, given any two points on $K3 \times T^2$ with identical $K3$ coordinates, we can integrate $D(\Phi)(\gamma)\dot{\gamma}$ for a path $\gamma$ connecting them, and conclude that $\Phi$ maps these points to points with the same $K3$ coordinate. So, $\pi_{K3} \circ \Phi$ does not depend on the $T^2$ coordinate, only on the $K3$ coordinate, while $\pi_{T2} \circ \Phi$ does not depend on the $K3$ coordinate, only on the $T^2$ coordinate. This is exactly the desired result. So, $M$ is isometric to a quotient of $K3 \times T^2$ by an isometry. Keeping track of complex structure implies the stronger result that $M$ is isomorphic to a quotient of $K3 \times T^2$ by a symplectic automorphism, simply because $M$ is isomorphic to a quotient of $\tilde{M}$ by a symplectic automorphism.

We now need to show, or at least argue, that, up to diffeomorphism, this quotient depends only on the isomorphism class of $G$. Here is a sketch, working with the case when $G \cong \mathbb{Z}_N$, and the case when $G$ is a direct product of two cyclic groups is similar. Let $S$ and $S'$ be two K3 surfaces admitting $G$ as a group of symplectic automorphisms, and let $\sigma_N$ generate $G$. We cite, without proof, the result of Nikulin that $K3/\sigma_N$ can be desingularized, and the resulting manifold is a K3 surface. Let $F \subset S$ and $F' \subset S'$ be the sets of fixed points of $\sigma_N$. The preceding sentence should imply that $(S - F)/\sigma_N$ is diffeomorphic to $(S' - F')/\sigma_N$. This is a modification of the original diffeomorphism that we modify to map blown up singularities to blown up singularities. Since

$$ (S - F) \times T^2/\sigma_N \cong ((S - F)/\sigma_N) \times (T^2/\sigma_N), $$

this implies that, excepting fixed points, $S \times T^2/\sigma_N$ is diffeomorphic to $S' \times T^2/\sigma_N$. The diffeomorphism can surely be extended to these fixed points as well, yielding a global diffeomorphism $S \times T^2/\sigma_N \cong S' \times T^2/\sigma_N$. This completes the argument modulo handwaving.

4.2. Examples. The preceding analysis took the existence of a group $G$ acting on $K3$ by symplectic automorphisms as a given. In general, given a group $G$, it is difficult to construct
a K3 surface admitting $G$ as symplectic automorphisms. Here, largely to convince the reader that we can, we construct a few symplectic automorphisms on the quartic, which shows that at least some of the groups identified by Nikulin can appear as the group of symplectic automorphisms of some K3 surface. For a list of examples, covering all the groups mentioned previously, consult [20].

Consider the K3 surface $S \subset \mathbb{C}P^3$ given by the vanishing set of the quartic $q(z_0, \ldots, z_3)$,

$$q(z_0, z_1, z_2, z_3) = z_0^4 + z_1^4 + z_2^4 + z_3^4. \tag{99}$$

That is,

$$S = \{[z_0, z_1, z_2, z_3] \in \mathbb{C}P^3 : q(z_0, \ldots, z_3) = 0\}. \tag{100}$$

This surface admits a number of symplectic automorphisms. The canonical $(2, 0)$-form on $S$, up to a global amplitude, is given on $U^n$, where $U^n$ for $n = 0, 1, 2, 3$ is the portion of $\mathbb{C}P^3$ on which $z_n$ is nonzero, by

$$\Omega = \frac{1}{4z_{m_3/n}} dz_{m_1/n} \wedge dz_{m_2/n}, \tag{101}$$

where $n, m_1, m_2, m_3$ is any even permutation of $0, 1, 2, 3$ (in order) and $z_{m/n} = z_m/z_n$. The surface $S$ admits a number of manifestly symplectic actions. Given any even permutation $\sigma$ of $\{0, 1, 2, 3\}$, the corresponding action on $\mathbb{C}P^3$,

$$\sigma : [z_0, z_1, z_2, z_3] \mapsto [z_{\sigma(0)}, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}], \tag{102}$$

descends to a symplectic automorphism on $S$. Similarly, if we multiply two elements of \{z_0, z_1, z_2, z_3\} by a fourth root of unity and its conjugate respectively, this descends to a symplectic automorphism on $S$. We can combine these in various ways.

- First consider the action of $\mathbb{Z}_2$ on $\mathbb{C}P^3$ where the generator $\sigma_2 \in \mathbb{Z}_2$ acts by

  $$\sigma_2 : [z_0, z_1, z_2, z_3] \mapsto [-z_0, -z_1, z_2, z_3].$$

  This descends to a symplectic automorphism on $S$. This has eight fixed points, four when the first two coordinates are zero: $[0, 0, 1, e^{\pi i/8}]$, $[0, 0, 1, e^{3\pi i/8}]$, $[0, 0, 1, e^{5\pi i/8}]$, and four when the second two coordinates are zero: $[1, e^{\pi i/8}, 0, 0]$, $[1, e^{3\pi i/8}, 0, 0]$, $[1, e^{5\pi i/8}, 0, 0]$, $[1, e^{7\pi i/8}, 0, 0]$.

- Consider the action of $\mathbb{Z}_3 = (\sigma_3)$ on $\mathbb{C}P^3$ given in homogeneous coordinates by

  $$\sigma_3 : [z_0, z_1, z_2, z_3] \mapsto [z_0, z_3, z_1, z_2].$$

  This descends to an action on $S$, and is a symplectic automorphism. Let $p = [z_0, z_1, z_2, z_3]$ be a fixed point of this action on $S$. First suppose that $z_0 \neq 0$. We may then suppose that $z_0 = 1$. Note that $z_1$, $z_2$, and $z_3$ must all be equal, and must and may in fact be equal to one of $3^{-1/4} \cdot e^{\pi i/4}$, $3^{-1/4} \cdot e^{3\pi i/4}$, $3^{-1/4} \cdot e^{5\pi i/4}$, or $3^{-1/4} \cdot e^{7\pi i/4}$. Therefore, there exist four
such fixed points. Now suppose that $z_0 = 0$. This implies that $z_1 = \lambda z_3$, $z_2 = \lambda z_1$, and $z_3 = \lambda z_2$ for some $\lambda \in \mathbb{C}^\times$. Since $z_1, z_2, z_3$ cannot all be zero by the definition of projective space, this implies that $\lambda$ is a third root of unity and $z_1 \neq 0$. We may then assume that $z_1 = 1$. The requirement that $p \in S$ is then $1 + \lambda^4 + \lambda^8 = 0$. Therefore $\lambda = e^{4\pi i/3}, e^{2\pi i/3}$. Therefore, there exist two such distinct points.

- Consider the action of $\mathbb{Z}_4$ on $\mathbb{CP}^3$ where the generator $\sigma_4 \in \mathbb{Z}_4$ acts by $\sigma_4 : [z_0, z_1, z_2, z_3] \mapsto [iz_0, -iz_1, z_2, z_3]$. This descends to a symplectic automorphism on $S$. This has four fixed points, $[0, 0, 1, e^{n\pi i/4}]$ for $n = 1, 3, 5, 7$. Note that this induces an action of $\mathbb{Z}_2$ on $S$, the action given earlier with the identification $\sigma_2 = \sigma_4^2$.

### 4.3. Symplectic Automorphisms of K3 surfaces

At the beginning of this section, we mentioned the importance of symplectic automorphisms in constructing all threefolds of SU(2) holonomy. Here, we will examine a bit more closely the possible symplectic automorphisms of a K3 surface $S$, paying close attention to the induced map on cohomology. It should be noted that a general K3 surface admits no nontrivial symplectic automorphisms. However, some K3 surfaces do, and we listed a couple in the preceding subsection.

We can decompose the de Rham cohomology group $H^2(S, \mathbb{C}) \cong H^2(S, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ into a direct sum of Dolbeault cohomology groups:

$$H^2(S, \mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S). \hspace{1cm} (103)$$

Identify $H^2(S, \mathbb{Z}) \cong \mathbb{Z}^{22}$ with the image of the natural embedding $H^2(S, \mathbb{Z}) \hookrightarrow H^2(S, \mathbb{C})$. We define the Picard of Néron-Severi lattice $Pic(S)$ and the transcendental lattice $T_S$ as

$$Pic(S) = H^{1,1}(S) \cap H^2(S, \mathbb{Z}) \quad \text{and} \quad T_S = Pic(S)^\perp \subset H^2(S, \mathbb{Z}). \hspace{1cm} (104)$$

The rank of the Picard lattice $Pic(S)$ is denoted as $\rho(S)$. A symplectic automorphism $g \in Aut(S)$ is an automorphism which preserves the canonical line bundle on $S$ or equivalently the canonical $(2,0)$-form $\Omega$:

$$g^*|_{H^{2,0}(S)} = id_{H^{2,0}(S)} \hspace{1cm} (105)$$

where $g^* : H^*(S) \rightarrow H^*(S)$ is the induced map on cohomology. It can be shown that

**Theorem 4.5** (Nikulin). An automorphism $g \in Aut(S)$ is symplectic if and only if it acts as the identity on the transcendental lattice: $g^*|_{T_S} = id_{T_S}$. \hspace{1cm} ■

In addition, Nikulin showed that the order of a (finite order) symplectic automorphism can only be one of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. This will partially follow from the analysis below.
4.3.1. Fixed Points. Any symplectic automorphism of K3 has fixed points. The number of fixed points $f(N)$ for $N > 1$ of a symplectic automorphism $g$ of order $N$ depends only on $N$:

$$f(N) = 24 \left( N \prod_{p|N} \left( 1 + \frac{1}{p} \right) \right)^{-1}.$$  

This section will be devoted to proving this formula. By convention, we define $f(1)$ by this equation, so that $f(1) = 24$. Concretely, for $n \in \{1, \ldots, 8\}$ the values of $f(n)$ are

<table>
<thead>
<tr>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(N)$</td>
<td>24</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

We can derive this result by computations involving orbifold Euler characteristics. Suppose that $G$ is a group of symplectic automorphisms of a K3 surface $S$. The orbifold $S/G$ may be desingularized by attaching Ricci flat manifolds. The holonomy of the desingularized surface must contain the holonomy we began with. $A$ priori, the holonomy could be strictly larger. This is what occurs when K3 is constructed as a $T^4$ orbifold in the usual way. There is holonomy “hidden” in the singularities. Since the desingularized surface admits a Ricci flat metric, it is Calabi-Yau. By the scarcity of Calabi-Yau fourfolds, the desingularized surface is either K3 or $T^4$. By the consideration regarding holonomy, it is a K3 surface. The orbifold Euler characteristic of $S/G$, which we denote by $\chi(S; G)$ is equal to the Euler characteristic of the desingularized surface, which is $\chi(K3) = 24$. By definition, the orbifold Euler characteristic is

$$\chi(S; G) = \frac{1}{|G|} \sum_{gh=hg} \chi(S^{(g,h)})$$

where the sum is over pairs of commuting elements of $G$ and $S^{(g,h)}$ is the submanifold of $S$ fixed by the group generated by $g$ and $h$. Note that this is not in general equal to the usual topological Euler characteristic of the orbifold, due to homology concentrated at singularities. Since the fixed submanifold of a nontrivial symplectic automorphism of K3 is a collection of points $\chi(S^{(g,h)})$ is the number of points fixed by $(g, h)$. We are interested in the case $G = \mathbb{Z}_N$. Let $\sigma_N$ generate $\mathbb{Z}_N$. Since $\langle \sigma_N^{m_1}, \sigma_N^{m_2} \rangle = \langle \sigma_N^{\gcd(m_1,m_2)} \rangle = \langle \sigma_N^{\gcd(m_1,m_2,N)} \rangle$
We can write the sum on the right hand side in terms of the Möbius function \( \mu \), where
\[
\sum_{m_1, m_2} \chi \left( S_{\gcd(m_1, m_2)} \right)
\]
(108)
\[
= \frac{1}{N} \sum_{d|N} f(d) \cdot \# \{([m_1], [m_2]) \in \mathbb{Z}_N^2 : \langle [m_1], [m_2] \rangle = (N/d)\mathbb{Z}_N \}.
\]
(109)

Given such an \((m_1, m_2)\in \mathbb{Z}_N^2\), the map \(((m_1, [m_2]) \mapsto ([m_1d/N], [m_2d/N]) \in \mathbb{Z}_d^2\) is a bijection with the set above and the set \( \{([n_1], [n_2]) \in \mathbb{Z}_d^2 : \langle [n_1], [n_2] \rangle = \mathbb{Z}_d \} \). Let \( p_1, p_2, \ldots \) be the distinct primes dividing \( d \). Let \( S_{p_i} \) be the subset of pairs of elements of \( \mathbb{Z}_d \) such that both elements lie in \( p_i\mathbb{Z}_d \). By inclusion-exclusion,
\[
\# \{([n_1], [n_2]) \in \mathbb{Z}_d^2 : \langle [n_1], [n_2] \rangle = \mathbb{Z}_d \} = d^2 + \sum_{n=1}^\infty \sum_{d|p_1, \ldots, p_n|d} (-1)^n \# S_{p_1} \cap \cdots \cap S_{p_n}
\]
where the sum on the right hand side is over finite sets of distinct primes dividing \( d \). Note that
\[
\# S_{p_1} \cap \cdots \cap S_{p_n} = \left( \frac{d}{p_1 \cdots p_n} \right)^2.
\]
for any \( i_1, \ldots, i_n \). Therefore
(110)
\[
\# \{([n_1], [n_2]) \in \mathbb{Z}_d^2 : \langle [n_1], [n_2] \rangle = \mathbb{Z}_d \} = \sum_{n=0}^\infty \sum_{d|p_1, \ldots, p_n|d} (-1)^n \left( \frac{d}{p_1 \cdots p_n} \right)^2.
\]

We can write the sum on the right hand side in terms of the Möbius function \( \mu(d) \):
(111)
\[
\sum_{n=0}^\infty \sum_{d|p_1, \ldots, p_n|d} (-1)^n \left( \frac{d}{p_1 \cdots p_n} \right)^2 = \sum_{j|d} \mu(j) \left( \frac{d}{j} \right)^2 = d \cdot \varphi(d) \cdot \prod_{p|d} \left( 1 + \frac{1}{p} \right),
\]
where \( \varphi \) is the Euler totient. So,
(112)
\[
24 = \frac{1}{N} \sum_{d|N} f(d) \cdot d \cdot \varphi(d) \cdot \prod_{p|d} \left( 1 + \frac{1}{p} \right).
\]

By the Möbius inversion formula, we can invert this summation to find \( f(N) \). The result is that
(113)
\[
f(N) = \frac{24}{N \cdot \varphi(N)} \prod_{p|N} \left( 1 + \frac{1}{p} \right)^{-1} \sum_{d|N} (N/d) \mu(d) = 24 \left( \frac{N}{\prod_{p|N} \left( 1 + \frac{1}{p} \right)} \right)^{-1},
\]
which is exactly what we wanted to show.
4.3.2. **Action on Cohomology.** Let $G \subset \text{Aut}(S)$ be a group of symplectic automorphisms.

We denote the group of induced maps $g^* : H^o(S, R) \to H^o(S, R)$ for $g \in G$ by $G^*$, where $R$ can be any one of $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ and $H^o(S, R)$ can be any preserved cohomology subspace of sublattice. These induced maps are all $R$-linear. We consider, in particular, the induced maps $g^* : H^*(S, \mathbb{Z}) \to H^*(S, \mathbb{Z})$ and $g^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$. These maps preserve the intersection product on cohomology, and are therefore lattice automorphisms: $G^* \subset \text{Aut}(\Gamma_{K3})$. It is natural to ask how $G^*$ depends on the isomorphism class $G$. It cannot be the case in general that $G^*$ is uniquely determined by the isomorphism class of $G$, since if $h \in \text{Aut}(S)$ is any symplectic automorphism of $S$ then we can define a group action of $G$ on $S$ by symplectic automorphisms where $g \in G$ acts as $h^{-1}gh$. Since $(h^{-1}gh)^* = (h^*)g^*(h^*)^{-1}$, the new group is conjugate to the old group. In fact, this is all that can happen.

**Theorem 4.6** (Nikulin). If $G \cong H$ are two finite abelian groups acting on $S$ by symplectic automorphisms, then $G^*$ and $H^*$ are related by conjugation.

So, in this sense, the action of a group $G$ of symplectic automorphisms on cohomology is determined by the isomorphism class of $G$.

4.4. **Cohomology Lattices.** In this section we will investigate the action of $G^*$ on $H^*(S, \mathbb{Z})$.

In particular, we will aim to understand the sublattice $H^2(S, \mathbb{Z})^G$ of $H^2(S, \mathbb{Z}) \cong \Lambda_{K3}$ which is left invariant by $G$. The physical motivation for this is to understand the lattice of electromagnetic charges in the corresponding CHL model. The second cohomology lattice $\Lambda_{K3}$ is given by

$$\Lambda_{K3} \cong H \oplus H \oplus H \oplus E_8(-1) \oplus E_8(-1),$$

(114)

where $H$ is the two-dimensional hyperbolic lattice and $E_8(-1)$ is the $E_8$ root lattice with negative inner product. Automatically, $H^2(S, \mathbb{Z})^G$ is primitively embedded in $\Lambda_{K3}$. Note that $G^*$ automatically preserves the terms in the Hodge decomposition. Furthermore, $G^*$ is a group of (linear) isometries and lattice isomorphisms. Consequently, the space of self-dual forms $\mathcal{H}^+(S, \mathbb{C}) \subset H^*(S, \mathbb{C})$ is fixed by $G^*$. This includes the subspace $H^0(S, \mathbb{C}) \oplus H^{0,2}(S) \oplus H^{2,0}(S) \oplus H^4(S, \mathbb{C})$. The analogous statement with integral cohomology holds. It follows that $\dim(H^*(S, \mathbb{Z})^G) \geq 6$, where $\dim(H^*(S, \mathbb{Z})^G)$ is the rank of $H^*(S, \mathbb{Z})$. In fact, the rank of $H^*(S, \mathbb{Z})^G$ is given by the following theorem.

---

$^3$Note that $\text{Aut}(S)$ can be nonabelian [20].
Theorem 4.7. Let $m(n)$ be the number of elements of $G$ of order $n$. Then

$$\dim(H^*(S,\mathbb{Z})^G) = \frac{1}{|G|} \left[ 24 + \sum_{n=2}^8 m(n) f(n) \right].$$

Concretely, the values of this rank for the relevant groups in Nikulin’s classification are given in the following table:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\dim(H^*(S,\mathbb{Z})^G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_1$</td>
<td>24</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>16</td>
</tr>
<tr>
<td>$\mathbb{Z}_3$</td>
<td>12</td>
</tr>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>10</td>
</tr>
<tr>
<td>$\mathbb{Z}_5$</td>
<td>8</td>
</tr>
<tr>
<td>$\mathbb{Z}_6$</td>
<td>8</td>
</tr>
<tr>
<td>$\mathbb{Z}_7$</td>
<td>6</td>
</tr>
<tr>
<td>$\mathbb{Z}_8$</td>
<td>6</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_4$</td>
<td>6</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_6$</td>
<td>6</td>
</tr>
</tbody>
</table>

Consider $g \in G$ of order $n$. Note that the isometry $g^* : H^2(S) \to H^2(S)$ is diagonalizable over $\mathbb{C}$. Consequently, we can decompose $H^2(S)$ into a direct sum of eigenspaces of $g^*$:

$$H^2(S) = \bigoplus_{i=0}^{n-1} H^2(S)_{\zeta_i^n},$$

where $\zeta_n \in \text{U}(1)$ is a primitive $n$th root of unity and $H^2(S)_{\zeta_i^n}$ is the eigenspace of $g^*$ in $H^2(S)$ with eigenvalue $\zeta_i^n$. Similarly, since $G$ is abelian, given two elements $g, h \in G$ of order $n$ and $m$ respectively, we can decompose $H^2(S)$ into a direct sum of intersections of eigenspaces of $g^*$ and $h^*$:

$$H^{1,1}(S) = \bigoplus_{r=0}^{n-1} \bigoplus_{j=0}^{m-1} H^2(S)_{\zeta_i^n \zeta_j^m},$$

where $H^2(S)_{\zeta_i^n \zeta_j^m} = H^2(S)_{\zeta_i^n} \cap H^2(S)_{\zeta_j^m}$. We will compute these decompositions when $g$ generates $G$ or $g$ and $h$ together generate $G$.

4.4.1. Representation Decomposition. Of course, the map $G \to G^*$ given by $g \mapsto g^*$, where $g^* : H^2(S) \to H^2(S)$, is a real twenty-two dimensional representation of $G$. We can then ask for the decomposition of this representation into irreducible representations. Since $G$ is abelian, the irreducible representations of $G$ are one-dimensional. The character $\chi(g)$ of an element $g \in G$ in our representation is given by the Lefschetz fixed point theorem, which relates the character of a map to the number of its fixed points:

$$f(|g|) = \sum_{r=0}^{\infty} (-1)^r \text{Tr}(g^r|_{H^r(S,\mathbb{C})}) = 2 + \text{Tr}(g^*|_{H^2(S)}) = 2 + \chi(g),$$

where $|g|$ is the order of $g$ and $f$ is defined by (Eq. 106), so that

$$\chi(g) = f(|g|) - 2.$$

Since $f(|g|)$ is known, $\chi(g)$ is known. By producing a complete set of characters of $G$ and comparing them to $\chi$, we can determine the relevant decompositions. We first assume that
4.4.2. One Cyclic Factor. First suppose that $G \cong \mathbb{Z}_N$ is cyclic with generator $\sigma_N$. Consider the character $\chi^{(m)}$ defined by $\chi^{(m)}(\sigma_N) = \zeta_N^m$. The characters $\{\chi^{(1)}, \ldots, \chi^{(N)}\}$ form a complete set of characters for $G$. By direct computation,

$$\langle \chi, \chi^{(m)} \rangle = \frac{1}{N} \sum_{d | N} (f(d) - 2) \sum_{(j,d) = 1} \zeta_d^{mj}.$$  

The second sum on the right hand side is over integers $j \in \{1, \ldots, d\}$ such that $\gcd(j, d) = 1$. This sum is given by

$$\sum_{(j,d) = 1} \zeta_d^j = \gcd(d, m) \cdot \mu(d/\gcd(d, m)) \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right).$$

**Proof.** First consider the case when $m = 1$, which is the identity

$$\sum_{(j,d) = 1} \zeta_d^j = \mu(d).$$

This can be proven in a number of different ways. One of the quickest is to extract it from the Möbius inversion formula. Denote the expression on the left-hand side by $\tilde{\mu}(d)$. We “complete” the sum:

$$\sum_{\delta | d} \tilde{\mu}(\delta) = \sum_{j=1}^{d} \zeta_d^j = \begin{cases} 1 & (d = 1) \\ 0 & (d > 1) \end{cases}$$

where we have used the geometric series formula, whence the Möbius inversion formula yields $\tilde{\mu}(d) = \mu(d)$.

Now consider the case when $m > 1$. We can write

$$\sum_{(j,d) = 1} \zeta_d^{mj} = \sum_{(j,d) = 1} \zeta_d^{j/\gcd(m,d)}.$$ 

Given an integer coprime to $d$, that integer is coprime to $d/\gcd(m,d)$, but the converse does not hold. If $d/\gcd(m,d)$ is divisible by the same set of primes as $d$, that is if $m$ has no prime divisors with higher multiplicity than in $d$, the converse does hold, and consequently the right-hand side is precisely

$$\sum_{(j,d) = 1} \zeta_d^{j/\gcd(m,d)} = \gcd(m, d) \mu(d/\gcd(m,d)).$$
Otherwise, it suffices to note that there are \( \gcd(m, d) \prod_{1 < (p, d) \leq (p, m)} (1 - 1/p) \) invertible residues modulo \( d \) equivalent to an invertible residue modulo \( d / \gcd(m, d) \), which follows from an argument using inclusion-exclusion. 

\[
\prod_{p \text{ prime}} \left( 1 - \frac{1}{p} \right).
\]

This is not the most elegant formula, but it is straightforward to evaluate. The results are contained in the following table, encoded in the characteristic polynomial of \( \sigma_N \). We also list the decomposition of \( \Lambda_{K3} \) into representations \([n] \), where \([n] \) is a representation in which a basis is cyclically interchanged by \( G \). It should be noted that these are, in general, reducible. The entries in the rightmost column display a certain symmetry. The given decomposition is often called a frame shape, and this symmetry is expressed by saying that the frame shape is balanced.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( \det(x \cdot \id_{\Lambda_{K3}} - g^*\vert_{\Lambda_{K3}}) )</th>
<th>Representation on ( \Lambda_{K3} )</th>
<th>Representation on ( H^*(K3, \mathbb{C}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_1 )</td>
<td>( (x - 1)^{22} )</td>
<td>22 ([1])</td>
<td>24 ([1])</td>
</tr>
<tr>
<td>( \mathbb{Z}_2 )</td>
<td>( (x - 1)^6(x^2 - 1)^8 )</td>
<td>6 ([1]) (\oplus) 8 ([2])</td>
<td>8 ([1]) (\oplus) 8 ([2])</td>
</tr>
<tr>
<td>( \mathbb{Z}_3 )</td>
<td>( (x - 1)^2(x^3 - 1)^6 )</td>
<td>4 ([1]) (\oplus) 6 ([3])</td>
<td>6 ([1]) (\oplus) 6 ([3])</td>
</tr>
<tr>
<td>( \mathbb{Z}_4 )</td>
<td>( (x - 1)^2(x^2 - 1)^2(x^4 - 1)^4 )</td>
<td>2 ([1]) (\oplus) 2 ([2]) (\oplus) 4 ([4])</td>
<td>4 ([1]) (\oplus) 2 ([2]) (\oplus) 4 ([4])</td>
</tr>
<tr>
<td>( \mathbb{Z}_5 )</td>
<td>( (x - 1)^2(x^5 - 1)^4 )</td>
<td>2 ([1]) (\oplus) 4 ([5])</td>
<td>4 ([1]) (\oplus) 4 ([5])</td>
</tr>
<tr>
<td>( \mathbb{Z}_6 )</td>
<td>( (x^2 - 1)^2(x^3 - 1)^2(x^6 - 1)^2 )</td>
<td>2 ([2]) (\oplus) 2 ([3]) (\oplus) 2 ([6])</td>
<td>2 ([1]) (\oplus) 2 ([2]) (\oplus) 2 ([3]) (\oplus) 2 ([6])</td>
</tr>
<tr>
<td>( \mathbb{Z}_7 )</td>
<td>( (x - 1)(x^7 - 1)^3 )</td>
<td>1 ([1]) (\oplus) 3 ([7])</td>
<td>3 ([1]) (\oplus) 3 ([7])</td>
</tr>
<tr>
<td>( \mathbb{Z}_8 )</td>
<td>( (x^2 - 1)(x^4 - 1)(x^8 - 1)^2 )</td>
<td>1 ([2]) (\oplus) 1 ([4]) (\oplus) 2 ([8])</td>
<td>2 ([1]) (\oplus) 1 ([2]) (\oplus) 1 ([4]) (\oplus) 2 ([8])</td>
</tr>
</tbody>
</table>

4.4.3. Two Cyclic Factors. We can perform a similar analysis when \( G \) is a product of two cyclic factors. Suppose that \( G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \) with generators \( \sigma_{n_1} \) and \( \sigma'_{n_2} \), and let \( \chi^{(m_1, m_2)}(\sigma_{n_1}) = \zeta_{m_1}^{m_1} \) and \( \chi^{(m_1, m_2)}(\sigma'_{n_2}) = \zeta_{m_2}^{m_2} \). The characters \( \chi^{(m_1, m_2)} : m_1 \in \mathbb{Z}_{n_1}, m_2 \in \mathbb{Z}_{n_2} \) take on indices \( (m_1, m_2) \). We will use the same notation for the familiar quadratic characters, which we denote by \( \chi^m \), taking indices \( m \).
\( \mathbb{Z}_{n_2} \) form a complete set of characters of \( g \). Note that

\[
\langle \chi, \chi^{(m_1, m_2)} \rangle = \frac{1}{n_1 n_2} \sum_{d_1 | n_1} \sum_{d_2 | n_2} (f(lcm(d_1, d_2)) - 2) \left( \sum_{(j_1, d_1) = 1} \zeta_{d_1}^{m_1 j_1} \right) \left( \sum_{(j_2, d_2) = 1} \zeta_{d_2}^{m_2 j_2} \right).
\]

The character inner product on the left hand side is

\[
\langle \chi, \chi^{(m_1, m_2)} \rangle = \dim(H^2(S)_{\zeta_{n_1}^{m_1}, \zeta_{n_2}^{m_2}}),
\]

and the right-hand side may be computed as before. This is not particularly fun to compute, but it is straightforward. We will not need these going forward, and so we leave their computation to the enthusiastic or otherwise masochistic reader.

4.4.4. Detailed Description. The preceding analysis only managed to determine certain dimensions related to the action of \( G \) on cohomology. We would like more substantive information. An explicit matrix representation of the elements of \( G \) would be ideal, but this is rather difficult to determine. However, even without an explicit representation in hand, some additional nontrivial information can be determined.

Since the action on cohomology depends only on \( G \), up to isomorphism, relevant properties of this action can be computed by working with a sufficiently nice K3 surface admitting \( G \) as a group of symplectic automorphisms. This is the strategy followed by Garbagnati and Sarti [22][23], using elliptic fibrations. Here we list a few relevant results from [23]. The sublattice of \( \Lambda_{K3} = H^2(K3, \mathbb{Z}) \) fixed by \( G \) is given in the following table for the thirteen relevant groups in Nikulin’s classification. In the following table, a lattice \((\mathbb{Z}^n, L)\), with \( L \) an \( n \times n \) symmetric non-degenerate bilinear form, is abbreviated by \( L \), since \( n \) can be
determined by its size.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\text{rank}(\Lambda^G_{\mathbb{K}_3})$</th>
<th>$\Lambda^G_{\mathbb{K}_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_1$</td>
<td>22</td>
<td>$H \oplus H \oplus H \oplus E_8(-1) \oplus E_8(-1)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>14</td>
<td>$H \oplus H \oplus H \oplus E_8(-2)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3$</td>
<td>10</td>
<td>$H \oplus H(3) \oplus H(3) \oplus A_2 \oplus A_2$</td>
</tr>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>8</td>
<td>$Q_4$</td>
</tr>
<tr>
<td>$\mathbb{Z}_5$</td>
<td>6</td>
<td>$H \oplus H(5) \oplus H(5)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_6$</td>
<td>6</td>
<td>$H \oplus H(6) \oplus H(6)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_7$</td>
<td>4</td>
<td>$H(7) \oplus (\frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td>$\mathbb{Z}_8$</td>
<td>4</td>
<td>$H(8) \oplus (\frac{0}{4}, \frac{0}{4})$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>10</td>
<td>$H(2) \oplus H(2) \oplus Q_{2,2}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times \mathbb{Z}_3$</td>
<td>6</td>
<td>$U(3) \oplus H(3) \oplus (\frac{3}{2}, \frac{3}{2})$</td>
</tr>
<tr>
<td>$\mathbb{Z}_4 \times \mathbb{Z}_4$</td>
<td>4</td>
<td>$Q_{4,4}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_4$</td>
<td>6</td>
<td>$Q_{2,4}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_6$</td>
<td>4</td>
<td>$Q_{2,6}$</td>
</tr>
</tbody>
</table>

The bilinear forms, $Q_4$, $Q_{2,2}$, $Q_{2,4}$, $Q_{2,6}$, $Q_{4,4}$ haven’t been defined yet. The form $Q_4$ is

\[
Q_4 = \begin{pmatrix}
0 & +4 & 0 & +2 & 0 & -1 & 0 & 0 \\
+4 & 0 & +4 & +4 & -4 & 0 & 0 & -4 \\
0 & +4 & 0 & 0 & 0 & 0 & 0 & 0 \\
+2 & +4 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -4 & 0 & 0 & -2 & -1 & 0 & -2 \\
-1 & 0 & 0 & -1 & -1 & -2 & +1 & +1 \\
0 & 0 & 0 & 0 & +1 & -2 & 0 & 0 \\
0 & -4 & 0 & 0 & -2 & +1 & 0 & -2
\end{pmatrix}
\]

The forms $Q_{2,2}$ and $Q_{2,4}$ are

\[
Q_{2,2} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & -4 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & -4 & 4 & 0 \\
0 & 0 & 0 & 0 & 4 & -8 & 0 
\end{pmatrix}, \quad
Q_{2,4} = \begin{pmatrix}
4 & -2 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & -64 & -4 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & -4 & 80 & 4 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0
\end{pmatrix}
\]
Finally, the forms $Q_{2,6}$ and $Q_{4,4}$ are

\begin{equation}
Q_{2,6} = \begin{pmatrix} 0 & 6 & 0 & 0 \\ 6 & 0 & -3 & 0 \\ 0 & -3 & 6 & 6 \\ 0 & 0 & 6 & 8 \end{pmatrix}, \quad Q_{4,4} = \begin{pmatrix} 4 & 6 & 0 & 0 \\ 6 & 4 & 6 & 4 \\ 0 & 6 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}.
\end{equation}

We will later compute the integral homology of the relevant spaces, which will be expressed in terms of $\Lambda_{K3}^G$. The preceding table will then give a very explicit description of the relevant charge lattices.

**4.5. Homology of CHL Manifolds.** While it is difficult to visualize $K3 \times T^2 / G$ directly as an orbifold, we can visualize it as a smooth $K3$ fibration of $T^2$. First consider the case when $G \cong \mathbb{Z}_N$ is a cyclic group. Write $T^2 = S^1 \times \hat{S}^1$, where $H_1$ and $H_2$ are two overlapping open (extended-)half-circles, and split $T^2 = H_1 \times \hat{S}^1 \cup H_2 \times \hat{S}^1$. Our fibration is composed of two trivializations, one on $H_1 \times \hat{S}^1$ and one on $H_2 \times \hat{S}^1$. In other words, on $H_1 \times \hat{S}^1$ the fibration is $K3 \times H_1 \times \hat{S}^1$ and on $H_2 \times \hat{S}^1$ the fibration is $K3 \times H_2 \times \hat{S}^1$. The overlap between the two charts is a disjoint union of two copies of $K3 \times (0,1)$. On one of these copies, the transition map $K3 \rightarrow K3$ relating the two trivializations is trivial. On the other one of these copies, the transition map $K3 \rightarrow K3$ relating the two trivializations is the generator $\sigma_N \in \mathbb{Z}_N$. This fibration is an alternate description of $(K3 \times T^2) / G$, where the $T^2$ base in the fibration corresponds to an $N$th of the original torus, specifically a fundamental domain of the action of $\mathbb{Z}_N$. If instead $G \cong \mathbb{Z}_N \times \mathbb{Z}_M$ for some $N,M \geq 2$, we take $S^1$ to be the circle along which $\mathbb{Z}_N$ translates and $\hat{S}^1$ to be the circle along which $\mathbb{Z}_M$ translates. Split both $S^1$ and $\hat{S}^1$ into two halves and proceed analogously.

**Proposition 4.8.** The (smooth) manifold $(K3 \times T^2) / G$ has the structure of a (smooth) $K3$ fibration over $T^2$, given as above.  

The Hodge numbers and Betti numbers of $M = (K3 \times T^2) / G$ are straightforward to compute using de Rham and Dolbeault cohomology, since a differential form on $(K3 \times T^2) / G$ lifts to a unique $G$-invariant differential form on $K3 \times T^2$, and this lifting commutes with exterior differentiation and respects the Hodge decomposition. Furthermore, (i) a form down below is closed if and only if the lifted form up top is closed and (ii) a form down below is exact if and only if the form up top is exact. Statement (i) and the “only if” direction of statement (ii) are immediate consequences of the commutation of exterior
Figure 3. A construction of $K3 \times T^2/\mathbb{Z}_N$ as a fibration of $K3$ over $T^2$. The two-dimensional torus $T^2$ is drawn as a one-dimensional circle, while the six-dimensional $M$ is drawn as three-dimensional. The striped coloring represents the $G$-twisting of $K3$ as the base torus is traversed.

differentiation with lifting, while the “if” direction of statement (ii) holds because $G$ is finite (\(^4\)). Combining (i) and (ii), the cohomology ring $H^*((K3 \times T^2)/G, \mathbb{C})$ is naturally isomorphic to the $G$-invariant subring of $H^*(K3 \times T^2, \mathbb{C})$, denoted by $H^*(K3 \times T^2, \mathbb{C})^G$, with the isomorphism furnished by lifting. By keeping track of Hodge structure – which is preserved by $G$ – in the preceding analysis, we can make a slightly stronger statement:

**Proposition 4.9.** For every $i$ and $j$, $H^{i,j}((K3 \times T^2)/G, \mathbb{C})$ is naturally isomorphic to $H^{i,j}(K3 \times T^2, \mathbb{C})^G$. ■

\(^4\)We can define the projection map $\Pi_G = (1/|G|)\sum_{g \in G} g^*$ in order to convert arbitrary forms up top to forms which descend down below, and $\Pi_G$ commutes with exterior differentiation. (Compare with Maschke’s theorem.) Together these yield the “if” direction. If $G$ is infinite, the “if” direction can fail. Take, for example, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The form $dx$ down below lifts to the form $dx$ up top. $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$ but $H_1(\mathbb{R}^2, \mathbb{Z}) \cong 1$. 
Corollary 4.10. The Hodge diamond of \((K3 \times T^2)/G\) is given by

\[
\begin{array}{cccc}
  & h^{0,0} & & \\
 h^{1,0} & h^{0,1} & & \\
 h^{2,0} & h^{1,1} & h^{0,2} & 1 \\
 h^{3,0} & h^{2,1} & h^{1,2} & h^{0,3} = 1 \\
   & h^{3,1} & h^{2,2} & h^{1,3} & 1 \\
   & h^{3,2} & h^{2,3} & 1 & 1 \\
   & h^{3,3} & & & 1 \\
\end{array}
\]

where \(h^{1,1} = h^{2,1}\) are given by \(n - 1\), where \(n = \dim(H^2(K3, \mathbb{R})^G)\).

The preceding analysis sheds little light on integral homology and cohomology, however. Using the aforementioned fibration construction, it is relatively straightforward to determine the integral homology of \((K3 \times T^2)/G\) using Mayer-Vietoris and the Künneth formula. To be slightly more concrete, we will describe the singular homology directly in terms of (i) the integral homology of \((K3 \times T^2)/G\), the group of maps induced on homology by \(G\), and (ii) in terms of the projection map \(\Pi_G : H_2(K3, \mathbb{Z}) \to H_2(K3, \mathbb{Z})^G \otimes \mathbb{Q}\), given by \(\Pi_G = (1/|G|) \sum_{g \in G} g_*\). For simplicity, we restrict to the case when \(G = \mathbb{Z}_N\) for \(N \in \{2, \ldots, 8\}\). The case when \(G\) is a product of two cyclic groups is similar. As before, let \(T^2 = S^1 \times \hat{S}^1\), with \(G\) acting on \(T^2\) by translations along \(S^1\), and let \(\sigma_N \in \mathbb{Z}_N\) be a generator.

- The group \(H_0((K3 \times T^2)/G, \mathbb{Z})\) is isomorphic to \(\mathbb{Z}\), with a generator given by the homology class of an arbitrary point in \(M\).
- The group \(H_1((K3 \times T^2)/G, \mathbb{Z})\) is isomorphic to \(\mathbb{Z}^2\), with a basis given by the homology classes of \(\{x_0\} \times S^1\) and \(\{x_0\} \times \hat{S}^1\).
- The group \(H_2((K3 \times T^2)/G, \mathbb{Z})\) is isomorphic to \(\mathbb{Z}^{n+1}\), with generators given by the homology classes of \(\{x_0\} \times T^2\) and \(\gamma \times \{t_0\}\), where \(x_0 \in K3\) and \(t_0 \in T^2\) are arbitrary and \(\gamma\) is an arbitrary 2-cycle in \(K3\). Note that \(g_\sigma \gamma \times \{t_0\}\) is cohomologous in \(M\) to \(\gamma \times \{t_0\}\), and consequently \(\Pi_G(\gamma) \times \{t_0\}\) is cohomologous to \(\gamma \times \{t_0\}\). A basis is given by the homology class of \(\{x_0\} \times T^2\) along with the homology classes of \(\gamma_a \times \{t_0\}\) for \(a \in \{1, \ldots, n\}\), where \(\gamma_1, \ldots, \gamma_n\) is an integral basis of \(\Pi_G(H_2(K3, \mathbb{Z}))\). The basis elements \(\gamma_1, \ldots, \gamma_n\) do not in general lie within \(H_2(K3, \mathbb{Z})\) and therefore define “twisted” branes. To summarize,

\[
H_2(K3 \times T^2/G, \mathbb{Z}) \cong H_2(T^2, \mathbb{Z}) \oplus \Pi_G(H_2(K3, \mathbb{Z}))
\]
with the isomorphism as above.

- The group $H_3((K3 \times T^2)/G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2n}$, with generators given by the homology classes of $\gamma \times S^1$ and $\hat{\gamma} \times \hat{S}^1$, where $\gamma$ is arbitrary and $\gamma$ is $G$-invariant. As in the previous case, a basis is given by the homology classes of $\gamma_a \times S^1$ for $a \in \{1, \ldots, n\}$ and the homology classes of $\hat{\gamma}_b \times \hat{S}^1$ for $b \in \{1, \ldots, n\}$, where $\{\gamma_1, \ldots, \gamma_n\}$ is an integral basis for $H_2(K3, \mathbb{Z})^G$ and $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_n\}$ is an integral basis for $\Pi_G(H_2(K3, \mathbb{Z}))$.

- The group $H_4((K3 \times T^2)/G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{n+1}$, with generators given by the homology classes of $K3 \times \{t_0\}$ and $\gamma \times T^2$, where $t_0 \in T^2$ is arbitrary and $\gamma \in H_2(K3, \mathbb{Z})^G$ is a $G$-invariant 2-cycle in a K3-fiber. A basis is then given by the homology classes of $K3 \times \{t_0\}$ along with the homology classes of $\gamma_a \times T^2$ for $a \in \{1, \ldots, n\}$, where $\{\gamma_1, \ldots, \gamma_n\}$ is an integral basis of $H_2(K3, \mathbb{Z})^G$. To summarize,

$$H_4(K3 \times T^2/G, \mathbb{Z}) \cong H_4(K3, \mathbb{Z}) \oplus H_2(K3, \mathbb{Z})^G$$

with the isomorphism as above.

- The group $H_5((K3 \times T^2)/G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$, with a basis given by the homology classes of $K3 \times S^1$ and $K3 \times \hat{S}^1$.

- The group $H_6((K3 \times T^2)/G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, with a generator given by the homology class of the entire manifold $M$.

We can see that, as groups, $H_i((K3 \times T^2)/G, \mathbb{Z}) \cong H_{6-i}((K3 \times T^2)/G, \mathbb{Z})$, as required by Hodge duality. We introduce lattice structure on these $\mathbb{Z}$-modules, descended from the intersection product of $K3 \times T^2$, in which the maps given above are lattice isomorphisms. Then, the lattices $\Pi_G(H_2(K3, \mathbb{Z}))$ and $H_2(K3, \mathbb{Z})^G$ can both be thought of as naturally lying in $H_2(K3, \mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}^{3,19}$. In general, $\Pi_G(H_2(K3, \mathbb{Z}))$ contains $H_2(K3, \mathbb{Z})^G$, strictly if $G$ is nontrivial. This will, for nontrivial $G$, introduce an asymmetry between electric and magnetic charges, which originates in the topological considerations above.

4.5.1. Computation from Mayer-Vietoris. Let’s now check the assertions just made using Mayer-Vietoris, assuming that we can ignore torsion. First consider the case when $G$ is cyclic, with generator $g$. Then we can write $(K3 \times T^2)/G$ naturally as $\hat{S}^1 \times (K3 \times S^1)/G$. We can therefore apply the Künneth formula in order to compute the integral homology of $(K3 \times T^2)/G$ in terms of the integral homology of $M' = (K3 \times S^1)/G$. As before, split $(K3 \times S^1)/G = X \cup Y$ into two trivializations, where $X \cong K3 \times H_1$ and $Y \cong K3 \times H_2$ as before. Each of $X$ and $Y$ is homeomorphic to $K3 \times (0, 1)$, which deformation retracts to $K3 \times \{1/2\} \cong K3$. The overlap $X \cap Y$ is a disjoint union of two submanifolds each
The maps (133)
\[ \homology{\alpha}{\negation{\congruenceclasses}} \]
Consequently, by the definition of exactness, \( H_{n+1}(M', \mathbb{Z}) \) is isomorphic to the kernel of the middle map, with an explicit isomorphism given by \( \kappa_* \). Since \( X \) and \( Y \) both deformation retract to \( K3 \), \( H_n(X, \mathbb{Z}) \cong H_n(K3, \mathbb{Z}) \) for every \( n \in \mathbb{N} \). The exact sequence in question is naturally isomorphic to
\[
\cdots \rightarrow H_1(M', \mathbb{Z}) \rightarrow H_0(K3, \mathbb{Z}) \oplus H_0(K3, \mathbb{Z}) \rightarrow H_0(K3, \mathbb{Z}) \oplus H_0(K3, \mathbb{Z}) \rightarrow H_0(M', \mathbb{Z}) \rightarrow 0.
\]
Since \( H_n(K3, \mathbb{Z}) \) is zero unless \( n \in \{0, 2, 4\} \), this sequence contains only three nontrivial sections.

\[
\begin{align*}
(130) \quad 0 & \rightarrow H_1(M', \mathbb{Z}) \rightarrow H_0(X \cap Y, \mathbb{Z}) \rightarrow H_0(X, \mathbb{Z}) \oplus H_0(Y, \mathbb{Z}) \rightarrow H_0(M', \mathbb{Z}) \rightarrow 0, \\
(131) \quad 0 & \rightarrow H_3(M', \mathbb{Z}) \rightarrow H_2(X \cap Y, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}) \oplus H_2(Y, \mathbb{Z}) \rightarrow H_2(M', \mathbb{Z}) \rightarrow 0, \\
(132) \quad 0 & \rightarrow H_5(M', \mathbb{Z}) \rightarrow H_4(X \cap Y, \mathbb{Z}) \rightarrow H_4(X, \mathbb{Z}) \oplus H_4(Y, \mathbb{Z}) \rightarrow H_4(M', \mathbb{Z}) \rightarrow 0.
\end{align*}
\]
The maps \( H_{n+1}(X, \mathbb{Z}) \oplus H_{n+1}(Y, \mathbb{Z}) \rightarrow H_{n+1}(M', \mathbb{Z}) \) are all surjective, while the maps
\[ H_{n+1}(M', \mathbb{Z}) \rightarrow H_2n(X \cap Y, \mathbb{Z}) \]
are all injective. Consequently, the kernel of the middle map, with an explicit isomorphism given by \( \partial_* \), while \( H_{n+1}(M', \mathbb{Z}) \) is isomorphic to the quotient of \( H_{n+1}(X, \mathbb{Z}) \oplus H_{n+1}(Y, \mathbb{Z}) \) by the image of the middle map, with an explicit isomorphism descending from \( (k_* - \ell_*) : \)
\[
\begin{align*}
(133) \quad H_{n+1}(M', \mathbb{Z}) & \cong \ker(\iota) \quad \text{and} \quad H_{n+1}(M', \mathbb{Z}) \cong (H_{n+1}(X, \mathbb{Z}) \oplus H_{n+1}(Y, \mathbb{Z}))/\image(\iota).
\end{align*}
\]
Written naturally \(^5\) as a map \( \iota : H_{2n}(K3, \mathbb{Z}) \oplus H_{2n}(K3, \mathbb{Z}) \rightarrow H_{2n}(K3, \mathbb{Z}) \oplus H_{2n}(K3, \mathbb{Z}) \),
\[
(134) \quad \iota(\alpha, \beta) = (\alpha - \beta, (g_*)(\alpha - \beta)).
\]
Consequently, \( \ker(\iota) \) contains exactly those pairs \( (\alpha, \alpha) \in H_{2n}(K3, \mathbb{Z}) \oplus H_{2n}(K3, \mathbb{Z}) \) for \( \alpha \in H_{2n}(K3, \mathbb{Z}) \). Similarly, the elements in \( (H_{2n}(K3, \mathbb{Z}) \oplus H_{2n}(K3, \mathbb{Z}))/\image(\iota) \) are equivalence classes \( [(\alpha, \beta)] \) of elements \( \alpha, \beta \in H_{2n}(K3, \mathbb{Z}) \) where each equivalence class has

\(^5\)There is some unimportant ambiguity here regarding how we define the isomorphisms between the homology of \( X, Y, X \cap Y \) and \( K3 \).
a representative of the form \((\alpha, 0)\), and \([(\alpha_1, 0)] = [(\alpha_2, 0)]\) if and only if \(\alpha_2 = (g_*)^m(\alpha_1)\) for some \(m \in \mathbb{N}\). Under the given isomorphisms,

\[
H_{2n+1}(M', \mathbb{Z}) \cong H_{2n}(K3, \mathbb{Z})^G \quad \text{and} \quad H_{2n}(M', \mathbb{Z}) \cong \Pi_G(H_{2n}(K3, \mathbb{Z})).
\]

Therefore, by the Künneth formula,

\[
H_{2n+1}((K3 \times T^2)/G, \mathbb{Z}) \cong H_{2n}(K3, \mathbb{Z})^G \oplus \Pi_G(H_{2n}(K3, \mathbb{Z}))
\]

and

\[
H_{2n+2}((K3 \times T^2)/G, \mathbb{Z}) \cong H_{2n}(K3, \mathbb{Z})^G \oplus \Pi_G(H_{2n+2}(K3, \mathbb{Z}))
\]

naturally. We deduce:

**Proposition 4.11.** The integral homology lattice \(H_*(M, \mathbb{Z})\) of \(M = (K3 \times T^2)/G\) is given by

\[
H_*(M, \mathbb{Z}) = \bigoplus_{n=0}^6 H_n(M, \mathbb{Z}),
\]

where \(H_n(M, \mathbb{Z})\) is as above.

The important point here is not the group structure conveyed by the previous result, but rather the geometric description of the homology elements given implicitly in the preceding analysis.
5. \( \mathcal{N} = 4 \) Compactification Revisited

We will now briefly revisit the specifics of four-dimensional \( \mathcal{N} = 4 \) compactifications of string theory, with the classification of threefolds of SU(2) holonomy in hand. There are only thirteen possible compactification manifolds up to diffeomorphism giving rise to a Type II theory with \( \mathcal{N} = 4 \) supersymmetry in four dimensions. These are of the form \((K3 \times T^2)/G\), where \( G \) is one of

\[
\{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_2^2, \mathbb{Z}_3^2, \mathbb{Z}_4^2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_6\}.
\]

(138)

So, consider Type II string theory compactified on \( M \cong K3 \times T^2/G \), where \( G \) is as above. As we have mentioned, the four-dimensional field content of the supergravity theory contains two types of multiplets, the supergravity multiplet and the matter multiplet. In order to give a geometric interpretation for the fields in these multiplets, it is useful to break the full \( R \)-symmetry by truncating the \( \mathcal{N} = 4 \) super-Poincaré algebra to \( \mathcal{N} = 2 \). As we saw in the first section, under this truncation the \( \mathcal{N} = 4 \) supergravity multiplet splits into a direct sum of one \( \mathcal{N} = 2 \) supergravity multiplet, one \( \mathcal{N} = 2 \) vector multiplet, and two \( \mathcal{N} = 2 \) gravitino multiplets,

\[
\mathcal{M}_{D=4,N=4}^{\text{supergravity}} = \mathcal{M}_{D=4,N=2}^{\text{supergravity}} \oplus 2 \mathcal{M}_{D=4,N=2}^{\text{gravitino}} \oplus \mathcal{M}_{D=4,N=2}^{\text{vector}},
\]

(139)

and the \( \mathcal{N} = 4 \) matter multiplet splits into a direct sum of one \( \mathcal{N} = 2 \) vector multiplet and one \( \mathcal{N} = 2 \) hypermultiplet,

\[
\mathcal{M}_{D=4,N=4}^{\text{matter}} = \mathcal{M}_{D=4,N=2}^{\text{vector}} \oplus \mathcal{M}_{D=4,N=2}^{\text{hyper}}.
\]

(140)

At generic points in moduli space, the field content of the four-dimensional supergravity theory consists of one supergravity multiplet and \( n \) matter multiplets, where \( n \) is determined by the topology of the compactification manifold:

\[
n = \text{rank}(H^2(K3, \mathbb{Z}))^G
\]

(141)

as before. Since each matter multiplet contains six scalars and the supergravity multiplet contains two scalars, the theory contains \( n_s = 2 + 6n \) scalars at generic points in moduli space. Since each matter multiplet contains one gauge fields and the supergravity multiplet contains six gauge field, the theory contains \( r_g = 6 + n \) gauge fields at generic points in moduli space, with corresponding gauge group \( U(1)^r_g \). At non-generic points in moduli space, otherwise massive gauge fields of the full string theory can become massless and consequently enter in the supergravity theory \([5][8]\).
Ten-dimensional Type IIA supergravity compactified on $K3 \times T^2$ is most easily understood as eleven-dimensional Type IIA supergravity compactified on $K3 \times T^2 \times S^1 = K3 \times T^3$, where the final factor of $S^1$ is assumed to be small. There are three special matter multiplets, corresponding to the three factors of $S^1$ in $T^3 = S^1 \times S^1 \times S^1$, which each contain the component of the metric corresponding to the volume of the corresponding circle. The special matter multiplet corresponding to the circle upon which we reduce the eleven-dimensional theory to obtain the ten-dimensional theory is the usual four-dimensional universal matter multiplet, containing the usual dilaton and the axion. The other two special matter multiplets have an analogous interpretation in the eleven-dimensional theory, which is obscured when we describe them using the ten-dimensional theory.

After the truncation to $\mathcal{N} = 2$ supersymmetry, in both the Type IIA and Type IIB theories, we end up with $n + 1$ vector multiplets, $n - 1$ non-universal hypermultiplets, 1 universal hypermultiplet, 1 supergravity multiplet, and 2 gravitino multiplets. Note that $n - 1 = h^{1,1}(M) = h^{2,1}(M)$, so that we end up with two more vector multiplets than we would expect from our experience with generic Calabi-Yau compactifications. We will provide a geometric interpretation of this splitting in the Type IIA case, using dimensional reduction in order to understand what happens. Analogous results can certainly be derived in the Type IIB case. We might expect that one of our additional vector multiplets is that which descends from the $\mathcal{N} = 4$ supergravity multiplet. For lack of a better label, I will call the resultant vector multiplet the gravivector multiplet. But, as we shall see, this vector is actually described by the $\mathcal{N} = 2$ formalism. Another possibility is that at least one of the additional vector multiplets originates in the matter multiplet containing the dilaton and axion. I will call this matter multiplet the universal matter multiplet and the descendant vector multiplet the universal vector multiplet. Actually, it seems that the additional vector multiplets come from matter multiplets involving the Ramond-Ramond one-form reduced on the two independent cycles of $M$. I will call these matter multiplets the special matter multiplets and the resultant vector multiplets special vector multiplets. Before proceeding further, let’s determine the relevant fields. We did this a tad abstractly in section 3, but we can now be a bit more concrete.

5.1. Scalar Fields. In both the Type IIA theory and the Type IIB theory,

- The moduli space of Einsteinian metrics on the internal manifold decomposes into the moduli space of $G$-invariant Einsteinian metrics on the $K3$ fiber, which is $3n - 8$
dimensional (6), and the moduli space of \( T^2 \), which is a three-dimensional space with two complex structure moduli and one Kähler modulus.

- Reducing the \( B \)-field on the internal torus base yields a single scalar \( B_{ij} \), while reducing the \( B \)-field on the internal K3 fiber, corresponding to the scalars \( B_{mn} \), yields \( n \) distinct scalars, corresponding to the \( n \) distinct cohomology classes of \( G \)-invariant two-forms on K3.
- The dilaton \( \Phi \) yields a single scalar.
- The axion \( d_4^{\dagger} \star_4 d_4 B \) yields a single scalar as well.

These comprise \( 4n - 2 \) distinct scalars. In Type IIA

- Reducing the Ramond-Ramond 1-form \( C^{(1)} \) on a cycle in \( T^2 \) yields two scalars \( C_i \).
- Reducing the Ramond-Ramond 3-form \( C^{(3)} \) on a \( G \)-invariant 2-cycle in the K3 fibers and a 1-cycle in \( T^2 \) yields \( 2n \) scalars \( C_{mnk} \).
- Reducing the spacetime-dual \( d_4^{\dagger} \star_4 d_4 C^{(3)} \) on either cycle in \( T^2 \) yields two scalars.

These comprise \( 2n + 4 \) scalars. In the Type IIB theory,

- The Ramond-Ramond 0-form \( C^{(0)} \) yields a single scalar.
- Reducing the Ramond-Ramond 2-form \( C^{(2)} \) on a \( G \)-invariant 2-cycle in a K3 fiber, corresponding the the terms \( C_{mn} \), yields \( n \) distinct scalars. Reducing \( C^{(2)} \) on the \( T^2 \) base yields a single scalar \( C_{ij} \).
- The spacetime-dual \( d_4^{\dagger} \star_4 d_4 C^{(2)} \), which has no internal indices, yields a single scalar.
- Reducing the Ramond-Ramond 4-form \( C^{(4+)} \) on the K3 fiber yields a single scalar. Reducing it on a self-dual 2-cycle of K3 and the \( T^2 \) base yields \( n/2 \) scalars.
- Reducing \( d_4^{\dagger} \star_4 d_4 C^{(4+)} \) on a \( G \)-invariant 2-cycle of a K3 fiber yields \( n/2 \) scalars.

These comprise \( 2n + 4 \) scalars as before, as required by \( T \)-duality. In total, in either theory, there are \( 2 + 6n \) scalars, justifying the assertion above that the theory contains \( n \) matter multiplets. In the preceding analysis, we implicitly used Proposition 4.11.

Approximate Type IIA descriptions of the scalar fields and their locations within the supermultiplets of the theory, arrived at by a combination of a bit of guesswork and the 11D to 10D reduction, are included in the following table, as well as their parities, since we

---

\(^6\)One of these dimensions parametrizes the manifold volume, while the others parameterize the Grassmannian of \( G \)-invariant 3-planes in \( H^2(K3, \mathbb{R})^G \otimes_{\mathbb{Z}} \mathbb{R} \). Therefore the moduli space is essentially \( \mathbb{R}_+ \times \text{SO}(3, n - 3)/\text{SO}(3) \times \text{SO}(n - 3) \).
have really been counting pseudoscalars with scalars.

<table>
<thead>
<tr>
<th>10D Field</th>
<th>Terms</th>
<th>Modes</th>
<th>$N = 4$ Multiplet</th>
<th>$N = 2$ Multiplet</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber Complex</td>
<td>$G_{mn}$</td>
<td>2($n - 3$)</td>
<td>Matter</td>
<td>Hyper</td>
<td>+1</td>
</tr>
<tr>
<td>Fiber Kähler</td>
<td>$G_{m\pi}$</td>
<td>$n - 3$</td>
<td>Matter</td>
<td>Vector</td>
<td>+1</td>
</tr>
<tr>
<td>Fiber Volume</td>
<td>$G_{m\pi}$</td>
<td>1</td>
<td>(Universal) Matter</td>
<td>(Universal) Vector</td>
<td>+1</td>
</tr>
<tr>
<td>Fiber $B$-field Mixed (-)</td>
<td>$B_{m\pi}$</td>
<td>$n - 3$</td>
<td>Matter</td>
<td>Vector</td>
<td>-1</td>
</tr>
<tr>
<td>Fiber $B$-field Mixed (+)</td>
<td>$B_{m\pi}$</td>
<td>1</td>
<td>(Universal) Matter</td>
<td>(Universal) Vector</td>
<td>-1</td>
</tr>
<tr>
<td>Base Kähler</td>
<td>$G_{i1}$</td>
<td>1</td>
<td>Supergravity</td>
<td>(Gravi)-Vector</td>
<td>+1</td>
</tr>
<tr>
<td>Base $B$-field</td>
<td>$B_{i1}$</td>
<td>1</td>
<td>Supergravity</td>
<td>(Gravi)-Vector</td>
<td>-1</td>
</tr>
<tr>
<td>Base Dilatons</td>
<td>$G_{i1}$</td>
<td>2</td>
<td>(Special) Matter (I,II)</td>
<td>(Special) Hyper</td>
<td>+1</td>
</tr>
<tr>
<td>Fiber $B$-field Pure</td>
<td>$B_{mn}, \overline{B_{mn}}$</td>
<td>2</td>
<td>(Special) Matter (I,II)</td>
<td>(Special) Vector</td>
<td>-1</td>
</tr>
<tr>
<td>Dilaton</td>
<td>$\Phi$</td>
<td>1</td>
<td>(Universal) Matter</td>
<td>(Universal) Hyper</td>
<td>+1</td>
</tr>
<tr>
<td>Axion</td>
<td>$d_4^I \star_4 d_4 B$</td>
<td>1</td>
<td>(Universal) Matter</td>
<td>(Universal) Hyper</td>
<td>+1</td>
</tr>
<tr>
<td>R-R 3-Form Pure</td>
<td>$C_{mnk}, \overline{C_{mnk}}$</td>
<td>2</td>
<td>(Universal) Matter</td>
<td>(Universal) Hyper</td>
<td>-1</td>
</tr>
<tr>
<td>R-R 3-Form Mixed (+)</td>
<td>$C_{mnk}, \overline{C_{mnk}}$</td>
<td>2</td>
<td>(Special) Matter (I,II)</td>
<td>(Special) Hyper</td>
<td>-1</td>
</tr>
<tr>
<td>R-R 3-Form Mixed (-)</td>
<td>$C_{mnk}, \overline{C_{mnk}}$</td>
<td>2($n - 3$)</td>
<td>Matter</td>
<td>Hyper</td>
<td>-1</td>
</tr>
<tr>
<td>R-R 1-Form</td>
<td>$C_1$</td>
<td>2</td>
<td>(Special) Matter (I,II)</td>
<td>(Special) Vector</td>
<td>+1</td>
</tr>
<tr>
<td>Dual R-R 3-Form</td>
<td>$(d_4^I \star_4 d_4 C^{(3)})_i$</td>
<td>2</td>
<td>(Special) Matter (I,II)</td>
<td>(Special) Hyper</td>
<td>+1</td>
</tr>
</tbody>
</table>

The scalars parametrize the scalar manifold

$$\mathcal{M}_{\text{scalar}}(G) = \text{SO}(6, n)/(\text{SO}(6) \times \text{SO}(n)) \times \mathbb{H},$$

where $\mathbb{H} \cong \text{SL}(2, \mathbb{R})/\text{SO}(2, \mathbb{R})$ is the upper-half of the complex plane. The scalar manifold $\mathcal{M}_{\text{scalar}}(G)$ can be considered as the $G$-invariant submanifold of the scalar manifold $\mathcal{M}_{\text{scalar}}(1)$ of the unorbifolded theory. Here the supergravity multiplet moduli $\tau$ parametrizes $\mathbb{H}$ and the remaining scalars, lying within the matter multiplets, parametrize the Grassmannian factor $\text{SO}(6, n)/(\text{SO}(6) \times \text{SO}(n))$. As we can see in the table above, the supergravity multiplet modulus $\tau \in \mathbb{H}$ has a pleasant geometrical interpretation in terms of the fibration interpretation of $(K3 \times T^2)/G$. In Type IIA, $\tau$ is the complexified Kähler modulus of the base torus. In Type IIB, $\tau$ is the complex structure modulus of the base torus. The matter multiplet moduli can also be encoded in an enlightening form. Let $L$ be the inner product on $H^*(K3, \mathbb{Z})^G \oplus H^*(T^2, \mathbb{Z})^G$ inherited from the intersection products on $H^*(K3, \mathbb{Z})$ and $H^*(T^2, \mathbb{Z})$. We can encode the matter multiplet moduli within a $(n + 6) \times (n + 6)$ matrix $M$ such that

$$MLM^T = L \quad \text{and} \quad M = M^T.$$
These moduli can also be encoded in the frame $\mu$, a $(n + 6) \times (n + 6)$ matrix such that $M = \mu^T \mu$. The choice in frame is – as usual – somewhat arbitrary. The matter multiplet moduli can be viewed as the Narain moduli of the dual heterotic theory.

5.2. Gauge Fields. We can also interpret the gauge fields in terms of the geometry of the K3 fibration (over $T^2$) in question, given a truncation of our $(N = 4)$ super-Poincaré algebra to the $(N = 2)$ super-Poincaré algebra. On the Type IIA side,

- The gauge fields in the $(N = 4)$ matter multiplets originate in the Ramond-Ramond 3-form $C^{(3)}$, specifically the components $C^{(3)}_{\mu m n}$ with $m, n$ internal indices corresponding to the K3 fiber.
- The one gauge field in the $(N = 4)$ supergravity multiplet which ends up in a $(N = 2)$ vector multiplet after truncation also comes from $C^{(3)}$, but with the internal indices corresponding to the $T^2$ base.
- The one gauge field in the $(N = 4)$ supergravity multiplet which ends up in the $(N = 2)$ supergravity multiplet after truncation comes from the Ramond-Ramond one form, $C^{(1)}$, which has no internal indices ($^7$).
- The remaining four gauge fields in the $(N = 4)$ supergravity multiplet, which end up in the $(N = 2)$ gravitino multiplets after truncation, come from the $B$-field reduced on either circle in the $T^2$ base and the Kaluza-Klein gauge fields $G_{\mu i}$ on either circle in the $T^2$ base.

The Type IIB side seems a bit more complicated.

As before, we implicitly used Proposition 4.11. The CHL electromagnetic charge lattice $\Lambda_{EM} = \Lambda_{EM}(G)$ is given by $\Lambda_{EM} = \Lambda_E(G) \oplus \Lambda_M(G)$ where in Type IIA

$$\Lambda_E \otimes \mathbb{C} = \begin{cases} H_0(K3 \times T^2/G) \oplus H_1(K3 \times T^2/G) \oplus H_2(K3 \times T^2/G, ) , \\ \star \circ C^{(3)}_{\mu i j} \end{cases}$$

$$\Lambda_M \otimes \mathbb{C} = \begin{cases} H_0(K3 \times T^2/G) \oplus H_5(K3 \times T^2/G) \oplus H_4(K3 \times T^2/G) , \\ \star \circ B_{\mu i} \end{cases}$$

where we labeled the gauge fields under which the given sublattices are charged and $\star \circ \mathbb{C}^{(3)}$ denotes the internal – that is, with respect to the compactification manifold – Hodge star.

---

$^7$There is some apparent disagreement between [17] and [18][5] in this regard. The former claims it might be the case that we really need to correct this naïve guess by adding some linear combination of the other gauge fields.
operation. On the Type IIB side

\[
\begin{align*}
\Lambda^E \otimes \mathbb{C} & = \mathbb{H}_1(K3 \times T^2/G) \oplus \mathbb{H}_1(K3 \times T^2/G) \oplus \mathbb{H}_3(K3 \times T^2/G), \\
\Lambda^M \otimes \mathbb{C} & = \mathbb{H}_5(K3 \times T^2/G) \oplus \mathbb{H}_5(K3 \times T^2/G) \oplus \mathbb{H}_3(K3 \times T^2/G). 
\end{align*}
\]

The lattices \( \Lambda^E \) and \( \Lambda^M \) are as above, but with integral homology. The \( \pm \) on the final terms are to distinguish self-dual versus anti-self-dual forms.

5.3. Charge Lattices. Let \( G \cong \mathbb{Z}_N \times \mathbb{Z}_M \), where \( N, M \) can be equal to one. In the previous section we described the topological interpretation and group structure of the electromagnetic charge lattice \( \Lambda_{EM} = \Lambda_{EM}(G) \). In this section we will be a bit more explicit and give the lattice structure. The electromagnetic charge lattice is a direct sum of the electric charge lattice and the magnetic charge lattice: \( \Lambda_{EM} = \Lambda^E \oplus \Lambda^M \).

On the Type IIA side, the lattice of magnetic charges \( \Lambda^M = \Lambda^M(G) \) is given by

\[
\Lambda^M \cong \Gamma^{1,1}(N) \oplus \Gamma^{1,1}(M) \oplus H^*(K3, \mathbb{Z})^G
\]

while lattice of electric charges \( \Lambda^E = \Lambda^E(G) \) is given by its dual \( \Lambda^E \cong \Lambda^*_M \). More concretely,

\[
\Lambda^E \cong \Gamma^{1,1}(1/N) \oplus \Gamma^{1,1}(1/M) \oplus (H^*(K3, \mathbb{Z})^G)^*.
\]

Since \( (H^*(K3, \mathbb{Z})^G)^* \cong \Pi_G(H^*(K3, \mathbb{Z})) \) naturally, this is consistent with the earlier statement. On the Type IIB side, the lattice of magnetic charges \( \Lambda^M = \Lambda^M(G) \) is given by

\[
\Lambda^M \cong \Gamma^{1,1}(N) \oplus \Gamma^{1,1}(1/M) \oplus \Pi_{\mathbb{Z}_M} H^*(K3, \mathbb{Z})^{\mathbb{Z}_N}
\]

while lattice of electric charges \( \Lambda^E = \Lambda^E(G) \) is given by its dual \( \Lambda^E \cong \Lambda^*_M \). More concretely,

\[
\Lambda^E \cong \Gamma^{1,1}(1/N) \oplus \Gamma^{1,1}(M) \oplus \Pi_{\mathbb{Z}_N} H^*(K3, \mathbb{Z})^{\mathbb{Z}_M}.
\]

The results provided in the table in the previous section provide an explicit description of the charge lattices in the Type IIA case. Note that if \( M \) or \( N \) is 1, then up to electric-magnetic duality the lattices agree on the Type IIA and Type IIB side, as required by T-duality. When \( N \) and \( M \) are both greater than 1, then the charge lattices in the Type IIA and Type IIB theories disagree, which is consistent because orbifolding on both circles of \( T^2 \) prevents us from appealing to T-duality.
Charges \((q,p) \in \Lambda_{\text{EM}}\) satisfy the following charge quantization condition:
\[
MNq^2 \in 2\mathbb{Z} \quad \text{and} \quad p^2 \in 2\mathbb{Z} \quad \text{and} \quad q \cdot p \in \mathbb{Z}.
\]
I would just like to say here that the factors on the first term are a source of much grief.

We can encode certain features of the charges in the binary form
\[
Q_{q,p} = \frac{1}{2} \left( q^2 \quad q \cdot p \quad p^2 \right)
\]
where the inner products are to be interpreted within the ambient lattice \(\Lambda_{\text{EM}}(1)\).

5.4. **Dualities and Moduli Spaces.** We will finish this section by noting some facts about the dualities and moduli spaces of these theories, a topic first investigated by Chaudhuri and Polchinski in \([24]\). This is a very complex topic, as shown in \([9]\), and we will just scratch the surface.

5.4.1. **Descended Duality.** The moduli space of the unorbifolded theory descends to the moduli space of the orbifolded theory, up to a quotient by additional discrete dualities which arise after orbifolding. The descendant \(T\)-duality group \(T_0(\mathbb{Z})\) is defined to contain exactly those elements of \(SO(\Gamma^6,22)\) which commute with the action of \(G\), and the descendant \(S\)-duality group \(S_0(\mathbb{Z})\) is defined to contain those elements of \(SL(2,\mathbb{Z})\) which commute with the action of \(G\). The descendant \(U\)-duality group is \(U_0(\mathbb{Z}) = T_0(\mathbb{Z}) \times S_0(\mathbb{Z})\). In general, the full \(U\)-duality group of our theory will be larger than \(U_0(\mathbb{Z})\).

Essentially, the subgroup of the restrictions to \(\Lambda^G = \Lambda^G_{K3}\) of elements of \(SO(\Gamma^6,22)\) which commute with \(G\) is equal to the group \(SO(\Lambda^G)\), where \(\Lambda^G = H^*(K3,\mathbb{Z})^G \oplus H^*(T^2,\mathbb{Z})^G\) \([13]\). Therefore \(T_0(\mathbb{Z}) \cong SO(\Lambda^G)\). Furthermore \(S_0(\mathbb{Z}) \cong \Gamma_1(N) \cap \Gamma_1(M)^T\). So, the moduli space of the orbifolded theory \(\mathcal{M}_{K3 \times T^2/G}\) is given as the quotient of
\[
SO(\Lambda^G) \backslash SO(6,n)/(SO(6) \times SO(n)) \times \mathbb{H}/\Gamma_1(N) \cap \Gamma_1(M)^T.
\]
by the group of duality transformations which arise post-orbifold. Ignoring this technicality,
\[
\mathcal{M}_{K3 \times T^2/G} \cong SO(\Lambda^G) \backslash SO(6,n)/(SO(6) \times SO(n)) \times \mathbb{H}/\Gamma_1(N) \cap \Gamma_1(M)^T.
\]

5.4.2. **Action on Charges.** An element \(\Upsilon\) in the \(T\)-duality group acts on the matter multiplet moduli space as \(\Upsilon : M \mapsto \Upsilon M \Upsilon^T\) and on the charges as
\[
\Upsilon : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} \Upsilon q \\ \Upsilon p \end{pmatrix}.
\]
So, the electric and magnetic charges transform as vectors under the $T$-duality group. A matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \cap \Gamma_1(M)^T \) acts on the scalar \( \tau \) and the electric and magnetic charges as

\[
\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} +d & -b \\ -c & +a \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.
\]

Note that it is the inverse of the given matrix which shows up.
6. Black Holes in $\mathcal{N} = 4$ Supergravity

Given the models described in the previous two sections, it is natural to ask how simple physical systems look within them. This section will investigate exactly this question: what do black holes look like in CHL models of string theory? This question is in general quite complicated, as a generic black hole state will tend to break the supersymmetries of the ambient theory. We will focus on black holes which preserve a large degree of supersymmetry, namely half-BPS black holes. These are extremal. In the first part of this section, we will review some facts regarding extremal Reissner-Nordström black holes in Einstein-Maxwell theory, largely taken from [25][26]. In the second part of this section, following Moore [27][28], we will analyze the attractor equations describing the near horizon geometry of such black holes, and consequently determine the resultant attractor geometries. In the final part of this section, following Kachru and Tripathy [10][29], we will define a particular count of attractor geometries, and remark on one of their peculiar arithmetic properties.

6.1. Review of Reissner-Nordström. In this section we will very briefly review some properties of extremal black-holes in four-dimensional spacetime in the context of classical supergravity. An extremal black hole, in the near-horizon limit, develops an $\text{AdS}_2$ throat, which means that the near-horizon geometry is described by the metric

$$
\text{(154)}
\begin{align*}
\text{ds}^2 &= a^2 \text{ds}^2_{\text{AdS}_2} + r^2 \text{ds}^2_{S^2}
\end{align*}
$$

where

$$
\text{(155)}
\begin{align*}
\text{ds}^2_{\text{AdS}_2} &= -r^2 \text{dt}^2 + \frac{\text{dr}^2}{r^2}
\end{align*}
$$

is the metric for $\text{AdS}_2$ and $\text{ds}^2_{S^2}$ is the metric for $S^2 \subset \mathbb{R}^3$. Recall that anti-de Sitter space $\text{AdS}_2$ can be defined as a quasi-sphere in $\mathbb{R}^{2,1}$, where $\mathbb{R}^{2,1}$ is the three-dimensional space with metric

$$
\text{(156)}
\begin{align*}
\text{ds}^2 &= \text{dx}^2 - \text{dt}^2_1 - \text{dt}^2_2.
\end{align*}
$$

That is, $\text{AdS}_2$ consists of the points $(x, t_1, t_2) \in \mathbb{R}^{2,1}$ satisfying $x^2 - t_1^2 - t_2^2 = -1$. The type of extremal black hole with which we concern ourselves is the Reissner-Nordström black hole with critical mass.

6.1.1. The Reissner-Nordström Black Hole. Rather then dividing headfirst into Reissner-Nordström black holes in supergravity, with charges A through Z, we review the Reissner-Nordström black hole in classical Einstein-Maxwell theory in $(3 + 1)$-dimensions, in which...
there is only a single gauge field under which things can be charged. The action and
Lagrangian density are given by

\[ S = \int d^4x \sqrt{-g} \mathcal{L} \quad \text{and} \quad \mathcal{L} = \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

respectively. The Reissner-Nordström solution is given by the Reissner-Nordström metric and corresponding electromagnetic field strength. The metric is given by

\[ ds^2 = -\Delta(r) \, dt^2 + \Delta(r)^{-1} \, dr^2 + r^2 d\Omega^2, \]

where \( d\Omega^2 = r^2 \left( d\theta^2 + \sin^2(\theta) \, d\phi^2 \right) \) is the usual spherical metric and

\[ \Delta(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right) = 1 - \frac{2GM}{r} + \frac{G}{4\pi r^2} (Q^2 + P^2). \]

Here \( r_\pm \) are the radii at which \( \Delta(r) \) vanishes, corresponding to the inner and outer horizons of the black hole. By the quadratic formula,

\[ r_\pm = GM \pm \sqrt{G^2 M^2 - \frac{G}{4\pi} (Q^2 + P^2)}. \]

The nonzero components of the electromagnetic field strength are

\[ F_{\rho t} = \frac{Q}{4\pi r^2} \quad \text{and} \quad F_{\theta\phi} = \frac{P}{4\pi} \sin(\theta). \]

The black hole is extremal when \( M^2 = (Q^2 + P^2)/4\pi G \), that is when \( r_\pm = R \) are both equal to \( GM \). Perform the coordinate change \( \tau = \lambda t/R^2 \) and \( \rho = \lambda^{-1} (r - R) \). In these coordinates, the metric and field strengths are given by

\[ ds^2 = -\frac{\rho^2 R^4}{(R + \lambda \rho)^2} \, d\tau^2 + \frac{(R + \lambda \rho)^2}{\rho^2} \, d\rho^2 + (R + \lambda \rho)^2 \left( d\theta^2 + \sin^2(\theta) \, d\phi^2 \right), \]

\[ F_{\rho\tau} = \frac{QR^2}{4\pi (R + \lambda \rho)^2}, \quad F_{\theta\phi} = \frac{P}{4\pi} \sin(\theta). \]

The near horizon limit is the limit \( \lambda \to 0 \), in which

\[ ds^2 = R^2 \left( -\rho^2 \, d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + R^2 \left( d\theta^2 + \sin^2(\theta) \, d\phi^2 \right), \]

\[ F_{\rho\tau} = \frac{Q}{4\pi}, \quad F_{\theta\phi} = \frac{P}{4\pi} \sin(\theta). \]

Therefore, the near horizon geometry is isometric to a product \( \text{AdS}_2 \times S^2 \), with the horizon at \( \rho = 0 \) having area \( A = 4\pi R^2 \). In terms of the charges,

\[ \frac{A}{4G} = \frac{G}{4\pi} (Q^2 + P^2). \]
The existence of the AdS$_2$ throat implies an associated horizon isometry group SO(2, 1) $\times$ SO(3), which is an enhancement of the SO(3) isometry group of the full solution.

The story in supergravity is similar [26], although we must of course take into account the additional fields. One important class of black holes is that of small black holes, which have vanishing horizon area in the supergravity approximation. This singularity is resolved when higher order terms in the curvature tensor are taken into account [30][31][32][33].

6.2. Attractor Geometries. In supergravity, the attractor mechanism fixes some components of the near horizon geometry of a supersymmetric black hole in terms of its charge. When the supergravity under consideration is Type II supergravity with $\mathcal{N} = 4$ supersymmetries, corresponding to Type II string theory compactified on $(K3 \times T^2)/G$, where $G$ is one of the thirteen groups identified by Nikulin, the resultant attractor geometry possesses additional algebraic structure, and the collection of all attractor geometries possesses additional arithmetic structure. When $G = 1$, this algebraic structure was first identified by Moore in [28] and this arithmetic structure was first identified by Kachru and Tripathy in [29][10]. Following them, this section classifies the CHL attractor geometries in the case when $G$ is cyclic of prime order, and in the process slightly extends the analysis in [28] and [10] to this case. So, in all that follows, let $G \in \{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8\}$ and $\mathcal{N} = |G|$.

In $\mathcal{N} = 2$ supersymmetry, an attractor geometry corresponding to a charge $\gamma \in \Lambda_{\text{EM}}$ is one which locally maximizes the black hole central charge

$$|Z(\Omega; \gamma)|^2 = \frac{1}{i \int \Omega \wedge \bar{\Omega}} \left| \int_{\gamma} \Omega \right|^2 = \frac{1}{i \int \Omega \wedge \bar{\Omega}} \left| \int_M \Omega \wedge \bar{\gamma} \right|^2,$$

where $\Omega : \mathcal{M} \to H^3(M)$ is the canonical $(3,0)$-form implicitly dependent on moduli. Actually this doesn’t make literal sense, as the moduli only specify a line bundle in which the canonical $(3,0)$-form lies, but the central charge factors through to a function of this line bundle. The moduli space $\mathcal{M}$ can be written as a union of the closures of pairwise disjoint sets $U_i$, $\mathcal{M} = \cup_i \overline{U}_i$ such that if the scalar fields in the theory take asymptotic values at infinity within $U_i$, there is a unique corresponding attractor point to which the scalar fields radially flow. A similar picture holds in $\mathcal{N} = 4$ supersymmetry, although the relation between the attractor point and the central charges is now a bit more complicated. Note that, in the $\mathcal{N} = 2$ case, the attractor value $\Omega^*(\gamma)$ of the canonical $(3,0)$-form depends only on $\text{span}_\mathbb{R}\{\gamma\} \subseteq \Lambda_{\text{EM}} \otimes_\mathbb{Z} \mathbb{R}$. The same is true within $\mathcal{N} = 4$ supersymmetry, and this is most easily seen from the version of the attractor equations in [31]. Consequently, rescaling the given charge does not change the corresponding attractor point. We will consider only charges that support BPS states, and consequently admit at least one attractor value.
We can consider the moduli space $\mathcal{M}$ as the $G$-invariant portion of the moduli space of the unorbifolded theory and can consider a suitable rescaling of the electromagnetic charge lattice $\Lambda_{\text{EM}}$ as lying within the $G$-invariant portion of the electromagnetic charge lattice of the unorbifolded theory. Given a charge vector $\gamma \in \Lambda_{\text{EM}}$, we can consider the corresponding attractor points in $\mathcal{M}$, those moduli which within $\mathcal{M}$ locally maximize the central charge, or the corresponding attractor points in the unorbifolded theory. I claim that these coincide.

The classification of attractor points in the unorbifolded theory includes the observation that given a charge supporting at least one BPS state, there is a unique corresponding attractor point. This attractor point is specified by a few explicit equations, given in $[28][27]$ for example. From these equations it can be seen that if the charge $\gamma$ is $G$-invariant, then $\Omega^* (\gamma)$ is $G$-invariant and consequently corresponds to the attractor point of the orbifolded theory corresponding to $\gamma \in \Lambda_{\text{EM}}$. Conversely, given a $G$-invariant $\Omega \in H^3(M)$ and $G$-invariant $\gamma$ and central charge $Z$, $Z(\Omega + \delta\Omega; \gamma) = Z(\Omega; \gamma)$ for any $\delta\Omega \in (H^3(M)^G)^\perp \subseteq H^3(M)$. This follows from the observation that given these quantities, for any $g \in G$

$$\int_M \delta\Omega \wedge \hat{\gamma} = 0 \quad \text{and} \quad \int_M \delta\Omega \wedge \overline{\Omega} = \int_M \Omega \wedge \overline{\delta\Omega} = 0.$$ 

Since any $\delta\Omega \in H^3(M)$ can be decomposed as $\delta\Omega_{\parallel} + \delta\Omega_{\perp}$ for some $\delta\Omega_{\parallel} \in H^3(M)^G$ and some $\delta\Omega_{\perp} \in (H^3(M)^G)^\perp$, an attractor point in the orbifolded theory lifts to an attractor point in the unorbifolded theory. This proves the claim. Consequently:

**Proposition 6.1.** Let $\Lambda'_{\text{EM}}$ denote those charges supporting BPS states. The following diagram commutes:

$$\Lambda'_{\text{EM}}(G) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow H^3(M)$$

where the vertical maps are the natural inclusions and the horizontal maps send charges to the unique attractor forms $\Omega^*$ in the corresponding theory. Alternatively, the diagram

$$\Lambda'_{\text{EM}}(G) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathcal{P}(\mathcal{M}(G))$$

commutes, where $\mathcal{P}(\mathcal{M}(G))$ is the power set of $\mathcal{M}(G)$, $\mathcal{P}(\mathcal{M}(\mathbf{1})^G)$ is the power set of the $G$-invariant subspace of $\mathcal{M}(\mathbf{1})$, and the horizontal maps send charges to the subvariety of $G$-invariant moduli space solving the attractor equations. ■
Remark 6.2. It is important to restrict attention to the $G$-invariant portion of moduli space in the second half of the preceding proposition. The hypermultiplet moduli in the unorbifolded theory are not fixed by the attractor mechanism. Since we have more hypermultiplet moduli in the unorbifolded theory, we find some moduli values solving the attractor equation which are not $G$-invariant.

Fix $G$, and let $n$ be the rank of the lattice $H^2(K3,\mathbb{Z})^G$ as before. The Narain moduli $\Upsilon \in \text{SO}(n,6)/(\text{SO}(n) \times \text{SO}(6))$ determine an orthogonal decomposition $\mathbb{R}^{n,6} = \mathbb{R}^{n,0} \oplus \mathbb{R}^6$, under which electric and magnetic charge vectors $p, q \in \mathbb{R}^{n,6}$ decompose as $p = p^L + p^R$ and $q = q^L + q^R$, where $p^L, q^L \in \mathbb{R}^{n,0}$ and $p^R, q^R \in \mathbb{R}^6$. The following version of the attractor equations is given by Moore \cite{28} in the unorbifolded case, and we will assume that an analogous result holds here.

Proposition 6.3. Given a black hole of charge $(p, q) \in \Lambda_{\text{EM}}$, $\Upsilon^*$ solves the attractor equations if and only if it determines an orthogonal decomposition such that $p^L$ and $q^L$ are both zero.

The subvariety $\mathcal{V}(p, q) \subset \text{SO}(n,6)/(\text{SO}(n) \times \text{SO}(6))$ of Narain moduli $\Upsilon$ which satisfy this equation is $4n$-dimensional. This is exactly the number of hypermultiplet degrees of freedom, which then locally parametrize this variety. This is the $\mathcal{N} = 4$ analogue of the statement that the $\mathcal{N} = 2$ attractor mechanism does not fix hypermultiplet degrees of freedom. Following Moore, we give a Type IIB geometric interpretation in the unorbifolded case of the attractor equations in terms of the fields of Type IIB supergravity. For shorthand, let $\text{Gr}_+(k, V)$ be the space of spacelike $k$-planes in a real vector space $V$ with quadratic form of signature $(r, k)$. We utilize the following result:

Theorem 6.4 \cite{28}. Suppose that $e, e^*$ together span a hyperbolic plane in $\mathbb{R}^{q+1, p+1}$ and satisfy $e^2 = 0 = e^{*2}$. There exists a diffeomorphism

$$
\Phi_{e,e^*} : \text{Gr}_+(p + 1, \mathbb{R}^{q+1, p+1}) \to \text{Gr}_+(p, \text{span}_\mathbb{R}\{e\}^\perp/\text{span}_\mathbb{R}\{e\}) \times \text{span}_\mathbb{R}\{e, e^*\}^\perp \times \mathbb{R}_+.
$$

This theorem, whose proof is rather straightforward and computational, can be used to describe certain Grassmannians in terms of smaller Grassmannians. Fix some hyperbolic plane $\langle e_1, e_1^* \rangle$ within $H^2(K3, \mathbb{R})$, which yields an orthogonal decomposition of $H^2(K3, \mathbb{R})$ as $\mathbb{R}^{18,2} \oplus \text{span}_\mathbb{R}\{e_1, e_1^*\}$. The preceding result with $p = 2$ gives a diffeomorphism

$$(168) \quad \text{Gr}_+(3, H^2(K3, \mathbb{R})) \to \text{Gr}_+(2, \text{span}_\mathbb{R}\{e_1\}^\perp/\text{span}_\mathbb{R}\{e_1\}) \times \text{span}_\mathbb{R}\{e_1, e_1^*\}^\perp \times \mathbb{R}_+.$$
The left-hand side of the preceding equivalence is the moduli space of Ricci flat metrics on K3 [8]. Let the associated 3-plane be denoted $\Sigma$. We can decompose $\Sigma = \Xi \oplus K$, where $\Xi$ is a spacelike 2-plane with $\Xi = \text{span}_\mathbb{R}\{\text{Re} \Omega, \text{Im} \Omega\}$. Therefore $\Xi$ determines a complex structure such that $e_1 \in H^{1,1}(K3)$. Now choose another $e_2, e_2^*$ spanning a hyperbolic plane with $e_2^2 = e_2^{*2} = 0$ such that $H^*(K3, \mathbb{R}) \cong H^2(K3, \mathbb{R}) \oplus \text{span}_\mathbb{R}\{e_2, e_2^*\}$ is an orthogonal decomposition. Given a spacelike 4-plane $\Pi \in \text{Gr}_+(4, H^*(K3, \mathbb{R}))$, we choose an orthogonal decomposition $\Pi = \Sigma' \oplus (Ve_2 + e_2^* + B)$ where $B \in H^2(K3, \mathbb{R})$ represents the B-field and $V$ represents the volume of the K3. The spacelike 3-plane $\Sigma'$ does not lie within $H^2(K3, \mathbb{R})$, but does uniquely determine such a $\Sigma'$. Now choose $e_3, e_3^*$ spanning a hyperbolic plane with $e_3^2 = e_3^{*2} = 0$ such that $\mathbb{R}^{21,5} \cong H^*(K3, \mathbb{R}) \oplus \text{span}_\mathbb{R}\{e_3, e_3^*\}$ is an orthogonal decomposition. We then decompose $\Theta \in \text{Gr}_+(5, \mathbb{R}^{21})$ as $\Theta = \Pi' \oplus \text{span}_\mathbb{R}\{\alpha e_3 + e_3^* + C\}$, where the decomposition is orthogonal. Here $C \in H^*(K3, \mathbb{R})$ are the IIB RR forms, while $\alpha = e^{-\phi}$ is the dilaton. The spacelike 4-plane $\Pi'$ does not lie within $H^*(K3, \mathbb{R})$, but does uniquely determine such a $\Pi$. We finally consider IIB on K3 $\times T^2$. We decompose $\mathbb{R}^{6,22} = \mathbb{R}^{5,21} \oplus \text{span}_\mathbb{R}\{e_4, e_4^*\}$, where the given decomposition is orthogonal. An element $\Upsilon \in \text{Gr}_+(6, \mathbb{R}^{6,22})$ decomposes as $\Upsilon = \Theta' \oplus (ae_4 + e_4^* + A)\mathbb{R}$, where $A \in \mathbb{R}^{21,5}$ corresponds to the Wilson line around a fixed cycle in $T^2$ while the factor $a$ corresponds to the area of $T^2$. We now reverse the logic. Given $p, q \in \mathbb{R}^{6,22}$, we pick vectors $e_1, e_1^*, e_2, e_2^*, e_3, e_3^*, e_4, e_4^*$ yielding an orthogonal decomposition

\begin{equation}
\mathbb{R}^{22,6} = \mathbb{R}^{18,2} \oplus \text{span}_\mathbb{R}\{e_1, e_1^*\} \oplus \text{span}_\mathbb{R}\{e_2, e_2^*\} \oplus \text{span}_\mathbb{R}\{e_3, e_3^*\} \oplus \text{span}_\mathbb{R}\{e_4, e_4^*\}
\end{equation}

so that $p, q$ lie within the factor of $\mathbb{R}^{18,2}$. By the geometric interpretation above, this implies that [28]

\begin{equation}
\Pi^{2,0} B = \Pi^{2,0} C^2 = \Pi^{2,0} \int_{T^2} C^4 = 0,
\end{equation}

where $\Pi^2 : H^2(M) \to H^{2,0}(M)$ is projection. The attractor equations regarding the Narain moduli are supplemented by the final attractor equation, given by Moore in the unorbifolded case in [28]. We will assume that the following analogous result holds more generally:

**Proposition 6.5.** The attractor value $\tau_* = \tau(p, q)$ of the supergravity multiplet modulus $\tau \in \mathbb{H}$ solves the equation $Np^2 \tau_*^2 - 2p \cdot q \tau_* + Nq^2 = 0$. Consequently

\[\tau_* = \frac{p \cdot q + i \sqrt{-D(p, q)}}{Np^2},\]

where $D(p, q) = (p \cdot q)^2 - N^2p^2q^2 < 0$.  \[\blacksquare\]
It should be remarked that appropriate rescalings of the charge lattices in question, while keeping independent the relation between the electric and magnetic charge lattices, can make this true. Alternatively, rather than consider the electric and magnetic charge lattices sitting within some ambient lattice, we can imagine them separately, and then rescale the action of one on another. This rescaling is ad hoc, and not well understood, but seems reasonable on the basis that it also appears in [12] in some form.

6.3. Counting Attractors. Given a charge \((p, q) \in \Lambda_{\text{EM}}\) with \(D(p, q) < 0\), the attractor value \(\tau_*(p, q)\) of the supergravity multiplet scalar \(\tau\) is the unique root of the quadratic \(f_{p,q}(\tau) = (N/2)p^2\tau^2 - p \cdot q\tau + (N/2)q^2\) in the upper half-plane, assuming the preceding proposition. In particular, \(\tau_*\) is a quadratic integer in the upper half plane, and the discriminant \(D(p, q)\) of the charge is exactly the discriminant of \(f_{p,q}\). The quadratic \(f_{p,q}\) is naturally associated with a quadratic form, namely

\[
Q_{p,q} = \frac{1}{2} \begin{pmatrix} Np^2 & p \cdot q \\ p \cdot q & Nq^2 \end{pmatrix}
\]

If we rescale the inner products on \(\Lambda_E\) and \(\Lambda_M\) appropriately, then this transforms by conjugation under \(S\)-duality. We are interested in counting \(U\)-duality classes of attractor values \(\tau_*(p, q)\) for \((p, q) \in \Lambda_{\text{EM}}(G)\). We will grade by discriminant, which is a \(U\)-duality invariant. So, consider

\[
N(D) = \#\{[\tau_*(p, q)]_{U(\mathbb{Z})} : (p, q) \in \Lambda_{\text{EM}}(G) \text{ with } D(p, q) = D\}
\]

for \(D < 0\), where \([\tau_*(p, q)]_{U(\mathbb{Z})}\) is the \(U\)-duality class of \(\tau_*(p, q)\), or equivalently

\[
N(D) = \#\{[Q_{p,q}]_{U(\mathbb{Z})} : (p, q) \in \Lambda_{\text{EM}}(G) \text{ with } D(p, q) = D\}
\]

Every element in the duality group \(U(\mathbb{Z})\) acts on the supergravity multiplet scalar manifold \(\mathbb{H}\) by automorphisms. Recall that \(\text{Aut}(\mathbb{H}) \cong \text{PSL}_2(\mathbb{R})\), with an explicit isomorphism given by identifying elements of \(\text{SL}_2(\mathbb{R})\) with the corresponding linear fractional transformations. The given action then induces a homomorphism \(U(\mathbb{Z}) \to \text{PSL}_2(\mathbb{R})\). Let \(S(\mathbb{Z})\) be the image of this homomorphism. By construction, naturally \(S(\mathbb{Z}) \cong U(\mathbb{Z})/K\), where \(K \subseteq U(\mathbb{Z})\) is the kernel of the given homomorphism. By construction,

\[
N(D) = \#\{[\tau_*(p, q)]_{S(\mathbb{Z})} : (p, q) \in \Lambda_{\text{EM}}(G) \text{ with } D(p, q) = D\}
\]

and

\[
N(D) = \#\{[Q_{p,q}]_{S(\mathbb{Z})} : (p, q) \in \Lambda_{\text{EM}}(G) \text{ with } D(p, q) = D\}.
\]
Figure 4. A fundamental domain $\mathcal{F}$ for $\text{SL}_2(\mathbb{Z})$, given explicitly in Appendix A. The red point is on a conical singularity of order three, and so is counted with a factor of $1/3$ in $H(D)$. The blue point is on a conical singularity of order two, and so is counted with a factor of $1/2$ in $H(D)$. The green point is a generic point in $\mathcal{F}$, where there is no singularity, and so is counted without an extra factor.

Consider

$$H(D) = \sum_{[\tau(\gamma)]_{\mathcal{F}}(\mathbb{Z})} \frac{1}{\# \text{Stab}_{\mathcal{F}}(\tau(\gamma))} = \sum_{[\tau(\gamma)]_{\mathcal{F}}(\mathbb{Z})} \frac{1}{\# \text{Stab}_{\mathbb{Z}}(\tau(\gamma))}$$

where the sum is over all equivalence classes of attractor points of charges $\gamma$ such that $D(\gamma) = D$. We should consider $H(D)$ as $N(D)$ suitably modified. We can arrange the constants $H(D)$ into the generating function

$$Z(q) = \sum_{D=1}^{\infty} H(-D)q^D.$$  

While the definition of $H(D)$ and $N(D)$ presented in the preceding section was purely algebraic, we can also present a more geometric notion. Consider the orbifold $\mathcal{F} = \mathbb{H}/\mathbb{S}(\mathbb{Z})$. By definition, every $\tau \in \mathbb{H}$ has a unique representative of its duality class in $\mathcal{F}$. The count $N(D)$ then counts the number of elements in $\mathcal{F}$ with discriminant $D$, while $H(D)$ counts the number of points in $\mathcal{F}$ with discriminant $D$ weighted by the inverse of the order of the conical singularity there (Fig. 4).
6.3.1. The Unorbifolderd Theory. As an example, let us consider the case when \( G = 1 \), that is the unorbifolded theory. Then \( S(\mathbb{Z}) = \text{PSL}_2(\mathbb{Z}) \) and \( H(D) \) is the Hurwitz class number corresponding to discriminant \( D \), counting \( \text{SL}_2(\mathbb{Z}) \)-equivalence classes of binary quadratic forms weighted by the inverse order of the corresponding stabilizer. Then as noted in [10], \( Z(q) \) is a weight 3/2 mock modular form:

**Theorem 6.6** (Zagier [34]). The function

\[
I(\tau, 1) = -\frac{1}{12} + \sum_{D=1}^{\infty} H(-D)q^D + \frac{1}{8\pi \sqrt{\text{Im}(\tau)}} \sum_{n \in \mathbb{Z}} \beta(4\pi \text{Im}(\tau)n^2)q^{-n^2}
\]

is a modular form of weight 3/2 under the congruence subgroup \( \Gamma_0(4) \). Here \( \beta(s) = \int_1^\infty e^{-st}t^{-3/2}dt \) is the usual \( \beta \)-function.

6.3.2. The Orbifolded Theory for Prime Order. A similar story holds in the CHL models considered. Let \( G = \mathbb{Z}_N \) for \( N \in \{2, 3, 5, 7\} \). Unlike in the unorbifolded theory, the \( S \)-duality group \( S(\mathbb{Z}) \) is now rather complicated. Of course, \( S(\mathbb{Z}) \) contains that portion of \( \text{PSL}_2(\mathbb{Z}) \) which commutes with the action of \( G \), namely \( \Gamma_1(N)/\{\pm 1\} \). In fact, it can be shown [13][9] that \( S(\mathbb{Z}) \) contains the significantly larger group \( \Gamma_0(N)^+ / \{\pm 1\} \), where \( \Gamma_0(N)^+ \) is the subgroup of \( \text{SL}_2(\mathbb{R}) \) generated by \( \text{SL}_2(\mathbb{Z}) \) and the Fricke involution

\[
W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.
\]

**Remark 6.7.** This definition of \( \Gamma_0(N)^+ \) is only correct when \( N \) is prime. For composite \( N \) there are additional Fricke involutions to consider.

**Remark 6.8.** It can be shown that for the values of \( N \) considered, namely \( N = 2, 3, 5, 7 \), \( \Gamma_0(N)^+ \) is precisely the normalizer of \( \Gamma_0(N) \) in \( \text{SL}_2(\mathbb{R}) \) [35].

This motivates the following conjecture [9]:

**Conjecture 6.9.** Given the preceding setup, \( S(\mathbb{Z}) \cong \Gamma_0(p)^+ / \{\pm 1\} \).

Assuming that the charge lattices are sufficiently large, we can eliminate all mention of charges from our definition of \( N(D) \) and \( H(D) \), writing

\[
N(D) = \# \{ [\tau]_{S(\mathbb{Z})} \in \mathcal{F} : D(\tau) = D \}
\]

and

\[
H(D) = \sum_{[\tau]_{S(\mathbb{Z})}} \frac{1}{\#\text{Stab}_{S(\mathbb{Z})}(\tau)}
\]
where we consider all \([\tau]_{S(\mathbb{Z})}\) in \(\mathcal{F}\) with \(D(\tau) = D\), where \(\tau\) solves an integral quadratic with leading and constant coefficient divisible by \(N\), and \(D\) is the discriminant of the least such quadratic. Assuming that the correct duality group to consider is in fact \(\Gamma_0(p)^+\),

\[
N(D) = \#\{[\tau]_{\Gamma_0(p)^+} \in \mathcal{F} : D(\tau) = D\}
\]

and

\[
H(D) = \sum_{[\tau]_{\Gamma_0(p)^+}} \frac{1}{#\text{Stab}_{S(\mathbb{Z})}(\tau)}
\]

where the sum is as before. The counting function

\[
Z(q) = \sum_{D=1}^{\infty} H(-D)q^D
\]

is, up to a constant, the holomorphic lift \(I(\tau, 1)\) considered by Funke in [36][37], in which the following is shown.

**Theorem 6.10** (Funke). The function

\[
I(\tau, 1) = \text{vol}(X) + \sum_{D=1}^{\infty} H(-D)q^D + \frac{1}{8\pi \sqrt{\text{Im}(\tau)}} \sum_{n \in \mathbb{Z}} \beta(4\pi \text{Im}(\tau)n^2)q^{-n^2}
\]

is a weight 3/2 modular form, where \(\text{vol}(X)\) is the normalized volume of a particular modular curve.

In particular, \(Z(q)\) is a weight 3/2 mock modular form, as before.
7. Microscopic Entropy

Beginning with the pioneering work of Bekenstein \[38][39][40]\ and Hawking \[41][42][43]\, the last few decades have witnessed an explosion of research related to the topic of black hole thermodynamics, which is now the cynosure of many physicists. Information-theoretic considerations related to semi-classical calculations regarding quantum field theory in classical black hole backgrounds \[42]\ required that black holes carry an entropy – now known as the Bekenstein-Hawking entropy – proportional to their surface area, in apparent contradiction to the classical no-hair theorem. The macroscopic analyses of Bekenstein, Hawking, and Wald \[44]\ necessitated a corresponding microscopic analysis, in which the Bekenstein-Hawking entropy is interpreted as the logarithm of some relevant degeneracy. It has fallen upon quantum gravity to provide such an analysis. Quantum mechanically, a black hole is – at least expected to be – a strongly interacting many-body system, and consequently carries many microscopic degrees of freedom. This yields a description of a black hole as a thermodynamic object, whose entropy corresponds to the logarithm of the number of accessible microstates in an appropriate ensemble. This microscopic entropy must agree with the macroscopic entropy considered by Bekenstein, Hawking, and Wald, and in particular must agree with the Bekenstein-Hawking entropy to leading order. This constitutes one marked success of string theory as a theory of quantum gravity, as it has provided such an analysis. Strominger and Vafa \[45]\ considered Type II string theory compactified on $K3 \times S^1$ or equivalently heterotic string theory compactified on $T^5$, and constructed BPS black holes as bound states of particular $D$-brane configurations. Within this context, they derived the relation between black hole entropy and horizon area, finding agreement with the Bekenstein-Hawking formula. A subsequent flurry of work extended and improved upon the original analysis of Strominger and Vafa. One particular line of work, that relevant to us here, is increasing the precision of the original calculation beyond the leading order. String theory predicts corrections to both the macroscopic Bekenstein-Hawking(-Wald) entropy and the microscopic Cardy entropy. That these corrections agree is a nontrivial mathematical prediction of string theory.

In this section we focus on exact computations of the microscopic entropy of small black holes in Type II string theory compactified on $(K3 \times T^2)/G$ for

$$G \in \{1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}.$$ 

Dabholkar, Denef, Moore, and Pioline \[30][31][32][33]\ provided an exact count of the number of microstates of small black holes in Type II string theory compactified on $K3 \times T^2$, settling
the case when $G \cong 1$. They also partially considered the case $G \cong \mathbb{Z}_2$. Here, we extend their ideas to settle all eight. This section is partially based on my own paper [11] and a recent paper by Nally [12].

7.1. Partition Functions. The strategy we follow is perennial. We encode the relevant microscopic degeneracies in a partition function

$$Z(q) = \sum_{n=0}^{\infty} d(n) q^n,$$

where $d(n)$ is the degeneracy of the $n$th energy level. This function will exhibit nontrivial analytic properties which will allow us to explicitly extract $d(n)$ through a carefully performed Fourier-Laplace transform. Before we begin, we must first determine what exactly $Z(q)$ is in the cases under consideration. Fortunately, this has already been done in the literature [46][47][48][49][50].

**Proposition 7.1.** Let $G \cong \mathbb{Z}_N$ for $N \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Given $g \in G$, let $\lambda_i(g)$ for $i \in \{1, \ldots, 24\}$ denote the eigenvalues of $\left(2\pi i\right)^{-1} \log(g^*): H^*(K3, \mathbb{C}) \to H^*(K3, \mathbb{C})$. The partition function counting half-BPS $g$-twisted states is given by $Z(g)(q)$ where

$$\frac{16}{Z(g)(q)} = q^{-a} \prod_{i=1}^{24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i \lambda_i(g)} q^{N(\lambda_i(\sigma_N)+n)}\right),$$

where $a$ is the zero-point energy. 

**Remark 7.2.** The factor of sixteen originates on the heterotic side in the degeneracy of the Ramond sector ground state. It will be ignored from here-on-out.

**Remark 7.3.** We will mainly be interested in the case when $g = 1$, in which case the given partition function encodes the degeneracies of untwisted half-BPS states.

By understanding $\{\lambda_1(g), \ldots, \lambda_{24}(g)\}$, we can simplify the preceding product representation. Recall that, as a $\mathbb{C}[G]$-module, $H^*(K3, \mathbb{Z})$ splits into a direct sum

$$H^*(K3, \mathbb{R}) = \bigoplus_{d|N} V_d^{\nu_d},$$

where for each $d$ dividing $N$, $\nu_d$ is some non-negative integer and $V_d$ is the standard $d$-dimensional representation of $\mathbb{Z}_N$ induced by the standard $d$-dimensional representation of $\mathbb{Z}_d$. That is, $V_d = \text{span}_\mathbb{C}\{e_0, \ldots, e_{d-1}\}$ and $\sigma_N: e_i \mapsto e_{i+1 \mod d}$. 

Corollary 7.4. Given the preceding setup,

\[ Z(q) = \prod_{d|N} \eta(d \cdot n)^{-\nu_d}, \]

where \( \eta \) is the Dedekind \( \eta \)-function and \( Z(q) = Z^{(1)}(q) \) is the partition function counting untwisted half-BPS states. ■

Proof. For \( g = 1 \), \( Z(q) = q^{-a} \prod_{i=1}^{24} \prod_{n=1}^{\infty} (1 - q^{N \lambda_i (\sigma_N + n)})^{-1} \). Group the terms according to the given decomposition of the cohomology lattice, we find \( Z(q) = q^{-a} \prod_{d|N} \prod_{n=1}^{\infty} (1 - q^{N/d} (j + dn))^{-\nu_d} \). The right-hand side can be rearranged as \( q^{-a} \prod_{d|N} \prod_{n=1}^{\infty} (1 - q^{n N/d})^{-\nu_d} = q^{-a} \prod_{d|N} \prod_{n=1}^{\infty} (1 - q^{nd})^{-\nu_d} \), since \( \nu_{N/d} = \nu_d \). It can be checked that \( a \) is precisely the power needed to convert the products to \( \eta \)-functions [46]. □

The partition functions in question can therefore be written as \( \eta \)-products. More generally, the partition functions counting small black holes in non-geometric CHL models are \( \eta \)-quotients [13]. A consequence is that \( Z(q) \) is a modular form with multiplier system for a particular congruence subgroup of the modular group. Referring back to section 4, the \( \eta \)-products in question are

\[
\begin{array}{|c|c|}
\hline
G & 1/Z(q) \\
\hline
1 & \eta(q)^{24} \\
\mathbb{Z}_2 & \eta(\tau)^8 \eta(2\tau)^8 \\
\mathbb{Z}_3 & \eta(\tau)^6 \eta(3\tau)^6 \\
\mathbb{Z}_4 & \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4 \\
\mathbb{Z}_5 & \eta(\tau)^4 \eta(5\tau)^4 \\
\mathbb{Z}_6 & \eta(\tau)^2 \eta(2\tau)^2 \eta(3\tau)^2 \eta(6\tau)^2 \\
\mathbb{Z}_7 & \eta(\tau)^3 \eta(7\tau)^3 \\
\mathbb{Z}_8 & \eta(\tau)^2 \eta(2\tau) \eta(4\tau) \eta(8\tau)^2 \\
\hline
\end{array}
\]

Remark 7.5. For \( G = 1 \), \( Z(q) \) has a nice topological interpretation:

\[ Z(q) = \frac{1}{q} \sum_{n=0}^{\infty} \chi(\text{Sym}^n(K3)) q^n, \]

where \( \chi(\text{Sym}^n(K3)) \) is the Euler characteristic of the \( n \)th symmetric product of K3. It is natural to wonder whether or not \( G \in \{ \mathbb{Z}_2, \ldots, \mathbb{Z}_8 \} \) yields \( Z(q) \) with some other nice topological interpretation.
There are two observations worth noting. The first is that $Z(q)$ is a modular form of weight $w$ under the group $\Gamma_1(N) \subseteq \text{SL}_2(\mathbb{Z})$. Here $w$ is one-half of the number of $\eta$-functions in the preceding products, and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \mod p \text{ and } c \equiv 0 \mod p \right\}. \quad (185)$$

This is of course expected from duality considerations. This means that, for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$,

$$Z(M \cdot \tau) = (c\tau + d)^w Z(\tau). \quad (186)$$

In fact, $Z(q)$ is a modular form of weight $w$ under the larger group $\Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \mod p \right\}. \quad (187)$$

However, these have in general some nontrivial multiplier system. So, there is some homomorphism $\epsilon : \Gamma_0(N) \to \mathbb{C}^\times$ such that

$$Z(M \cdot \tau) = (c\tau + d)^w \epsilon^w Z(\tau), \quad (188)$$

where [51]

$$\epsilon = \begin{cases} (-1)^{(d-1)/2} \left( \frac{N}{d} \right) & (d \equiv 1 \mod 2) \\ \left( \frac{d}{N} \right) & (d \equiv 0 \mod 2), \end{cases} \quad (189)$$

where $\left( \frac{\cdot}{\cdot} \right)$ denotes the Jacobi symbol. Note that $\epsilon = \pm 1$, and so if the weight $w$ is even then this multiplier system is trivial. In particular, if $N \in \{1, 2, 3, 4, 5, 6\}$ then this multiplier system is trivial. The duality group descending from the duality group of the unorbifolded theory is only $\Gamma_1(N)$, so this comes as a bit of a surprise. The second is that the $\eta$-products are balanced, so that $Z(q)$ is actually invariant under the larger group $\Gamma_0(N)^+ \subset \text{SL}_2(\mathbb{R})$ constructed by adjoining Fricke involutions and more generally Atkin-Lehner involutions to $\Gamma_0(N)$. This is a crucial component of Nally’s analysis in [12].

Given the partition function $Z(q)$, the degeneracy of the $n$th energy level of the untwisted half-BPS black hole is (8) the coefficient of $q^n$ in $Z(q)$. This coefficient can be extracted

8Depending on our convention for indexing, we might have to shift our initial index up or down by one.
using the Fourier-Laplace transform. This reduces the problem of computing microscopic black hole entropy to the problem of performing a rather complicated integral.

7.2. Rademacher Series. We would like to utilize the analytic properties of \( Z(q) \), as defined in the preceding section, in order to say something nontrivial about the degeneracies \( d(n) \). In order to carry out this maneuver, we must take a detour through a bit of number theory. This section is mostly taken from my paper [11].

A partition of an integer \( n \) is a multiset of positive integers whose sum is \( n \). Let \( p(n) \) denote the number of partitions of \( n \). The value of \( p(n) \) may be computed by brute force for sufficiently small \( n \) by simply enumerating all possible partitions and then counting. However, \( p(n) \) grows rapidly and brute force computation rapidly becomes intractible. Another technique, pioneered by Euler [52], is to study the properties of the generating function

\[
Z(q) = \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)}.
\]

He showed that

\[
\frac{1}{Z(q)} = \prod_{n=1}^{\infty} (1 - q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(3k-1)/2}.
\]

This is a result regarding formal series. However, we can interpret these series as complex valued functions in some appropriate domain. Viewed as complex functions, \( Z(q) \) and \( 1/Z(q) \) are both nonvanishing and holomorphic on the open unit disk \( \mathbb{D} \subset \mathbb{C} \) and cannot be analytically continued beyond \( \mathbb{D} \). One naturally asks (i) what are the analytic properties of these generating functions, and (ii) what properties of \( p(n) \) may be deduced from these analytic properties. With regards to (ii), if we know sufficiently many details regarding the analytic properties of \( Z(q) \), \( p(n) \) may simply be extracted by performing a Fourier-Laplace transform:

\[
p(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{Z(q)}{q^{n+1}} dq
\]

for some suitably chosen contour \( \gamma \). The difficulty lies in computing this contour integral.

Before continuing, a word on notation: Given a function \( f(q) : \mathbb{D} \to \mathbb{C} \), we may pull back \( f(q) \) by the map \( q = e^{2\pi i \tau} \) to get a new function \( f(e^{2\pi i \tau}) : \mathbb{H} \to \mathbb{C} \), where \( \mathbb{H} \) is the open upper-half of the complex plane. We will often denote \( f(e^{2\pi i \tau}) \) as simply \( f(\tau) \) when no confusion should arise. In order to keep track of which variable we are working with, we will often refer to two copies of \( \mathbb{C} \) as the \( q \)-plane and the \( \tau \)-plane.
The first progress with regards to (i) was due to Dedekind. Dedekind considered the eponymous function \( \eta : \mathbb{D} \to \mathbb{C} \)

\[
\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)
\]

where \( q = e^{2\pi i \tau} \). This is simply \( 1/\mathcal{Z}(q) \) with a mysterious additional factor of \( q^{1/24} \). Dedekind showed that \( \eta(\tau) \) is a modular form of weight \( 1/2 \). That is, for any matrix

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),
\]

where \( \text{SL}_2(\mathbb{Z}) \) is the group of integral matrices with determinant \(+1\),

\[
\eta \left( \frac{a\tau + b}{c\tau + d} \right) = \epsilon(a, b, c, d) \sqrt{c\tau + d} \cdot \eta(\tau)
\]

where \( \epsilon(a, b, c, d) = \epsilon(M) : \text{SL}_2(\mathbb{Z}) \to \mathbb{C} \) is a homomorphism. This phase factor is called a multiplier system. Dedekind computed \( \epsilon(M) \). It is given by

\[
\epsilon(a, b, c, d) = \begin{cases} 
\exp \left( \frac{\pi i b}{12} \right) & (c = 0, d = 1), \\
\exp \left( -\frac{\pi i b}{12} + \frac{\pi i}{4} \right) & (c = 0, d = -1), \\
\exp \left( \pi i \left[ \frac{a+d}{12c} - s(d, c) - \frac{1}{2} \right] \right) & (c > 0), \\
\exp \left( \pi i \left[ \frac{a+d}{12c} - s(-d, -c) \right] \right) & (c < 0).
\end{cases}
\]

Here

\[
s(h, k) = \sum_{n=1}^{k-1} \frac{n}{k} \left( \frac{hn}{k} - \left[ \frac{hn}{k} \right] - \frac{1}{2} \right)
\]

is known as a Dedekind sum. In retrospect, (Eq. 195) is rather remarkable, and allows us to extract asymptotics of \( \eta(\tau) \) for \( \tau \) near a given rational \( q \in \mathbb{Q} \subset \mathbb{C} \) in terms of the asymptotics of \( \eta(\tau) \) near \(+i\infty\), which are incredibly simple: As \( \tau \to +i\infty \), that is as \( q \to 0 \), \( \eta(q) \sim q^{1/24} \). Hardy and Ramanujan [54] followed by Rademacher [55][56] used this in order to carry out the Fourier transform in Eq. 192 and therefore compute \( p(n) \). This idea is rather general, and can be used to compute the Fourier coefficients of a wide variety of automorphic forms [57]. Modifications can be used in order to compute the Fourier coefficients of modular forms which are modular under a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). This idea was pioneered by Zuckerman [58]. In this section we will use such a modification in order to compute the Fourier coefficients of a finite product of modular forms precomposed...
with multiplication by different scalar factors $\mathcal{M} \subset \mathbb{N}$. These forms are modular forms under a congruence subgroup of the modular group, specifically

$$\Gamma_0(\text{lcm}(\mathcal{M})) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{\text{lcm}(\mathcal{M})} \right\}. $$

So, our result can be obtained using Zuckerman’s method applied to $\Gamma_0(\text{lcm}(\mathcal{M}))$. We instead modify Rademacher’s original method in a slightly different, but ultimately equivalent, way so that the calculation may be done for many $\eta$-quotients simultaneously. Another manner of performing this computation will be outlined in the next section.

We consider $\eta$-quotients, functions $Z(q)$ of the form

$$Z(\tau) = \prod_{m=1}^{\infty} \eta(m\tau)^{\delta_m}$$

where $\{\delta_m\}_{m=1}^{\infty}$ is a sequence of integers of which only finitely many are nonzero. As mentioned before, all of the partition functions relevant to us are of this form. Those with $\delta_m \geq 0$ for all $m$ are often called $\eta$-products. Those partition functions which arise in counting small untwisted black holes in CHL models are of this form.

In order to present our main formula, we need a few preliminary definitions. Let

$$n_0 = -\frac{1}{24} \sum_{m=1}^{\infty} m \cdot \delta_m.$$ 

The function $Z(q) \cdot q^{n_0}$ is holomorphic in the open unit disk $\mathbb{D}$, and so we may write

$$Z(q) = q^{-n_0} \sum_{n=0}^{\infty} d(n) q^n.$$ 

for some coefficients $d(n)$. From the product formula for $\eta(q)$ in (Eq. 193), each $d(n)$ is an integer. The main result of this paper is an explicit formula for $d(n)$ for a large class of sequences $\{\delta_m\}_{m=1}^{\infty}$. Let

$$c_1 = -\frac{1}{2} \sum_{m=1}^{\infty} \delta_m, \quad c_2(k) = \prod_{m=1}^{\infty} \left[ \frac{\gcd(m,k)}{m} \right]^{\delta_m/2}, \quad c_3(k) = -\sum_{m=1}^{\infty} \delta_m \frac{\gcd(m,k)^2}{m},$$

$$A_k(n) = \sum_{\substack{0 \leq h < k \ \text{gcd}(h,k) = 1}} \exp \left[ -2\pi i \left( \frac{h}{k} \cdot n + \frac{1}{2} \sum_{m=1}^{\infty} \delta_m \cdot s \left( \frac{mh}{\gcd(m,k)}, \frac{k}{\gcd(m,k)} \right) \right) \right].$$

Finally let $\mathcal{M}$ be the set of $m$ for which $\delta_m$ is nonzero. The quantities $c_1, c_2(k), c_3(k)$ are coefficients which appear in our calculation and formula. The coefficient $c_1$ is the negative of the weight of $Z(\tau)$ as a $\Gamma_0(\text{lcm}(\mathcal{M}))$-modular form. The sums $A_k(n)$ closely resemble,
and in some cases are Kloosterman sums. It can be shown that $A_k(n)$ is real for all $k$ and $n$. With these definitions in hand, we may state

**Theorem 7.6.** If $c_1 > 0$ and the periodic function $g(k) : \mathbb{N} \to \mathbb{R}$ given by

\[
g(k) = \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m, k)^2}{m} \right\} - \frac{c_3(k)}{24}
\]

is non-negative, then for $n \in \{1, 2, \ldots\}$ such that $n > n_0$,

\[
d(n) = 2\pi \left( \frac{1}{24(n - n_0)} \right)^{\frac{c_1 + 1}{2}} \sum_{k = 1}^{\infty} c_2(k) c_3(k) \frac{c_1 + 1}{2} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(n - n_0) \right]
\]

where $I_{1+c_1}$ is the $(1 + c_1)$th modified Bessel function of the first kind.

**Proof.** Relegated to Appendix B.

We would like to extract useful asymptotics from (Eq. 284). One reason is to compare with non-exact macroscopic computations of black hole entropy, such as the Bekenstein-Hawking entropy and its perturbative corrections. These asymptotics are contained in the following proposition. For this section we assume that the hypotheses of Theorem 7.6 are satisfied, so that (Eq. 205) applies.

**Proposition 7.7.** Let $\mathcal{K} \subset \mathbb{N}$ be the set of $k$ that maximize $c_3(k)/k^2$ and let $c_3 > 0$ be the maximum value. For any $\epsilon > 0$, there exists some constant $C > 0$ which may depend only on $\{\delta_m\}_{m=1}^\infty$ such that for all $n \in \mathbb{N}$ with $n > n_0$ and

\[
\left| \sum_{k \in \mathcal{K}} c_2(k) k^{c_1} A_k(n) \right| > \epsilon
\]

it is the case that

\[
d(n) = (1 + O(e^{-C\sqrt{n}})) \cdot 2\pi \left( \frac{c_3}{24(n - n_0)} \right)^{\frac{c_1 + 1}{2}} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(n - n_0) \right] \sum_{k \in \mathcal{K}} c_2(k) k^{c_1} A_k(n).
\]

The dependence on $\epsilon$ is in the bounding coefficient of the $O(e^{-C\sqrt{n}})$.

The proof of (Prop. 7.7) is straightforward and an exercise in using the asymptotics of the modified Bessel functions.
Proof. Note that $c_3(k)$ is periodic with period $\text{lcm}(\mathcal{M})$. So, $\mathcal{K} \subseteq \{1, \ldots, \text{lcm}(\mathcal{M})\}$. We first break up the Rademacher series in (Eq. 205) into $\text{lcm}(\mathcal{M})$ sums, one for each possible value of $k$ modulo $\text{lcm}(\mathcal{M})$. We then show that each of these smaller sums is exponentially dominated by the leading term. We then absorb the resulting Bessel functions with $k \not\in \mathcal{K}$ into those with $k \in \mathcal{K}$. In the following, we will use the result that if $0 < a < b$ then $I(ax)$ is exponentially dominated by $I(bx)$ for any positive $x$ and any positive weight modified Bessel function $I$ of the first kind. So, we first consider for fixed $b \in \{1, \ldots, \text{lcm}(\mathcal{M})\}$ with $c_3(b) > 0$

\begin{equation}
\sum_{\substack{k \in [b] \\
 k > 0}} c_2(k)c_3(k)^{(c_1+1)/2} A_k(n)k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n-n_0)} \right]
\end{equation}

where $[b]$ is the equivalence class of integers modulo $\text{lcm}(\mathcal{M})$. Because $c_2(k), c_3(k)$ have period $\text{lcm}(\mathcal{M})$, this is

\begin{equation}
c_2(b)c_3(b)^{(c_1+1)/2} \sum_{\substack{k \in [b] \\
 k > 0}} A_k(n)k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n-n_0)} \right].
\end{equation}

We wish to show that the sum above is dominated by the first term. Consider the rest of the terms,

\begin{equation}
\sum_{\substack{k \in [b] \\
 k > \text{lcm}(\mathcal{M})}} A_k(n)k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n-n_0)} \right],
\end{equation}

which is bounded above in absolute value by

\begin{equation}
\sum_{\substack{k \in [b] \\
 k > \text{lcm}(\mathcal{M})}} \left| I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n-n_0)} \right] \right|.
\end{equation}

Using the expansion of $I_{1+c_1}(z)$ [59],

\begin{equation}
I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n-n_0)} \right] = \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+c_1+2)} j! \left( \frac{\pi}{2k} \sqrt{\frac{2}{3} c_3(b)(n-n_0)} \right)^{2j+1+c_1}.
\end{equation}

Suppose that $k_0$ is a real number satisfying $0 < k_0 \leq k$. The quantity

\begin{equation}
\left| I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n-n_0)} \right] \right|
\end{equation}
is bounded above by
\[
\sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2) j!} \left( \frac{\pi}{2k_0} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right)^{2j+1+c_1} \left( \frac{k_0}{k} \right)^{2j+1+c_1},
\]
which is in turn bounded by
\[
\left( \frac{k_0}{k} \right)^{1+c_1} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2) j!} \left( \frac{\pi}{2k_0} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right)^{2j+1+c_1} = \left( \frac{k_0}{k} \right)^{1+c_1} I_{1+c_1} \left[ \frac{\pi}{k_0} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right].
\]

Summing over all relevant \( k \),
\[
\sum_{k \in [b]} \sum_{k > \text{lcm}(\mathcal{M})} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right] \leq k_0^{1+c_1} \zeta(1 + c_1) I_{1+c_1} \left[ \frac{\pi}{k_0} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right].
\]

Therefore, setting \( k_0 = b + 1/2 \) yields
\[
\sum_{k \in [b]} \sum_{k > \text{lcm}(\mathcal{M})} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right] = O(e^{-C_b \sqrt{n}})
\]
\[
\times I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right]
\]
for some constant \( C_b > 0 \) depending on \( b \). Therefore, the expression in Eq. 209 is
\[
c_2(b)c_3(b)^{(c_1+1)/2}(A_b(n)b^{-1} + O(e^{-C_b \sqrt{n}}))I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right].
\]

It follows that
\[
\sum_{k \in [b]} \sum_{k > 0} c_2(k)c_3(k)^{(c_1+1)/2} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3}} c_3(k)(n - n_0) \right]
\]
is
\[
c_2(b)c_3(b)^{(c_1+1)/2}(A_b(n)b^{-1} + O(e^{-C_b \sqrt{n}}))I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3}} c_3(b)(n - n_0) \right].
\]
We can sum this result for all \( b \in \{1, \ldots, \text{lcm}(\mathcal{M})\} \) with \( c_3(b) > 0 \). We can absorb the terms with \( b \not\in \mathcal{K} \) into the error term. So, for some constant \( C \) depending only on \( \{\delta_m\}_{m=1}^{\infty} \),

\[
(219) \quad d(n) = 2\pi \left( \frac{1}{24(n-n_0)} \right)^{c_1+1} \sum_{k\in\mathcal{K}} (A_k(n)k^{-1} + O(e^{-C\sqrt{n}}))c_2(k)c_3(k) \frac{c_1+1}{2} \cdot I_{1+c_1} \left[ \pi \sqrt{\frac{2}{3}} c_3(n-n_0) \right].
\]

Since for \( k \in \mathcal{K} \) it is the case that \( c_3(k) = k^2 c_3 \),

\[
(220) \quad d(n) = 2\pi \left( \frac{c_3}{24(n-n_0)} \right)^{c_1+1} \frac{c_1+1}{2} \cdot I_{1+c_1} \left[ \pi \sqrt{\frac{2}{3}} c_3(n-n_0) \right] \sum_{k\in\mathcal{K}} (A_k(n)k^{-1} + O(e^{-C\sqrt{n}}))c_2(k)k^{1+c_1}.
\]

Note that each \( O(e^{-C\sqrt{n}}) \) in the sum over \( k \) in (Eq. 220) is different. Using the assumption in (Eq. 206),

\[
(221) \quad d(n) = (1 + O(e^{-C\sqrt{n}})2\pi \left( \frac{c_3}{24(n-n_0)} \right)^{c_1+1} \frac{c_1+1}{2} \cdot I_{1+c_1} \left[ \pi \sqrt{\frac{2}{3}} c_3(n-n_0) \right] \sum_{k\in\mathcal{K}} A_k(n)k^{c_1} c_2(k),
\]

as claimed, where the \( O(e^{-C\sqrt{n}}) \) term is bounded in terms of \( \{\delta_m\}_{m=1}^{\infty} \) and \( \epsilon \). □

It should be noted that the hypothesis contained in (Eq. 206) is almost always satisfied, and is satisfied for all functions of interest to us.

### 7.3. Rademacher Sums for \( N = 4, 8 \)

Nally [12] observed that the sums in Theorem 7.6, for \( G \in \{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7\} \) can be understood as one-cusp Rademacher sums over the subgroups \( \Gamma_0(p)^+ \subset \text{SL}_2(\mathbb{R}) \). This observation enabled him to give a gravitational interpretation of these series, a macroscopic construction of the entropy. Based on the analysis in [60], the context of which was moonshine, it is expected that, for \( G \in \{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8\} \), the “right” group to sum over is \( N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(N)) \), the normalizer in \( \text{SL}_2(\mathbb{R}) \) of the congruence subgroup \( \Gamma_0(N) \subset \text{SL}_2(\mathbb{Z}) \). It is always the case that \( \Gamma_0(N)^+ \subseteq N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(N)) \), and in fact equality holds when \( N \in \{1, 2, 3, 5, 6, 7\} \), and so this is consistent with [12]. However, for \( N \in \{4, 8\} \), equality no longer holds, and the analysis of [12] no longer applies. This is manifested in the fact that \( \Gamma_0(N)^+ \) no longer possesses a single unique cusp when \( N \in \{4, 8\} \). However, \( N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(N)) \) does, and so the analogous computation to that given in [12] must go through, even if it is significantly more tedious, and it is significantly more tedious.
Now, the group structure of \(N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(N))\) is an interesting topic, first addressed by Atkin and Lehner [35] – incorrectly, as it turns out, for some large \(N\) \(^9\). Let’s just look at the case \(N = 4\) to get a sense for how this should go. Consider the element

\[
S_2 = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{R}).
\]

Given any \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)\), we have that

\[
S_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} S_2^{-1} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + c/2 & b + d/2 - a/2 - c/4 \\ c & d - c/2 \end{pmatrix}.
\]

The right-hand side would be manifestly in \(\Gamma_0(4)\), were it not for the pesky \((d - a)/2\) term, which is not manifestly integral. However, since \(ad - bc = 1\), \(ad \equiv 1 \mod 2\), and so \(a\) and \(d\) are both odd, so that this is in fact integral and the right-hand side above is in fact in \(\Gamma_0(4)\). Therefore \(S_2 \in N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(4))\), but it can be checked that \(S_2 \notin \Gamma_0(4)^+\). It can also be checked that \(S_2\) and \(\Gamma_0(4)^+\) generate \(N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(4))\). Note also that \(S_2\) maps \(1/2\) to zero, so that \(N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(4))\) does have only one cusp, as claimed. The only remaining piece of information needed to apply the analysis of [12] is how \(Z(q) = \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4\) transforms under \(S_2\). \(S_2\) acts on \(\tau\) by \(S_2 : \tau \mapsto \tau + 1/2\). Under this transformation, \(\eta(2\tau) \to \eta(2\tau)\zeta_{48}^2\) and \(\eta(4\tau) \to \eta(4\tau)\zeta_{48}^4\) where \(\zeta_{48}\) is a primitive 48th root of unity. But, \(\eta(\tau) \not\to \zeta_{48}\eta(\tau)\). Indeed,

\[
S_2 : \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \mapsto \zeta_{48} \cdot q^{1/24} \prod_{n=1}^{\infty} (1 - (-1)^n q^n) = \zeta_{48} \eta(2\tau)^3 / \eta(\tau) \eta(4\tau).
\]

So

\[
S_2 : \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4 \mapsto -\eta(2\tau)^{14} / \eta(\tau)^4.
\]

This is a bit surprising, as the frame shape on the right-hand side is not a CHL frame shape [9]. It is then natural to wonder about the gravitational interpretation of \(S_2\). Regardless, using this piece of information, we can write the Rademacher series in Theorem 7.6 for \(N = 4\) as a Rademacher sum over \(N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(4))\).

\(^9\)See [61] for the corrected statement.
Acknowledgements

First and foremost, I would like to express my gratitude to Professor Shamit Kachru for patiently advising and guiding me over the last two years, and for overseeing this senior thesis. I am also grateful to Richard Nally and Brandon Rayhaun, who took the time to answer my innumerable questions and to provide helpful advice. I would also like to thank the other students in SITP for contributing to its extraordinarily welcoming and collaborative atmosphere, as well as Rick Pam and Elva Carbajal for organizing the physics department’s summer research program. Some of the work which became this thesis was done under the physics department’s summer research program and under a Stanford UAR major grant.

Appendix A. The Modular Group and Congruence Subgroups

In this section we list a few definitions and properties pertaining to the modular group and congruence subgroups, following [62]. The modular group, denoted as $\text{SL}_2(\mathbb{Z})$, is the group of integral $2 \times 2$ matrices $M \in \mathbb{Z}^{2 \times 2}$ with determinant $\det(M) = 1$:

\begin{equation}
\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.
\end{equation}

The group $\text{PSL}_2(\mathbb{Z})$ is defined to be $\text{SL}_2(\mathbb{Z})/\{\pm 1\}$. We will identify elements of $\text{PSL}_2(\mathbb{Z})$ by representatives of their equivalence classes when no confusion should arise. Two elements in $\text{SL}_2(\mathbb{Z})$ are

\begin{equation}
S = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\end{equation}

The matrix $S$ has order 4 and the matrix $T$ has infinite order. The matrix $ST$ has order 6. Viewed as elements of $\text{PSL}_2(\mathbb{Z})$, $S$ has order 2 and $ST$ has order 3. It turns out that $S$ and $T$ generate $\text{SL}_2(\mathbb{Z})$ and consequently $\text{PSL}_2(\mathbb{Z})$:

\textbf{Theorem A.1.} The group $\text{SL}_2(\mathbb{Z})$ is generated by $S$ and $T$. \hfill \Box

There are two simple proofs of this result, one algebraic and one geometric. We first present the algebraic proof given in [62] with minor notational changes.

\textit{Algebraic Proof of Theorem A.1.} Note that for any matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2} \)

\begin{equation}
S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ +a & +b \end{pmatrix} \quad \text{and} \quad T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix} \quad \text{for any } n \in \mathbb{Z}.
\end{equation}
Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an arbitrary element of \( \text{SL}_2(\mathbb{Z}) \). We want to show that \( M \) can be written as a product of powers of \( T \) and \( S \). We may assume without loss of generality that the upper left entry of \( M \) is at least as large in magnitude as the lower left entry of \( M \), that is \( |a| \geq |c| \). Then divide \( a \) by \( c \) to find some integers \( q \) and \( r \) with \( 0 \leq r < |c| \) such that \( a = cq + r \). By (Eq. 224), the matrix \( T^{-q}M \) has upper left entry \( r \) and lower left entry \( c \). The matrix \( ST^{-q}M \) then has upper left entry \(-c\) and lower left entry \( r \). Since \( |c| > |r| \), we may repeat this process until we have a matrix \( ST^{-q_1} \cdots ST^{-q_n}M \) with lower left entry \( 0 \), where the process has been applied \( n \) times and \( q_n = q \). Since this matrix is in \( \text{SL}_2(\mathbb{Z}) \) it must be of the form
\[
(225) \quad ST^{-q_1} \cdots ST^{-q_n}M = \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}
\]
for some \( m \in \mathbb{Z} \). The right hand side is \( \pm T^m \). So, \( ST^{-q_1} \cdots ST^{-q_n}M = \pm T^m \). Therefore,
\[
(226) \quad M = S^{1 \pm 1} T^m S^{3 \cdots} T^{q_1} S^{3} T^m.
\]
This completes the proof. \( \square \)

The algebraic proof involved studying the action of \( \text{SL}_2(\mathbb{Z}) \) on itself by left multiplication. The geometric proof involves studying the action of \( \text{SL}_2(\mathbb{Z}) \) on the (open) upper-half of the Poincaré complex plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \). Given a matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \), where \( \text{GL}_2^+(\mathbb{R}) \) is the group of invertible \( 2 \times 2 \) real matrices with positive determinant, \( M \) acts on \( \tau \in \mathbb{H} \) as
\[
(227) \quad M \cdot \tau = M(\tau) = \frac{a\tau + b}{c\tau + d}.
\]
The fact that the right hand side of (Eq. 227) lies in \( \mathbb{H} \) follows from the formula
\[
(228) \quad \text{Im} \left( \frac{a\tau + b}{c\tau + d} \right) = \frac{(ad - bc)}{|c\tau + d|^2} \text{Im}(\tau).
\]
The fact that (Eq. 227) defines a group action of \( \text{GL}_2^+(\mathbb{R}) \) follows from the calculation
\[
(229) \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) \cdot \tau = \begin{pmatrix} \alpha \tau + \beta \\ \gamma \tau + \delta \end{pmatrix} \cdot \tau = \begin{pmatrix} (a\alpha + b\gamma) \tau + (a\beta + b\delta) \\ (c\alpha + d\gamma) \tau + (c\beta + d\delta) \end{pmatrix} = \begin{pmatrix} (a\alpha + b\gamma) \tau + (a\beta + b\delta) \\ (c\alpha + d\gamma) \tau + (c\beta + d\delta) \end{pmatrix}
\]
\[
(230) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \cdot \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha \tau + \beta \\ \gamma \tau + \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma \tau + (a\beta + b\delta) \\ c\alpha + d\gamma \tau + (c\beta + d\delta) \end{pmatrix}.
\]
Another way to understand this group action is as descending from the group action of \( \text{GL}_2(\mathbb{C}) \) on \( \mathbb{C}^P^1 = \mathbb{C}^2 / \mathbb{C}^x \). The group \( \text{GL}_2(\mathbb{C}) \) acts on \( \mathbb{C}^2 \) by definition, and this action
commutes with scalar multiplication. This group action consequently descends from one on \( \mathbb{C}^2 \) to one on \( \mathbb{C}P^1 \). Written using a standard coordinate chart on \( \mathbb{C}P^1 \equiv \mathbb{C} \cup \{ \infty \} \), this action is exactly a linear fractional transformation on \( \mathbb{C} \). These linear fractional transformations will, in general, not preserve the upper-half plane \( \mathbb{H} \subseteq \mathbb{C} \). Those in \( GL_2^+(\mathbb{R}) \) do. Note that the action defined in (Eq. 227) is identical for \( M \) and \( \lambda M \) for any \( \lambda \in \mathbb{C} \times \). This action of \( GL_2^+(\mathbb{R}) \) on \( \mathbb{H} \) therefore descends naturally to an action of \( SL_2(\mathbb{R}) \) and \( PSL_2(\mathbb{R}) \) on \( \mathbb{H} \). This can be seen as due to the fact that the action of \( GL_2(\mathbb{C}) \) on \( \mathbb{C}^2 / \mathbb{C} \times \) descends to an action of \( SL_2(\mathbb{C}) / \mathbb{C} \times = PSL_2(\mathbb{C}) \) on \( \mathbb{C}^2 / \mathbb{C} \times \). By restricting the action of \( SL_2(\mathbb{R}) \) and \( PSL_2(\mathbb{R}) \) on \( \mathbb{H} \) to the subgroups \( SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}) \) and \( PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R}) \), we get an action of \( SL_2(\mathbb{Z}) \) and \( PSL_2(\mathbb{Z}) \) on \( \mathbb{H} \). Under this action,

\[
S(\tau) = -1/\tau \quad \text{and} \quad T(\tau) = \tau + 1.
\]

The Poincaré upper-half plane \( \mathbb{H} = \{ x + iy : x \in \mathbb{R}, y \in \mathbb{R}^+ \} \) is given the metric

\[
ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|d\tau|^2}{\text{Im}(\tau)^2}
\]

where \( \tau = x + iy \). We can compute that for any \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \),

\[
\frac{|d(M \cdot \tau)|^2}{\text{Im}(M \cdot \tau)^2} = \frac{|d\tau|^2}{\text{Im}(\tau)^2}.
\]

So, the elements of \( SL_2(\mathbb{R}) \), or equivalently \( PSL_2(\mathbb{R}) \), define isometries of \( \mathbb{H} \). It can be shown that these are the only isometries of \( \mathbb{H} \) [63].

We would like to understand the set \( \mathbb{H} / SL_2(\mathbb{Z}) \) consisting of orbits of \( \mathbb{H} \) under the group action defined in (Eq. 227). To this end we would like to find some simple set \( \mathcal{F} \subseteq \mathbb{H} \) such that \( \mathcal{F} \) contains exactly one element of each orbit in \( \mathbb{H} / SL_2(\mathbb{Z}) \). A set with this property is called a fundamental domain for the modular group \( SL_2(\mathbb{Z}) \).

**Theorem A.2.** The set

\[
\mathcal{F} = \{ \tau \in \mathbb{H} : |\tau| > 1 \quad \text{and} \quad |\text{Re}(\tau)| < 1/2 \} \cup \{ \tau \in \mathbb{H} : |\tau| \geq 1 \quad \text{and} \quad -1/2 \leq \text{Re}(\tau) \leq 0 \}
\]

is a fundamental domain for the modular group \( SL_2(\mathbb{Z}) \).

This is Proposition 1 in [64]. The classical proof given by Zagier in [64] is replicated below, with minor changes.

**Proof.** Take any \( \tau \in \mathbb{H} \). The set \( \{ m\tau + n : m, n \in \mathbb{Z} \} \) is a lattice in \( \mathbb{C} \). Every lattice has a point different from the origin of minimum magnitude. Let this be \( c\tau + d \). The integers \( c \) and \( d \) must be relatively prime. So, by the Euclidean algorithm, for some integers \( a \) and
We end up with

This can only be satisfied for integral

we can replace

(234)

group of order three generated by the element

stabilizer in

unit circle and in

for some integer

Im

the non-strict inequality

contradicting our assumption. If

Im(τ) = Im(τ)/|τ|^2 > Im(τ^*), contradicting the maximality of Im(τ^*). If |τ^*| = 1 and Re(τ^*) > 0, we can replace τ^* by S · τ^*, which satisfies |S · τ^*| = 1/|τ^*| = 1 and Re(S · τ^*) = −Re(τ^*).

We end up with τ^* ∈ ℱ, with τ^* in the same SL_2(ℤ) orbit as τ.

Now suppose that τ_1 and τ_2 are two distinct SL_2(ℤ) equivalent points in ℱ, with M ∈ SL_2(ℤ) yielding τ_2 = M · τ_1. Note that M cannot be ±1 or T^n for any n. So, M is of the form M = (a b c d) for some nonzero c. Note that Im(τ) > √3/2 for all τ ∈ ℱ except e^{2πi/3}. Assume without loss of generality that τ_2 ≠ e^{2πi/3}. Then, using Eq. 228,

(234) \[ \frac{\sqrt{3}}{2} < \text{Im}(τ_2) = \frac{\text{Im}(τ_1)}{|cτ_1 + d|^2} \leq \frac{\text{Im}(τ_1)}{\text{Im}(cτ_1 + d)^2} = \frac{1}{c^2 \text{Im}(τ_1)} \leq \frac{2}{\sqrt{3}c^2}. \]

This can only be satisfied for integral c if c = 0, ±1 and we have already deduced that c is nonzero. So c = ±1. We may also assume without loss of generality that Im(τ_1) ≤ Im(τ_2). Note that, since τ_1 ∈ ℱ, |±τ_1 + d| ≥ |τ_1| ≥ 1 for any integer d, with the first inequality strict if and only if d is nonzero and the second inequality strict if and only if τ_1 is not on the unit circle. So, if either τ_1 is not on the unit circle or d is nonzero, by (Eq. 228),

(235) \[ \text{Im}(τ_2) = \frac{\text{Im}(τ_1)}{|±τ_1 + d|^2} < \text{Im}(τ_1), \]

contradicting our assumption. If d is zero and τ_1 lies on the unit circle, we can still deduce the non-strict inequality Im(τ_2) ≤ Im(τ_1) which, when combined with our assumption that Im(τ_1) ≤ Im(τ_2), implies that Im(τ_1) = Im(τ_2). Since c = ±1 it is the case that τ_2 = a − 1/τ_1 for some integer a. Since Re(a − 1/τ_1) = a − Re(τ_1), a must be zero. The only point on the unit circle and in ℱ which is mapped by S to ℱ is +i, but this is mapped to itself, so that τ_2 = τ_1, contradicting our assumption that τ_1 and τ_2 were distinct. Consequently, no such τ_1 and τ_2 exist.

This argument shows that most points in ℱ, namely all points in the interior of ℱ, are not fixed by any non-identity element of PSL_2(ℤ). A point in Γ\ℍ^* which has nontrivial stabilizer in PSL_2(ℤ) is called an elliptic point. The only points in ℱ which are fixed by some non-identity element of PSL_2(ℤ) are e^{2πi/3}, whose stabilizer in PSL_2(ℤ) is the cyclic group of order three generated by the element T^{−1}S, and +i, whose stabilizer is the cyclic group of order two generated by S. Consequently, excepting finitely many cusps, ℍ/SL_2(ℤ)
inherits the complex structure of \( \mathbb{H} \). These cusps may be smoothed out and the surface may be compactified, and the result is a genus zero Riemann surface, namely the sphere \( \mathbb{C}P^1 \).

Note that we could have chosen any translate of \( F \), that is \( M(F) \) for any \( M \in \text{SL}_2(\mathbb{Z}) \), as a fundamental domain for the modular group instead of \( F \). The choice in Theorem A.2 is merely one of convenience.

**Geometric Proof of Theorem A.1.** Embedded in the proof of Theorem A.2 is a proof that \( S \) and \( T \) generate \( \text{SL}_2(\mathbb{Z}) \). Consider \( \tau = 2i \). The first part of the preceding proof shows that there exists some matrix \( g \) in the subgroup of \( \text{SL}_2(\mathbb{Z}) \) generated by \( S \) and \( T \) and some matrix \( M \) in \( \text{SL}_2(\mathbb{Z}) \) such that \( (gM) \cdot (2i) \in F \). But since \( 2i \) is in \( F \) already, by the second part of the preceding proof \( (gM) \cdot (2i) = 2i \). Since \( 2i \) is in the interior of \( F \) it has trivial stabilizer in \( \text{PSL}_2(\mathbb{Z}) \). So, \( gM = \pm 1 \). So \( M = \pm g^{-1} \) is in the subgroup of \( \text{SL}_2(\mathbb{Z}) \) generated by \( S \) and \( T \). \( \square \)

Theorem A.1 has two useful group theoretic corollaries, the latter of which will help us understand modular multiplier systems.

**Corollary A.3.** The group \( \text{SL}_2(\mathbb{Z}) \) is generated by two elements of order 4 and 6, while the group \( \text{PSL}_2(\mathbb{Z}) \) is generated by two elements of order 2 and 3. \( \blacksquare \)

**Corollary A.4.** Any homomorphism \( \text{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times \) maps into the 12th roots of unity and any homomorphism \( \text{PSL}_2(\mathbb{Z}) \to \mathbb{C}^\times \) maps into the 6th roots of unity. \( \blacksquare \)

Congruence subgroups are particular subgroups of \( \text{SL}_2(\mathbb{Z}) \). These include

\[
(236) \quad \Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \text{ mod } n \text{ and } b, c \equiv 0 \text{ mod } n \right\},
\]

\[
(237) \quad \Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \text{ mod } n \right\},
\]

\[
(238) \quad \Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \text{ mod } n \text{ and } c \equiv 0 \text{ mod } n \right\}.
\]

The genus \( g(\Gamma) \) of a congruence subgroup \( \Gamma \) is defined to be the genus of the Riemann surface \( \Gamma \backslash \mathbb{H}^* \). The fundamental domain for \( \Gamma \) can be taken to be \( [\text{SL}_2(\mathbb{Z}) : \Gamma] \) copies of a fundamental domain for \( \text{SL}_2(\mathbb{Z}) \) such as \( F \).
A.1. A Couple Relevant Properties of $\Gamma_0(N)$. We list a couple properties of $\Gamma_0(N)$ that can be used to better understand the structure of the Rademacher sums in section 7. The formulas in this section are from [65]. The index of $\Gamma_0(N)$ in $\text{SL}_2(\mathbb{Z})$ is given by

$$
[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).
$$

The number of inequivalent cusps $\nu_\infty(\Gamma_0(N))$ of $\Gamma_0(N)$ is given by

$$
\nu_\infty(\Gamma_0(N)) = \sum_{d|N} \phi(\gcd(d,N/d))
$$

where $\phi : \mathbb{N} \to \mathbb{N}$ is Euler’s totient. The number of elliptic points $\nu_2(\Gamma_0(N))$ of order 2 in $\Gamma_0(N) \backslash \mathbb{H}^*$ is given by

$$
\nu_2(\Gamma_0(N)) = \begin{cases} 
0 & (d_2 > 1 \text{ or } d_3 > 0), \\
2^{d_1} & \text{(otherwise)}
\end{cases}
$$

where $d_1, d_2, d_3$ are the number of primes equivalent to 1, 2, 3 modulo 4 respectively that divide $N$. The number of elliptic points $\nu_3(\Gamma_0(N))$ of order 3 in $\Gamma_0(N) \backslash \mathbb{H}^*$ is given by

$$
\nu_3(\Gamma_0(N)) = \begin{cases} 
0 & (d_3 > 1 \text{ or } d_2 > 0), \\
2^{d_1} & \text{(otherwise)}
\end{cases}
$$

where $d_1, d_2, d_3$ are the number of primes equivalent to 1, 2, 3 modulo 3 respectively that divide $N$. Let $\#_1(N)$ be the number of solutions modulo $N$ to $x^2 + 1 \equiv 0 \pmod{N}$ and $\#_2(N)$ be the number of solutions modulo $N$ to $x^2 - x + 1 \equiv 0 \pmod{N}$. Then the genus $g(\Gamma_0(N))$ of $\Gamma_0(N) \backslash \mathbb{H}^*$ is given by Prop. 1.43 in [66]

$$
g(\Gamma_0(N)) = 1 + \frac{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}{12} - \frac{\#_1}{4} - \frac{\#_2}{3} - \frac{\nu_\infty(\Gamma_0(N))}{2}.
$$

Appendix B. Proof of Theorem 7.6

We will use Rademacher’s modification of the Hardy-Ramanujan-Littlewood circle method to compute $d(n)$. We present some lemmas which are contained in the original papers [56][55] without proof, or alternatively the exposition piece [67]. I will also skip some of the more tedious error bounding, which can be found in full in [11].

As in Rademacher’s computation of the Fourier coefficients of $1/\eta(\tau)$, we extract $d(n)$ by performing a Fourier-Laplace transform:

$$
d(n) = \frac{1}{2\pi i} \int_\gamma \frac{Z(q) \cdot q^{n_0}}{q^{n+1}} \, dq
$$
where $\gamma$ is a closed contour winding once around the origin. In essence, that is all there is to it. The rest of the proof is simply computing this integral.

We will define a sequence of suitable contours $\{\gamma_N\}_{N=1}^{\infty}$, compute the integral in (Eq. 244) for $\gamma = \gamma_N$ up to an error term, take $N \to \infty$ and show that the error term converges to zero. It is more convenient to define the contours in the $\tau$-plane and then map them into the $q$-plane. We will denote a pullback of some contour $\gamma$ in the $q$-plane to the $\tau$-plane as $\tau(\gamma)$.

Some preliminary definitions are in order. The $N$th Farey sequence $\mathcal{F}_N$ is the finite sequence containing all irreducible fractions in $[0, 1]$ of denominator at most $N$ in increasing order. The Ford circle $C(h/k)$ associated with the irreducible fraction $h/k$ is the circle in the $\tau$-plane with center $h/k + i/2k^2$ and radius $1/2k^2$. See (Fig. 5). We denote by $q(C(h/k))$ the mapping of $C(h/k)$ into the $q$-plane. Note that $q(C(0/1)) = q(C(1/1))$. It can be shown that the Ford circles corresponding to consecutive fractions $h_1/k_2$ and $h_2/k_2$ in some Farey sequence are tangent at the point

$$\tau(h_1/k_1, h_2/k_2) = \frac{h_1 k_1 + h_2 k_2 + i}{k_1^2 + k_2^2}.$$ (245)

For irreducible fractions $h_0/h_0 < h_1/k_1 < h_2/k_2$ with $C(h_0/k_0)$ and $C(h_2/k_2)$ tangent to $C(h_1/k_1)$ let $\tau(\gamma_{h_0/k_0, h_1/k_1, h_2/k_2})$ be the arc on $C(h_1/k_1)$ from the point of tangency with $C(h_0/k_0)$ to the point of tangency with $C(h_2/k_2)$ parametrized by arc length. We choose the contour to proceed around the Ford circle clockwise so that the arc does not touch the real line. Likewise for $h_2/k_2$ such that $C(h_2/k_2)$ is tangent to $C(0/1)$ let $\tau(\gamma_{0/1, h_2/k_2})$ be the arc on $C(0/1)$ from the point $+i$ to the point of tangency with $C(h_2/k_2)$ parametrized by arc length. Likewise for $h_0/k_0$ such that $C(h_0/k_0)$ is tangent to $C(1/1)$ let $\tau(\gamma_{h_0/k_0, 1/1})$ be the arc on $C(0/1)$ from the point of tangency with $C(h_0/k_0)$ to the point $1 + i$ parametrized by arc length. If we specify an $N \in \{1, 2, \ldots\}$ then we may define $\tau(\gamma_{N, h/k})$ for $h/k \in \mathcal{F}_N$ to be

$$\tau(\gamma_{N, h/k}) = \begin{cases} 
\tau(\gamma_{0/1, h_2/k_2}) & (h/k = 0/1) \\
\tau(\gamma_{h_0/k_0, h/k, h_2/k_2}) & (k \neq 1) \\
\tau(\gamma_{h_0/k_0, 1/1}) & (h/k = 1/1)
\end{cases}$$ (246)

where $h_0/k_0$ is the element in $\mathcal{F}_N$ immediately before $h/k$ if such an element exists and $h_2/k_2$ is the element in $\mathcal{F}_N$ immediately after $h/k$ if such an element exists.

We define $\tau(\gamma_N)$ to be the concatenation in order of each $\tau(\gamma_{N, h/k})$ for $h/k \in \mathcal{F}_N$. The contour $\gamma_N$ is then the mapping of $\tau(\gamma_N)$ into the $q$-plane. This is a concatenation of
the contours $\gamma_{N,h/k}$ for $h/k \in \mathcal{F}_N$. We redefine $\gamma_{N,0/1}$ to be the concatenation of what we used to call $\gamma_{N,0/1}$ and $\gamma_{N,1/1}$. These contours meet in the $q$-plane. The full contour $\gamma_N$ is piecewise smooth and has winding number one about the origin. See Fig. 6 for visualizations of the Rademacher contour $\gamma_N$ and $\tau(\gamma_N)$ for various values of $N$.

We now split up the contour integral in (Eq. 244) into a sum of integrals over subcontours:

\begin{equation}
(247) \quad d(n) = \frac{1}{2\pi i} \sum_{k=1}^{N} \sum_{1 \leq h < k} \int_{\gamma_{N,h/k}} Z(q) \frac{Z(q)}{q^{(n-n_0)+1}} dq
\end{equation}

where $(h, k)$ is shorthand for $\gcd(h, k)$. It is convenient to change coordinates within each subcontour integral in order to write them as integrals over similar contours. Note that for irreducible $h/k$ the coordinate transformation

\begin{equation}
(248) \quad z = -ik^2 \left( \tau - \frac{h}{k} \right) \quad \text{or equivalently} \quad \tau = i \cdot \frac{z}{k^2} + \frac{h}{k}
\end{equation}
maps the Ford circle \( C(h/k) \) in the \( \tau \)-plane to the circle \( B_{1/2}(1/2) \) in the \( z \)-plane with center \( 1/2 \) and radius \( 1/2 \). See Fig. 7. The point \( \tilde{\tau}(h/k, h_2/k_2) \) is mapped to the point

\[
(249) \quad \tilde{z}_{h/k}(h_2/k_2) = \frac{k^2}{k^2 + k_2^2} + i \left( hk - \frac{k^2}{k^2 + k_2^2} (hk + h_2k_2) \right).
\]

We moved the \( h/k \) into the subscript of \( \tilde{z} \) to emphasize that the coordinate transformation depends on \( h/k \) and that, for this reason, unlike \( \tilde{\tau} \), \( \tilde{z} \) is not symmetric under interchanging its arguments. The contour \( \tau(\gamma_{N,h/k}) = \tau(\gamma_{h_0/k_0,h/k,h_2/k_2}) \) is mapped to an arc along \( B_{1/2}(1/2) \) from \( \tilde{z}_{h/k}(h_0/k_0) \) to \( \tilde{z}_{h/k}(h_2/k_2) \), specifically the arc which does not contain the origin. Likewise, the contours \( \tau(\gamma_{0,1,h_2/k_2}) \) and \( \tau(\gamma_{h_0/k_0,1/1}) \) are mapped together to an arc along \( B_{1/2}(1/2) \) from \( \tilde{z}_{1/1}(h_0/k_0) \) to \( \tilde{z}_{0/1}(h_2/k_2) \), also the arc which does not contain the origin. Let \( \tilde{z}_{1,N,h/k} \) be \( \tilde{z}_{h/k}(h_0/k_0) \) where \( h_0/k_0 \) is the element of \( \mathcal{F}_N \) immediately before \( h/k \) if \( k \neq 1 \) and \( \tilde{z}_{1/1}(h_0/k_0) \) where \( h_0/k_0 \) is the element of \( \mathcal{F}_N \) immediately before \( 1/1 \) if \( k = 1 \). For irreducible \( h/k \) except \( 1/1 \) let \( \tilde{z}_{2,N,h/k} \) be \( \tilde{z}_{h/k}(h_2/k_2) \) where \( h_2/k_2 \) is the element of \( \mathcal{F}_N \) immediately after \( h/k \). It can be checked that

\[
(250) \quad \tilde{z}_{1,N,h/k} = \frac{k}{k^2 + k_0^2} (k + ik_0)
\]

\[
(251) \quad \tilde{z}_{2,N,h/k} = \frac{k}{k^2 + k_2^2} (k - ik_2)
\]

where \( h_0/k_0 < h/k < h_2/k_2 \) are consecutive fractions in \( \mathcal{F}_N \) or if \( h/k = 0/1 \) and \( h_0/k_0 \) is immediately preceding \( 1/1 \) or if \( h/k = 1/1 \) and \( h_2/k_2 \) is immediately following \( 0/1 \). See Fig. 7.

Performing the coordinate transformations to Eq. 247

\[
(252) \quad d(n) = i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k) = 1} \int_{z(\gamma_{N,h/k})} Z \left( \exp \left( 2\pi i \left( \frac{h}{k} - \frac{z}{k^2} \right) \right) \right) \exp \left( 2\pi (n - n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right) \right) \text{d}z.
\]

Here \( z(\gamma_{N,h/k}) \) is the mapping of \( \tau(\gamma_{N,h/k}) \) into the \( z \)-plane. That is, \( z(\gamma_{N,h/k}) \) is the arc of \( B_{1/2}(1/2) \) which avoids the origin and is from \( \tilde{z}_{1,N,h/k} \) to \( \tilde{z}_{2,N,h/k} \). Tracing through the definitions of \( z \) and \( Z(q) \), the integrands above are holomorphic in the right-half of the \( z \)-plane. We may therefore deform our subcontours from arcs on \( B_{1/2}(1/2) \) to chords through \( B_{1/2}(1/2) \). These chords begin at \( \tilde{z}_{1,N,h/k} \) and end at \( \tilde{z}_{2,N,h/k} \). We denote these
before we proceed, we state two geometric lemmas. the first concerns the properties of the chords $z_1z_2(N,h/k)$ and the second concerns the properties of arcs on $B_{1/2}(1/2)$. proofs of these are contained in [67].

**Lemma B.1.** The chord $z_1z_2(N,h/k)$ has length at most $2\sqrt{2}k/N$ and on this chord $|z| \leq \sqrt{2}k/N$. ■

**Lemma B.2.** In $B_{1/2}(1/2)/\{0\}$, Re$(z) \leq 1$ and Re$(1/z) \geq 1$ with Re$(1/z) = 1$ on the circle itself. On the arcs from 0 to $z_1,N,h/k$ and $z_2,N,h/k$ to 0, $|z| \leq \sqrt{2}k/N$ and the length of these arcs is at most $\pi \sqrt{2}k/N$. ■

By the previous two lemmas, for fixed $h/k$ as $N \to \infty$ the chords $z_1z_2(N,h/k)$ get shorter and nearer to the origin. As $z$ approaches the origin, $\tau = (h/k) + i(z/k^2)$ approaches $h/k$. We can calculate the asymptotics of $\eta(\tau)$ as $\tau$ approaches $+i\infty$ from the definition of $\eta(\tau)$. We can then calculate the asymptotics of $Z(\tau)$ near $h/k$ in terms of the asymptotics of each $\eta(m\tau)$ near $+i\infty$ using the modularity properties of $\eta(\tau)$. These asymptotics are sufficiently simple to integrate them. This is the key insight in the Hardy-Littlewood-Ramanujan circle
method. We now turn to expressing $\eta(m\tau)$ for $m \in \{1, 2, \ldots\}$ near $h/k$ in terms of $\eta(\tau)$ near $+i\infty$.

For irreducible fraction $h/k \in [0, 1]$, we may find by the Euclidean algorithm some integer $H(m, h, k) = H$ such that

$$mhH \equiv -\gcd(m, k) \mod k.$$ (254)

It follows that the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\text{SL}_2(\mathbb{Z})$, where

$$a = H, \quad b = -\frac{1}{k}(mhH + \gcd(m, k)), \quad c = \frac{k}{\gcd(m, k)}, \quad d = -\frac{mh}{\gcd(m, k)}.$$ (255)

As a member of the modular group, this matrix induces an action on the upper-half of the complex plane, given by $M(\tau) = \frac{a\tau + b}{c\tau + d}$ for any $\tau \in \mathbb{H}$. Under this action,

$$M(m\tau) = M \left( m \left( \frac{h}{k} + i \cdot \frac{z}{k^2} \right) \right) = \frac{\gcd(m, k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m, k) \right) = \frac{a\tau + b}{c\tau + d}.$$ (256)
As $z$ approaches $0$ – and therefore $\tau$ approaches $h/k$ – the right hand side goes to positive imaginary infinity, as desired. Using the modular transformation properties of $\eta(\tau)$,

$$\eta(m\tau) = [\epsilon(a,b,c,d)(cm\tau + d)^{1/2}]^{-1} \eta \left( \frac{am\tau + b}{cm\tau + d} \right).$$

In this case, $cm\tau + d = \im z/k \gcd(m,k)$ and by Eq. 196

$$\epsilon(a,b,c,d) = \exp \left( \pi i \left( \frac{H}{12k} \gcd(m,k) - \frac{mh}{12k} + s \left( \frac{mh}{\gcd(m,k)}, \frac{k}{\gcd(m,k)} \right) - \frac{1}{4} \right) \right),$$

so that

$$\eta \left( m \left( \frac{h}{k} + i \cdot \frac{z}{k^2} \right) \right) = \exp \left( \pi i \left( - \frac{H}{12k} \gcd(m,k) + \frac{mh}{12k} - s \left( \frac{mh}{\gcd(m,k)}, \frac{k}{\gcd(m,k)} \right) \right) \right) \times \sqrt{\frac{k \gcd(m,k)}{mz}} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m,k) \right) \right).$$

The constant $H = H(m,h,k)$ depends implicitly on $m, h,$ and $k$. Combining (Eq. 259) for all values of $m$

$$Z \left( \frac{h}{k} + i \cdot \frac{z}{k^2} \right) = \xi(h,k)\omega(h,k) \cdot z^{c_1} \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m,k) \right) \right) \delta_m,$$

where $c_1$ was defined in (Eq. 202) and we defined

$$\xi(h,k) = \prod_{m=1}^{\infty} \left[ \sqrt{\frac{k \gcd(m,k)}{m}} \exp \left( \pi i \left( \frac{mh}{12k} - s \left( \frac{mh}{\gcd(m,k)}, \frac{k}{\gcd(m,k)} \right) \right) \right) \right]^{\delta_m}$$

and

$$\omega(h,k) = \prod_{m=1}^{\infty} \exp \left( - \frac{\pi i}{12k} H \delta_m \gcd(m,k) \right).$$

Plugging in the previous formulas into (Eq. 253):

$$d(n) = i \sum_{k=1}^{N} k^{-2} \sum_{0 < h < k \atop (h,k)=1} \xi(h,k)\omega(h,k) \int_{\gamma_1} z^{c_1} \left[ \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m,k) \right) \right) \delta_m \right] \times \exp \left( 2\pi(n-n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right) \right) \, dz.$$
We expect to be able to replace each \( \eta(q) \) in the integrand with \( q^{1/24} \) and accrue a total \( o(1) \) error as \( N \to \infty \). Defining

\[
\begin{align*}
\text{Er}(N) &= i \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} k^{-2} \xi(h,k) \omega(h,k) \\
&\quad \int_{\mathbb{P}^1 \times \mathbb{P}^1(N,h/k)} z^{c_1} \exp \left( 2\pi(n - n_0) \left( \frac{z}{k^2} - i \frac{h}{k} \right) \right) \Delta_{h,k}(z) \, dz,
\end{align*}
\]

where \( \Delta_{h,k}(z) \) is the difference between the \( \eta \)-quotient and its leading order asymptotics

\[
\begin{align*}
\Delta_{h,k}(z) &= \left[ \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right) \right] \\
&\quad - \left[ \prod_{m=1}^{\infty} \exp \left( \frac{\pi i \gcd(m,k)}{12} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right) \right] \delta_m,
\end{align*}
\]

we can write

\[
\begin{align*}
d(n) &= \text{Er}(N) + i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k)=1} \xi(h,k) e^{-2\pi i (n-n_0) \frac{h}{k}} \\
&\quad \int_{\mathbb{P}^1 \times \mathbb{P}^1(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n - n_0)}{k^2} z + \frac{\pi c_3(k)}{12} \frac{z}{z} \right] \, dz.
\end{align*}
\]

The second term is the result of replacing each \( \eta \)-function by the appropriate asymptotics, and the first term is the accumulated error from doing so. It can be shown that \( \lim_{N \to \infty} \text{Er}(N) = 0 \).

Referring back to (Eq. 266), we have shown that

\[
\begin{align*}
d(n) &= o(1) + i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k)=1} \xi(h,k) e^{-2\pi i (n-n_0) \frac{h}{k}} \\
&\quad \int_{\mathbb{P}^1 \times \mathbb{P}^1(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n - n_0)}{k^2} z + \frac{\pi c_3(k)}{12} \frac{z}{z} \right] \, dz.
\end{align*}
\]
We now deform our contours back to arcs along $B_{1/2}(1/2)$:

\[(268)\quad d(n) = o(1) + i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k) = 1} \xi(h,k) e^{-2\pi i (n-n_0)h/k} \int_{\gamma_{N,h/k}} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2}z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] \, dz.\]

The next goal is to show that the main term on the right hand side above,

\[(269)\quad i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k) = 1} \xi(h,k) e^{-2\pi i (n-n_0)h/k} \int_{\gamma_{N,h/k}} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2}z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] \, dz,
\]

differs from

\[(270)\quad i \sum_{k=1}^{N} k^{-2} \sum_{0 \leq h < k \atop (h,k) = 1} \xi(h,k) e^{-2\pi i (n-n_0)h/k} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2}z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] \, dz\]

by an $o(1)$ term as $N \to \infty$. The contour in the second expression is traversed clockwise. In other words, we may replace our integrals over incomplete arcs of $B_{1/2}(1/2)$ by integrals over the complete circle $B_{1/2}(1/2)$ and only accrue a total $o(1)$ error as $N \to \infty$. The former is the latter minus $J_1 + J_2$, where $J_1 = J_1(N)$ and $J_2 = J_2(N)$ are defined by

\[(271)\quad J_1 = i \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k) = 1} k^{-2} \xi(h,k) e^{-2\pi i (n-n_0)h/k} \int_{\gamma_{1}(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2}z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] \, dz,
\]

and

\[(272)\quad J_2 = i \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k) = 1} k^{-2} \xi(h,k) e^{-2\pi i (n-n_0)h/k} \int_{\gamma_{2}(N,h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2}z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] \, dz.
\]
It can be shown that $J_1 = o(1)$ and $J_2 = o(1)$. Combining all of the previous results,

\begin{equation}
(273) \quad d(n) = o(1) + i \sum_{k=1}^{N} k^{-2} \sum_{\substack{0 \leq h < k \\ \gcd(h,k) = 1}} \xi(h,k) e^{-2\pi i (n-n_0) \frac{1}{k}}
\end{equation}

\[ \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi (n-n_0)}{k^2} \frac{1}{z} + \frac{\pi c_3(k)}{12} \right] \, dz. \]

Taking $N \to \infty$,

\begin{equation}
(274) \quad d(n) = i \sum_{k=1}^{\infty} k^{-2} \sum_{\substack{0 \leq h < k \\ \gcd(h,k) = 1}} \xi(h,k) e^{-2\pi i (n-n_0) \frac{1}{k}}
\end{equation}

\[ \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi (n-n_0)}{k^2} \frac{1}{z} + \frac{\pi c_3(k)}{12} \right] \, dz. \]

Referring to the definition of $\xi(h,k)$ in (Eq. 261) and of the Kloosterman-like sum $A_k(n)$ in (Eq. 203), this is exactly

\begin{equation}
(275) \quad d(n) = i \sum_{k=1}^{\infty} k^{-2+2c_1} c_2(k) A_k(n) \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi (n-n_0)}{k^2} \frac{1}{z} + \frac{\pi c_3(k)}{12} \right] \, dz,
\end{equation}

Now we just evaluate this integral

\begin{equation}
(276) \quad I = \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi (n-n_0)}{k^2} \frac{1}{z} + \frac{\pi c_3(k)}{12} \right] \, dz.
\end{equation}

First note that if $c_3(k) = 0$, then the integrand is everywhere holomorphic so that by the Cauchy integral formula $I = 0$. Otherwise, we can rewrite it as

\begin{equation}
(277) \quad I = \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{\pi}{k} \frac{|c_3(k)|}{6} (n-n_0) \left( \frac{z}{k} \frac{24(n-n_0)}{|c_3(k)|} \right) \right] \, dz
\end{equation}
with the ± given by \( \text{sign}(c_3(k)) \). We make the substitution

\[
\begin{align*}
    w &= \left( \frac{z}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1}, \\
    z &= \left( \frac{w}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1}, \\
    dz &= -\left( \frac{w^2}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1} \, dw,
\end{align*}
\]

whence

\[
(278) \quad I = - \left( k \sqrt{\frac{|c_3(k)|}{24(n - n_0)}} \right)^{c_1+1} \int_{1-i\infty}^{1+i\infty} w^{-(c_1+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) \left( w^{-1} \pm w \right) \right] \, dw.
\]

We now split into two cases depending on the sign of \( c_3(k) \). If \( c_3(k) < 0 \), then the integrand decays sufficiently rapidly in the right-half plane such that

\[
(279) \quad I = - \left( k \sqrt{\frac{|c_3(k)|}{24(n - n_0)}} \right)^{c_1+1} \lim_{R \to \infty} \int_{S(R)} w^{-(c_1+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) \left( w^{-1} - w \right) \right] \, dw
\]

where \( S(R) \) is the right semicircle of radius \( R \), centered at 1, traversed clockwise. The integrand is holomorphic inside this contour, since it does not contain the origin, so that by the Cauchy integral formula \( I = 0 \). Otherwise, if \( c_3(k) > 0 \), then the integrand decays sufficiently rapidly in the left-half plane such that

\[
(280) \quad I = - \left( k \sqrt{\frac{|c_3(k)|}{24(n - n_0)}} \right)^{c_1+1} \int w^{-(c_1+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) \left( w^{-1} + w \right) \right] \, dw
\]
for any positively oriented closed contour winding once around the origin. We rearrange the terms in the integral slightly:

\[ I = -2\pi i \left( k \sqrt{ \frac{|c_3(k)|}{24(n-n_0)} } \right)^{c_1+1} \frac{1}{2\pi i} \int w^{-(c_1+1)-1} \exp \left[ \frac{1}{2} \sqrt{ \frac{2}{3} } c_3(k)(n-n_0) \left( w^{-1} + w \right) \right] \, dw. \]

This integral is a standard form of the modified Bessel function of the first kind [68][59]:

\[ I = -2\pi i \left( k \sqrt{ \frac{|c_3(k)|}{24(n-n_0)} } \right)^{c_1+1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{ \frac{2}{3} } c_3(k)(n-n_0) \right] \]

where \( I_{c_1+1} \) is the modified Bessel function of the first kind of weight \( c_1 \). To summarize,

\[ I = \begin{cases} 0 & (c_3(k) \leq 0), \\ -2\pi i \left( k \sqrt{ \frac{|c_3(k)|}{24(n-n_0)} } \right)^{c_1+1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{ \frac{2}{3} } c_3(k)(n-n_0) \right] & (c_3(k) > 0). \end{cases} \]

Substituting \( I \) into (Eq. 275) and simplifying, we get our final expression

\[ d(n) = 2\pi \left( \frac{1}{24(n-n_0)} \right)^{c_1+1} \sum_{k=1}^{\infty} \sum_{c_3(k)>0} c_2(k)c_3(k)^{c_1+1} A_k(n) I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{ \frac{2}{3} } c_3(k)(n-n_0) \right]. \]

This completes the proof.

That being said, while proofs are nice, numerics are also nice. To convince the reader that the series given in (Eq. 205) is actually converging to \( d(n) \), I have included some numerics, originally computed (using Mathematica) for [11], in Figure 8. Here

\[ d(n, K) = 2\pi \left( \frac{1}{24(n-n_0)} \right)^{c_1+1} \sum_{k=1}^{K} \sum_{c_3(k)>0} c_2(k)c_3(k)^{c_1+1} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{ \frac{2}{3} } c_3(k)(n-n_0) \right]. \]

That is, \( d(n, K) \) is the \( K \)th partial sum of (Eq. 205). The convergence of \( d(n, K) \) to \( d(n) \) as \( K \to \infty \) is clear. The chaotic nature of the convergence, however, hints at the complexity of the number theory involved.
Figure 8. Convergence of the $K$th partial sums of (Eq. 205) for the listed $\eta$-quotients: $Z(q) = 1/\eta(4\tau)\eta(\tau)^3$ (upper left), $Z(q) = \eta(4\tau)/\eta(\tau)^3$ (upper right), $Z(q) = 1/\eta(2\tau)$ (middle left), $Z(q) = 1/\eta(11\tau)^2\eta(\tau)^2$ (middle right), $Z(q) = 1/\eta(\tau)\eta(22\tau)$ (bottom left), $Z(q) = 1/\eta(\tau)\eta(23\tau)$ (bottom right). The vertical axis is $|d(n, K) - d(n)|$ and the horizontal axis is $K$. The vertical axis is scaled logarithmically and the horizontal axis is scaled linearly. Each line is a plot of $d(n, K)$ for fixed $n$ and variable $K$. 
References


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