ORIENTABILITY OF MODULI SPACES AND OPEN 
GROMOV-WITTEN INVARIANTS

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Chapter 1

Introduction

The theory of $J$-holomorphic maps was introduced by Gromov and is one of the most profound methods in the study of symplectic manifolds. Applications in theoretical physics led to the development of the Gromov-Witten invariants. They are invariants of the symplectic manifold and can be interpreted as a count of $J$-holomorphic maps from a closed Riemann surface passing through prescribed constraints in the manifold. Open String Theory motivated the study of $J$-holomorphic maps from a bordered Riemann surface with boundary mapping to a Lagrangian submanifold, and predicts the existence of open Gromov-Witten invariants. Their mathematical definition, however, has proved to be a subtle problem. Two main obstacles are the question of orientability and the existence of real codimension one boundary strata of the moduli space of maps from a bordered Riemann surface. This work concentrates on these issues.

In the first part of this thesis, we show that the local system of orientations on the moduli space of $J$-holomorphic maps from a bordered Riemann surface is isomorphic to the pull-back of a local system defined on the product of the Lagrangian and its free loop space. The proof builds on the works of Fukaya et al. [FOOO], where the authors provide a canonical orientation of the moduli space of disks when the Lagrangian is relatively spin, and of Solomon [Sol], where this result was extended to surfaces of higher genus and a relatively pin$^\pm$ Lagrangian.
Before beginning a detailed discussion, we set up some necessary notation. Let \((M, \omega)\) be a symplectic manifold and \(L \subset M\) a Lagrangian submanifold. To define the moduli space we shall fix homology classes \(b = (b, b_1, \ldots, b_h) \in H_2(M, L) \oplus H_1(L)^{\oplus h}\), a bordered Riemann surface \((\Sigma, \partial \Sigma; j)\) with a fixed complex structure \(j\) and ordering of the boundary components \(\partial \Sigma = \bigsqcup_{i=1}^h (\partial \Sigma)_i \cong \bigsqcup_{i=1}^h S^1\), and finally integers \(l \in \mathbb{N}, k = (k_1, \ldots, k_h) \in \mathbb{N}^h\).

We consider the moduli space \(\mathcal{M}_{l,k}(\Sigma, b)\) of unparametrized \(J\)-holomorphic maps from the marked bordered Riemann surface \(\Sigma\) to \(M\), with boundary \(\partial \Sigma\) mapping to \(L\), which represent the class \(b\) in the relative homology \(H_2(M, L)\), and for which the restriction \(u_{|(\partial \Sigma)_i}\) represent the class \(b_i\) in the homology group \(H_1(L)\). The numbers \(l\) and \(k_i\) control the numbers of interior and boundary marked points. We shall denote by \(\overline{\mathcal{M}}_{l,k}(\Sigma, b)\) the Gromov compactification of \(\mathcal{M}_{l,k}(\Sigma, b)\).

Let \(\mathcal{B}(\Sigma, b)\) be the space of maps from the bordered Riemann surface \(\Sigma\) to \(M\), with boundary \(\partial \Sigma\) mapping to \(L\), which represent the class \(b \in H_2(M, L)\), and for which the restriction \(u_{|(\partial \Sigma)_i}\) represent the class \(b_i \in H_1(L)\). The determinant line of the linearization of the \(\bar{\partial}\)-operator forms a line bundle over \(\mathcal{B}(\Sigma, b)\), which we shall denote with \(\text{det}(D)\). The significance of this bundle comes from the fact that when the moduli space is cut transversely, the top exterior power of its tangent bundle is essentially the bundle \(\text{det}(D)\).

Let \(\mathcal{L}(L)\) denote the free loop space of \(L\). There are canonical maps

\[
ev_i^1 : \mathcal{B}(\Sigma, b) \to L \quad \text{assigning to a map } u, \text{ its evaluation at } x_1 \in S^1 \cong \partial \Sigma_i,
\]

\[
ev_{\mathcal{L}(L)}^1 : \mathcal{B}(\Sigma, b) \to \mathcal{L}(L) \quad \text{assigning to a map } u, \text{ the element } u_{|(\partial \Sigma)_i} : (\partial \Sigma)_i \to L,
\]

a forgetful map \(f : \mathcal{B}_{l,k}(\Sigma, b) \to \mathcal{B}(\Sigma, b)\) given by forgetting the marked points,

and a projection \(\pi : \mathcal{B}_{l,k}(\Sigma, b) \to \mathcal{B}_{l,k}(\Sigma, b)/\text{Aut}(\Sigma, \partial \Sigma; j) \supset \mathcal{M}_{l,k}(\Sigma, b)\).
We show that the first Stiefel-Whitney class of the determinant line bundle \( \text{det}(D) \), evaluated on a loop \( \gamma \), is equal to

\[
    w_1(\text{det} D) \cdot \gamma = \sum_{i=1}^{h} (w_1(TL) \cdot b_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^{h} w_2(TL) \cdot \beta_i
\]

where \( \alpha_i \) is the loop a marked point on \( \partial \Sigma_i \) traces in \( L \), and \( \beta_i \) is the torus \( \partial \Sigma_i \) traces in \( L \). We do this by showing that the determinant line of \( D \) is isomorphic to the tensor product of the determinant line of a \( \bar{\partial} \)-operator on a line bundle, and the determinant line of a \( \bar{\partial} \)-operator on an orientable bundle. The evaluation of their first Stiefel-Whitney classes on \( \gamma \) then gives the two parts in the formula.

The presence of \( w_2(L) \) in the above formula means that in the general case, the local system of orientations on \( \mathcal{M}_{l,k}(\Sigma, b) \) is not the pull-back of a system on \( L \), but rather of a system on the free loop space of \( L \). We construct a local system \( Z_{(w_1, w_2)} \) on the \( h \)-fold product of the Lagrangian and its free loop space, which traces the twisting coming from the right-hand side of the formula, and we show that its pull-back is isomorphic to the local system twisted by the first Stiefel-Whitney class of \( \text{det}(D) \).

Let us combine the evaluation maps in a map \( e\tilde{v}_L \times e\tilde{v}_{\mathcal{L}(L)} : \mathcal{B}(\Sigma, b) \to (L \times \mathcal{L}(L))^h \). The main result of the first part is Theorem 2.5.1 stating

**Theorem.** There is a local system \( Z_{(w_1, w_2)} \) on \( (L \times \mathcal{L}(L))^h \), such that the local system of orientations on \( \mathcal{M}_{l,k}(\Sigma, b) \) is isomorphic to \( \pi_1 \circ \jmath^* \circ (e\tilde{v}_L \times e\tilde{v}_{\mathcal{L}(L)})^* Z_{(w_1, w_2)} \). The isomorphism is canonical once we choose a trivialization of \( TL \) over a basepoint in \( L \), and a trivialization of \( TL \oplus 3\text{det}(TL) \) over those loops in \( L \) corresponding to a choice of a basepoint in each component of \( \mathcal{L}(L) \).

In the second part, we restrict ourselves to Lagrangian submanifolds which are the fixed locus of an anti-symplectic involution \( \tau \) on \( M \), and under certain topological restrictions, we define open Gromov-Witten invariants. The proof builds upon the works of Cho [Cho], and Solomon [Sol], who define such invariants in dimension 4.
and 6 using point constraints. We generalize this to higher dimensions and any type of constraints. The brief idea is to use the anti-symplectic involution to glue together the boundaries of several moduli spaces. In this way we obtain a space without a boundary. We determine the local system of orientations on this space, and when this system is a pull-back from the product of the Lagrangian and the analogue of the Deligne-Mumford moduli space, we define the new invariants similarly to the classical case.

The construction is largely motivated by the fact that the moduli space of real sphere maps, which has no boundary, is covered by moduli spaces of disk maps, representing classes whose double with respect to the involution $\tau$, is the fixed sphere class. These are the moduli spaces which we are going to glue together, simulating the matching of the real spheres they cover. The moduli space of real sphere maps $\mathcal{R}\mathcal{M}_{l+1,k}(\mathbb{C}P^1, A)$ is the space of unparametrized $J$-holomorphic maps from $\mathbb{C}P^1$ to $M$, with $l+1$ pairs of complex conjugate marked points, and $k$ real marked points, such that the maps satisfy $\tau \circ u \circ c_{\mathbb{C}P^1} = u$, where $c_{\mathbb{C}P^1}$ is the standard conjugation on $\mathbb{C}P^1$. This space is considered by Welschinger in [Wel05] and [Wel08], where the author defines enumerative invariants, which can be interpreted as counting the number of real spheres passing through prescribed points. We prove in Section 3.2, that the new moduli space $\tilde{\mathcal{M}}_{l+1,k}(A)$ is isomorphic to the moduli space of real spheres $\mathcal{R}\mathcal{M}_{l+1,k}(A)$. Moreover, the canonical evaluation maps on these spaces have the same image in $M^{l+1} \times L^k$, and therefore we expect the two invariants associated to these spaces to be the same.

In more detail, to establish the local equivalence between the pointed disk maps and the pointed sphere maps, we decorate with $+$ and $-$ the interior marked points of the disk, except a preferred one $z_0$, which we always assume has decoration $+$. This can be described by the decorated moduli space $\tilde{\mathcal{M}}_{l+1,k}(b) = \mathcal{M}_{l+1,k}(b) \times \mathbb{Z}_2$. There is a map $g$ defined on the boundary components formed from two bubbles, given as identity on the first bubble, and by conjugating the map, the marked points and the
decorations on the second. For a fixed class $A$ in $\pi_2(M)$, we use this map to identify the boundaries of the disjoint union of moduli spaces of disk maps, representing classes $b$, whose double with respect to $\tau$, is equal to $A$. In Theorem 3.2.1, we prove that the thusly obtained space, denoted by $\tilde{M}_{l+1,k}(A)$, has no boundary, and that we have canonical evaluation maps from this space to $L$ and $M$. Taking the target $M$ to be a point in the above construction, yields the analogue of the Deligne-Mumford moduli space, denoted by $\tilde{M}_{l+1,k}$.

The space $\tilde{M}_{l+1,k}(A)$ is not necessarily orientable, but heuristically, it will have a fundamental class with coefficients in the local system of orientations of $\tilde{M}_{l+1,k}(A)$. We would like to know when this system is a pull-back of a system on $L^k \times M^{l+1} \times \tilde{M}_{l+1,k}$, as in these cases we can define the open Gromov-Witten invariants as in the classical case – by pairing the image of the fundamental cycle with cohomology classes with coefficients in the corresponding local system on $L^k \times M^{l+1} \times \tilde{M}_{l+1,k}$. We prove in Theorem 3.4.1, that when the Lagrangian is relatively spin, and under certain admissibility condition defined in Section 3.4, the orientation system of $\tilde{M}_{l+1,k}(A)$ is in fact the pull-back of the orientation system on the analogue of the Deligne-Mumford moduli space $\tilde{M}_{k+1,l}$. Examples of admissible manifolds are the complex projective spaces $\mathbb{C}P^{4s-1}$, when we consider no boundary marked points. With this, we show that the image of the new moduli space $\tilde{M}_{l+1,k}(A)$ in the product $\tilde{M}^{l+1} \times L^k \times \tilde{M}_{l+1,k}$, carries a homology class with coefficients twisted by the first Stiefel-Whitney class of $\tilde{M}_{l+1,k}$, and we define the open Gromov-Witten invariants to be the absolute value of the evaluation of cohomology classes with twisted coefficients on this class. We calculate the invariant to be 1, when $(M, L) = (\mathbb{C}P^{4s-1}, \mathbb{R}P^{4s-1})$, the class $A$ has degree one, the number of boundary marked points is zero, and there are two interior marked points, constrained by a point and a hyperplane.

Further, in the general case, we express the first Stiefel Whitney class of $\tilde{M}_{l+1,k}(A)$ by classes pulled-back from $L$ and the moduli space of domains $\tilde{M}$, plus Poincare duals of boundary divisors in $\tilde{M}_{l+1,k}(A)$. This result is useful in determining when the local system of orientations on $\tilde{M}_{l+1,k}(A)$ is the pull-back of a local system on target.
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The presence of the Poincare duals of the boundary divisors in the formula, however, means that in general, the orientation system of $\widetilde{\mathcal{M}}_{l+1,k}(A)$ is not a pull-back from $L \times \widetilde{\mathcal{M}}_{l+1,k}$. One approach to overcome this is to use a more refined version of the moduli space of domains $\widetilde{\mathcal{M}}_{l+1,k}$, which will be adequate to capture the contribution from these classes. This will be the object of future work.
Chapter 2

Orientability of Moduli Spaces

In this part, we show that the local system of orientations on the moduli space of $J$-holomorphic maps from a bordered Riemann surface $\Sigma$ with a fixed complex structure is isomorphic to the pull-back of a local system defined on the product of the Lagrangian and its free loop space. The latter system is defined using only the first and second Stiefel-Whitney classes of the Lagrangian and the isomorphism allows us to determine whether the moduli space is orientable or not.

The orientability question in the case $\Sigma = D^2$ was previously studied in [FOOO]. The authors showed that the moduli space is not always orientable. However, in the case of a relatively spin Lagrangian they proved that the moduli space is orientable, and that a choice of a relatively spin structure determines a canonical orientation. This result was extended by Solomon [Sol] to relatively pin$^\pm$ Lagrangians and Riemann surfaces of higher genus with a fixed complex structure. Solomon constructed a canonical isomorphism between the determinant line bundle of the moduli space, and the pull-back by the evaluation maps of a certain number of copies of $\text{det}(TL)$. Other related works are [EES, WW] in the relatively spin setting, and in the context of real spheres [Wel08]. We extend these results to any Lagrangian.
This part is organized as follows. In Section 2.2, we show that the first Stiefel-Whitney class of \(\det(D)\) evaluated on a loop \(\gamma\) is equal to

\[
\begin{align*}
    w_1(\det(D)) \cdot \gamma &= \sum_{i=1}^{h} (w_1(\mathcal{L}) \cdot b_i + 1) \cdot (w_1(\mathcal{L}) \cdot \alpha_i) + \sum_{i=1}^{h} w_2(\mathcal{L}) \cdot \beta_i
\end{align*}
\] (2.0.1)

where \(\alpha_i\) is the loop a marked point on \(\partial \Sigma_i\) traces in \(L\) and \(\beta_i\) is the torus \(\partial \Sigma_i\) traces in \(L\). When \(L\) is relatively spin or pin\(\pm\) one can show that the term involving \(w_2(\mathcal{L})\) vanishes, and moreover the formula becomes that of [Sol] and [FOOO]. See Corollary 2.2.2. The presence of \(w_2(\mathcal{L})\) in the formula above means that the local system of orientations on \(\mathcal{M}_{l,k}(\Sigma, b)\) is not a pull-back of a system on \(L\). In Section 2.3, we construct a local system \(\mathcal{Z}_{(w_1, w_2)}\) on the \(h\)-fold product of the Lagrangian and its free loop space, which traces the twisting coming from the right-hand side in (2.0.1), and in Section 2.4, we show its pull-back is canonically isomorphic to the local system twisted by the first Stiefel-Whitney class of \(\det(D)\). Lastly, in Section 2.5 we show that the system pushes-down to the system of local orientations on \(\mathcal{M}_{l,k}(\Sigma, b)\).

## 2.1 Conventions and preliminaries

Let \(X, Y\) be Banach spaces and \(D : X \to Y\) a Fredholm operator. The determinant line of \(D\) is defined as

\[
\det(D) := \Lambda^{\text{max}} \ker(D) \otimes \Lambda^{\text{max}} \text{coker}(D)^{\vee}
\]

For \(D_1 : X_1 \to Y_1\) and \(D_2 : X_2 \to Y_2\) Fredholm operators, there is a canonical isomorphism

\[
\det(D_1 \oplus D_2) \cong \det(D_1) \otimes \det(D_2) \quad (2.1.1)
\]

Moreover, for a family of Banach spaces parametrized by a topological space \(B\), the determinant lines of a family \(D_b : X_b \to Y_b\) for \(b \in B\) form a line bundle over \(B\).

Let \(\Sigma\) be a compact Riemann surface with boundary, and \((E, F) \to (\Sigma, \partial \Sigma)\)
a bundle pair consisting of a complex bundle $E \to \Sigma$ and a maximally totally real subbundle $F \subset E|_{\partial \Sigma}$ over the boundary of $\Sigma$. A real Cauchy-Riemann operator on $(E, F) \to (\Sigma, \partial \Sigma)$ is the sum of a complex Cauchy-Riemann operator $D : \Omega^0(E, F) \to \Omega^{0,1}(E)$ and a zeroth order term taking values in $\text{End}_\mathbb{R}(E)$. The space of real Cauchy-Riemann operators is contractible.

More detailed discussion of the above may be found in [MS], [WW]. The following results appear in the works [FOOO], [EES], [Sol], [WW].

If $\Sigma$ is a nodal surface (with boundary), its normalization $\tilde{\Sigma}$ is obtained by resolving the nodes. $\tilde{\Sigma}$ is a disjoint union of Riemann surfaces $\Sigma_i$, together with two special points for each node. A bundle pair $(E, F)$ on $\Sigma$ is a bundle pair on $\tilde{\Sigma}$, together with an isomorphism of the fibers at pairs of special points corresponding to each node. A real Cauchy-Riemann operator $D_{(E, F)}$ on $\Sigma$ is a real Cauchy-Riemann operator $\tilde{D}_{(E, F)} = \bigoplus_i D^i$ on $\tilde{\Sigma}$, restricted to sections which match (under the given isomorphisms) at the relevant special points.

By gluing together punctured disks around the special points in $\tilde{\Sigma}$, we obtain a smooth surface $\Sigma^r$ for a gluing parameter $r$. We can also glue the bundle pairs, and the operators $D^i$ to obtain an operator $D^r$ on $\Sigma^r$.

For a sufficiently large $r$, there is a canonical isomorphism of determinant lines

$$\det(D^r) \cong \det(\tilde{D}_{(E, F)}) \otimes \Lambda^{max}(\bigoplus_i E_{z_i} \oplus \bigoplus_j F_{x_j})^\vee$$  \hspace{1cm} (2.1.2)$$

and by (2.1.1) we have

$$\det(\tilde{D}_{(E, F)}) \otimes \Lambda^{max}(\bigoplus_i E_{z_i} \oplus \bigoplus_j F_{x_j})^\vee \cong \bigotimes_i \det(D^i) \otimes \Lambda^{max}(\bigoplus_i E_{z_i} \oplus \bigoplus_j F_{x_j})^\vee$$

Here $z_i$ (respectively $x_i$) is one of the pair of special points corresponding to the $i$-th interior (resp. boundary) node.
Moreover, the gluing maps satisfy an associativity property. The isomorphism (2.1.2) is independent of the order in which we deform the nodes, as long as we have a fixed order of the components of $\tilde{\Sigma}$ and its boundary.

**Remark 2.1.1.** Let $(E,F) \to (\Sigma, \partial \Sigma)$ be bundle pair, and suppose we have a trivialization of $E$ over a curve $C_\epsilon \subset \Sigma$ homotopic to one of the boundary components of $\Sigma$. This trivialization can always be extended in a small neighborhood of the curve. We can pinch $\Sigma$ along $C_\epsilon$ and obtain a nodal curve $\Sigma^s$, with a diffeomorphism $(\Sigma - C_\epsilon) \to (\Sigma^s - \text{node})$. We can pull-back the bundle pair $(E, F)$ to $(\Sigma^s - \text{node})$. The trivialization of $E$ in the neighborhood of the curve induces trivializations in a punctured neighborhood of the node. They can be uniquely extended over the normalization of $\Sigma^s$. We say that the bundle pair $(E, F)$ descends to a bundle pair on the nodal surface.

### 2.2 The first Stiefel-Whitney class of $\det(D)$

Let $(M, \omega)$ be a symplectic manifold, $L \subset M$ a Lagrangian submanifold and $\Sigma$ a bordered Riemann surface with a fixed complex structure $j$ and ordered boundary components $(\partial \Sigma)_i \cong S^1$ for $i = 1, \ldots, h$. For a fixed tuple of homology classes $b = (b, b_1, \ldots, b_h) \in H_2(M, L) \oplus H_1(L)^{\oplus h}$, denote with $\mathfrak{B}(\Sigma, b)$ the Banach manifold of $W^{1,p}$-maps from $\Sigma$ to $M$, $p > 2$, with boundary mapping to $L$, which represent the class $b$ in the relative homology group $H_2(M, L)$, and whose restrictions to the boundary component $(\partial \Sigma)_i$ represent the class $b_i$ in the homology group $H_1(L)$ for every $i = 1, \ldots, h$.

Let $J$ be an $\omega$-compatible almost complex structure on $M$, and $\nabla$ a connection on $TM$ that preserves $J$. We can define a family of Fredholm operators $D$ over $\mathfrak{B}(\Sigma, b)$, which at a $J$-holomorphic map $u$ is given by the linearization of $\bar{\partial}_J u = \frac{1}{2} (du + J \circ du \circ j)$. The determinant line of the family forms a line bundle $\det(D)$ over $\mathfrak{B}(\Sigma, b)$. In this section we shall express the first Stiefel-Whitney class of $\det(D)$ in terms of the first and second Stiefel-Whitney classes of the Lagrangian $L$. 
For simplicity of notation assume that $\Sigma$ has only one boundary component. The case of several boundary components is a direct extension and is discussed at the end of the section.

**Theorem 2.2.1.** Let $\gamma$ be a loop in $\mathfrak{B}(\Sigma, b)$. The following formula for the first Stiefel-Whitney class of $\det(D)|_{\gamma}$ holds

$$w_1(\det(D)|_{\gamma}) = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha) + w_2(TL) \cdot \beta$$

where $b_1$ is the homology class $\partial \Sigma$ represents in $H_1(L)$, $\alpha$ is a loop in $L$ that a point on $\partial \Sigma$ traces along $\gamma$, and $\beta$ is the torus in $L$ that $\partial \Sigma$ traces along $\gamma$.

**Remark 2.2.1.** The homotopy class of the loop $\alpha$ is independent of the choice of a point on $\partial \Sigma$, used to define $\alpha$. Therefore the value of $w_1(TL) \cdot \alpha$ depends only on the loop $\gamma$.

**Proof.** A connection on $TM$ induces a family of real Cauchy-Riemann operators $\bar{\partial}_{(TM,TL)}$ over $\mathfrak{B}(\Sigma, b)$, which at a point $u \in \mathfrak{B}(\Sigma, b)$ is given by the real Cauchy-Riemann operator associated with the pulled-back connection. We can homotope $D$ to $\bar{\partial}_{(TM,TL)}$ by homotoping the zeroth-order term to zero. This gives an isomorphism of determinant lines $\det(D) \cong \det(\bar{\partial}_{(TM,TL)})$. Moreover, any two such homotopies are connected through a homotopy, and hence the isomorphism is canonical. Let

$$(E, F) = (TM \oplus 3 \det \mathcal{C} TM, TL \oplus 3 \det TL)$$

and

$$(E^1, F^1) = (\det \mathcal{C} TM, \det TL)$$

The connection on $TM$ induces connections on $E^1, E$ and $4E^1$, and we can consider the families of real Cauchy-Riemann operators $\bar{\partial}_{(E,F)}$, $\bar{\partial}_{(E^1,F^1)}$ and $\bar{\partial}_{(4E^1,4F^1)}$. Their determinant lines form bundles over $\mathfrak{B}(\Sigma, b)$, and by (2.1.1) we have

$$\det(\bar{\partial}_{(E,F)}) \otimes \det(\bar{\partial}_{(E^1,F^1)}) \cong \det(\bar{\partial}_{(E \oplus E^1, F \oplus F^1)}) =$$

$$= \det(\bar{\partial}_{(TM \oplus 4E^1, TL \oplus 4F^1)}) \cong \det(\bar{\partial}_{(TM,TL)}) \otimes \det(\bar{\partial}_{(4E^1,4F^1)})$$
Therefore
\[
\begin{align*}
  w_1(\det(D)) &= w_1(\det(\bar{\partial}_{TM,TL})) \\
  &= w_1(\det(\bar{\partial}_{E,F})) + w_1(\det(\bar{\partial}_{(E^1,F^1)})) + w_1(\det(\bar{\partial}_{(4E^1,4F^1)}))
\end{align*}
\]

We see in Corollary 2.2.4 that \( w_1(\det(\bar{\partial}_{E,F})|_\gamma) = w_2(TL) \cdot \beta \) and also that \( \det(\bar{\partial}_{(4E^1,4F^1)}) \) is orientable. In Proposition 2.2.6, we show that \( w_1(\det(\bar{\partial}_{(E^1,F^1)}|_\gamma) = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha) \). Combining these gives the formula
\[
  w_1(\det(D)|_\gamma) = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha) + w_2(TL) \cdot \beta
\]

\[\square\]

**Corollary 2.2.2.** Suppose the Lagrangian \( L \) is relatively pin\(^+\). Then the term involving \( w_2(TL) \) in the above formula vanishes, and the formula becomes
\[
  w_1(\det(D)|_\gamma) = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha)
\]

In particular, if \( L \) is also orientable, the formula implies that the moduli space is orientable.

**Proof.** The torus class \( \beta \in H_2(L) \) is the boundary of a class \( S \in H_3(M,L) \), given by following the image in \( (M,L) \) of the whole surface along the loop \( \gamma \), that is \( \beta = \partial S \). Let \( \delta : H^2(L;\mathbb{Z}_2) \to H^3(M,L;\mathbb{Z}_2) \). Then
\[
  w_2(L) \cdot \beta = \delta(w_2(L)) \cdot S \tag{2.2.1}
\]

There is an exact sequence
\[
  H^2(M;\mathbb{Z}_2) \xrightarrow{i^*} H^2(L;\mathbb{Z}_2) \xrightarrow{\delta} H^3(M,L;\mathbb{Z}_2)
\]

The relatively pin condition implies that \( w_2(L) \) or \( w_2(L) + w_1^2(L) \) is in the image of \( i^* \). Therefore \( \delta(w_2) = 0 \) or \( \delta(w_2 + w_1^2) = 0 \). Notice that \( w_1^2(L) \cdot \beta = 0 \) since \( \beta \) is the class of a torus, and thus \( 0 = \delta(w_2 + w_1^2) \cdot S = \delta(w_2(L)) \cdot S + w_1^2(L) \cdot \beta = \delta(w_2(L)) \cdot S. \)
Now (2.2.1) implies that $w_2(L) \cdot \beta$ is zero in both cases.

**Proposition 2.2.3.** Suppose that $(\tilde{E}, \tilde{F})$ is a bundle pair over $(M, L)$ with $\tilde{F}$ orientable. Then for every loop $\gamma$ in $\mathcal{B}(\Sigma, b)$, the first Stiefel-Whitney class of $\det(\tilde{\partial}(\tilde{E}, \tilde{F}))$ restricted to $\gamma$ satisfies

$$w_1(\det(\tilde{\partial}(\tilde{E}, \tilde{F}))|_\gamma) = w_2(\tilde{F}) \cdot \beta$$

where $\beta$ is the torus in $L$ the boundary $\partial \Sigma$ traces along $\gamma$.

**Proof.** A loop $\gamma$ in $\mathcal{B}(\Sigma, b)$ is a map $\Phi : (\Sigma, \partial \Sigma) \times S^1 \to (M, L)$ and we are interested in the pull-back bundle pair $\Phi^*(\tilde{E}, \tilde{F})$. We might not be able to trivialize $\Phi^*\tilde{F}$ over $\partial \Sigma \times S^1$ but we can trivialize it over $\partial \Sigma \times I$, since $\tilde{F}$ is orientable and $\partial \Sigma \times I$ is homotopic to a circle.

Let $U(\epsilon) = [0, \epsilon] \times \partial \Sigma \times I$ be an $\epsilon$-neighborhood of $\partial \Sigma \times I \subset \Sigma \times I$. With this notation $U(0) := \partial \Sigma \times I$. The trivialization $\Phi^*\tilde{F}|_{U(0)} \cong \mathbb{R}^{n+3} \times U(0)$ induces a trivialization $\Phi^*\tilde{E}|_{U(0)} \cong \Phi^*(\tilde{F} \oplus J\tilde{F})|_{U(0)} \cong \mathbb{C}^{n+3} \times U(0)$, which we can extend over the whole neighborhood $\Phi^*\tilde{E}|_{U(\epsilon)} \cong \mathbb{C}^{n+3} \times U(\epsilon)$.

At the two endpoints of the interval $I$, glue the trivial bundles $\mathbb{R}^{n+3} \times \partial \Sigma \times 1$ and $\mathbb{R}^{n+3} \times \partial \Sigma \times 0$ using a clutching map $g \in \pi_1(SO(n+3))$, so that the bundle we obtain $(\mathbb{C}^{n+3}, \mathbb{R}^{n+3}) \times_{(\tilde{g}, g)} I \to \partial \Sigma \times S^1$ is isomorphic to $\Phi^*\tilde{F}$. The map $g$ induces a clutching map $\tilde{g} \in U(n+3)$ for the complex bundles $\mathbb{C}^{n+3} \times \partial \Sigma \times 1$ and $\mathbb{C}^{n+3} \times \partial \Sigma \times 0$. Since the inclusion $SO(n+3) \to U(n+3)$ is nullhomotopic, the map $\tilde{g}$ can be extended to a
trivial map in a neighborhood, thus giving a trivialization \( \Phi^* \tilde{E}_{|C_\epsilon \times S^1} \cong \mathbb{C}^{n+3} \times C_\epsilon \times S^1 \), where \( C_\epsilon = \epsilon \times \partial \Sigma \subset [0, \epsilon] \times \partial \Sigma \subset \Sigma \).

For every \( t \in S^1 \) we can pinch \( \Sigma \times t \) along \( C_\epsilon \times t \) to obtain a nodal curve \( \Sigma^s \) with normalization consisting of a disk \( D^2 \) and a closed Riemann surface \( \Sigma^{cl} \), with special points \( 0 \in D^2 \) and \( p \in \Sigma^{cl} \). The bundle pair \( \Phi^*(\tilde{E}, \tilde{F}) \) descends to a bundle pair over the family of nodal curves in the sense of Remark 2.1.1. Therefore we have bundles \( \hat{E} \rightarrow \Sigma^{cl} \times S^1 \) and \( (\mathbb{C}^{n+3}, \mathbb{R}^{n+3}) \times (\tilde{g}, g) I \rightarrow D^2 \times S^1 \), with isomorphisms \( \hat{E}|_{p \times t} \cong \mathbb{C}^{n+3} \times p \times t \rightarrow \mathbb{C}^{n+3} \times 0 \times t \) at the special points \( (p, 0) \) for every \( t \in S^1 \).

We are interested in the first Stiefel-Whitney class of the family of real Cauchy-Riemann operators \( \bar{\partial}_{\Phi^*(\tilde{E}, \tilde{F})} \). Taking a family of complex linear Cauchy-Riemann operators \( D^{\Sigma^{cl}} \) on \( \hat{E} \rightarrow \Sigma^{cl} \times S^1 \) and gluing it to a family of real Cauchy-Riemann operators \( D^{D^2} \) on \( (\mathbb{C}^{n+3}, \mathbb{R}^{n+3}) \times (\tilde{g}, g) I \rightarrow D^2 \times S^1 \) induces a family of real Cauchy-Riemann operators \( D^r \) on \( \Phi^*(\tilde{E}, \tilde{F}) \rightarrow (\Sigma, \partial \Sigma) \times S^1 \). Since the space of real Cauchy-Riemann operators is contractible, there is a canonical isomorphism \( \det(\bar{\partial}_{\Phi^*(\tilde{E}, \tilde{F})}) \cong \det(D^r) \).

By (2.1.2) we have

\[
\det(D^r) \cong \det(D^{\Sigma^{cl}}) \otimes \det(D^{D^2}) \otimes \Lambda^{max}(\hat{E}|_{(p,t)}) , \quad \text{and thus} \quad w_1(\det(D^r)) = w_1(\det(D^{\Sigma^{cl}})) + w_1(\det(D^{D^2})) + w_1(\hat{E}|_{p \times S^1})
\]

The complex structure on the kernels and cokernels of the family of operators \( D^{\Sigma^{cl}} \) induces a canonical orientation on \( \det(D^{\Sigma^{cl}}) \), and in particular the first Stiefel-Whitney class of \( \det(D^{\Sigma^{cl}}) \) is zero. Moreover, \( \hat{E}|_{p \times S^1} \cong \mathbb{C}^{n+3} \times S^1 \) also has a canonical orientation and its first Stiefel-Whitney class is equal to zero. Therefore the problem reduces to the family of operators on \( (\mathbb{C}^{n+3}, \mathbb{R}^{n+3}) \times (\tilde{g}, g) I \rightarrow D^2 \times S^1 \).

If the clutching map \( g \in \pi_1(SO(n+3)) = \mathbb{Z}_2 \) is trivial, then we have a trivial bundle pair \( (\mathbb{C}^{n+3}, \mathbb{R}^{n+3}) \rightarrow D^2 \times S^1 \) and we can consider a trivial family of standard Cauchy-Riemann operators \( \bar{\partial}_0 \), restricted to real valued functions. It is obviously
orientable. Moreover, the operator is surjective and the kernel consists of real constant functions. Thus \( \text{Ker}(\partial_0) \cong \mathbb{R}^{n+3} \) by evaluation at a boundary point, inducing \( \det(\partial_0) \cong \det(\mathbb{R}^{n+3}) \). If the clutching map \( g \in \pi_1(SO(n+3)) \) is not trivial, it is shown in [FOOO] that the determinant line of a family of real Cauchy-Riemann operators on \((\mathbb{C}^{n+3}, \mathbb{R}^{n+3}) \times (\tilde{g}, g) \) is not orientable. Combining the two gives

\[
\text{w}_1(\det(D^{D^2})) = 0 \text{ if and only if } g = 0 \in \pi_1(SO(n + 3))
\]

On the other hand, the map \( g \) is the obstruction to trivializing \( \Phi^*\tilde{F} \) over \( \partial \Sigma \times S^1 \). Since \( \text{w}_1(\Phi^*\tilde{F}) = 0 \), this obstruction is equal to the second Stiefel-Whitney class of \( \Phi^*(\tilde{F}) \), and \( \text{w}_1(\det(D^{D^2})) = \text{w}_2(\Phi^*\tilde{F}) \).

To summarize

\[
\text{w}_1(\det(\tilde{\partial}(\tilde{E}, \tilde{F}))) \cdot \gamma = \text{w}_1(\det(\tilde{\partial}(\Phi^*\tilde{E}, \Phi^*\tilde{F}))) \\
= \text{w}_1(\det(D^{\Sigma^1})) + \text{w}_1(\det(D^{D^2})) + \text{w}_1(E|_{p \times S^1}) \\
= \text{w}_2(\Phi^*\tilde{F}) \cdot (\partial D^2 \times S^1) \\
= \text{w}_2(\tilde{F}) \cdot \Phi(\partial D^2 \times S^1)
\]

\[
\boxed{}
\]

**Corollary 2.2.4.** The first Stiefel-Whitney class of \( \det(\tilde{\partial}(\tilde{E}, \tilde{F}))|_{\gamma} \) is equal to \( \text{w}_2(F) \cdot \beta = \text{w}_2(TL) \cdot \beta \). The first Stiefel-Whitney class of \( \det(\tilde{\partial}(\tilde{A}E^1, \tilde{A}F^1))|_{\gamma} \) is 0 for any loop \( \gamma \) in \( \mathfrak{B}(\Sigma, b) \).

**Lemma 2.2.5.** Let \((\mathbb{L}, l) \to (D^2, \partial D^2)\) be a bundle pair where \( \dim_{\mathbb{C}} \mathbb{L} = \dim_{\mathbb{R}} l = 1 \), and let \( \mu = \mu(\mathbb{L}, l) \) be the Maslov index of the pair. If \( \mu \geq 0 \), for every complex linear Cauchy-Riemann operator \( D \) on \((\mathbb{L}, l) \to (D^2, \partial D^2)\) the following isomorphism holds

\[
\text{Ker}(D) \cong l_{x_1} \times \ldots \times l_{x_k}
\]

for any \( k \in \mathbb{N} \) satisfying \( k = \mu + 1 \), where \( x_i \) is a point on the boundary \( \partial D^2 \) for \( i = 1, \ldots, k \).
Proof. Following [MS] Appendix C.4, the bundle pair \((\mathbb{L}, l)\) is isomorphic to \((\mathbb{C} \times \partial D^2, \Lambda)\), where the fiber at \(e^{i\theta} \in \partial D^2 \cong S^1\) is given by

\[
\Lambda_{e^{i\theta}} = e^{i\theta \mu} \mathbb{R}
\]

The standard Cauchy-Riemann operator \(\bar{\partial}_0\) on \((\mathbb{C} \times D^2, \Lambda)\) associated to the zero connection is surjective when \(\mu \geq 0\), and thus \(\dim \text{Ker} \bar{\partial}_0 = \mu + 1\). Moreover, the elements of the kernel are polynomials \(\xi(z) = a_0 + \cdots + a_\mu z^\mu\) with \(a_i = \bar{a}_{\mu-i}\).

The kernel of the evaluation map

\[
\text{Ker}(\bar{\partial}_0) \rightarrow \Lambda_{x_1} \times \cdots \times \Lambda_{x_k} =: \times_i \Lambda_{x_i}
\]

consists of polynomials (of degree \(\mu\)) which vanish at all points \(x_i\), for \(i = 1, \ldots, k\). Therefore, if \(k = \mu + 1\) the kernel of the evaluation map is trivial. Since we have equality of dimensions

\[
\dim \text{Ker}(\bar{\partial}_0) = \mu + 1 = k = \dim(\times_i \Lambda_{x_i})
\]

the map \(ev\) must be an isomorphism.

Any other complex linear Cauchy-Riemann operator \(\bar{\partial}\) on \((\mathbb{C} \times D^2, \Lambda)\) has the form \(\bar{\partial} = W' \circ \bar{\partial}_0 \circ W\), where \(W\) is given by multiplication with a non-zero function \(w : D^2 \rightarrow \mathbb{C}\), and \(W'\) is given by multiplication with \(\frac{1}{w}\). Therefore, the operator \(\bar{\partial}\) is also surjective. Moreover, if \(\xi\) is in the kernel of \(\bar{\partial}\), then \(w\xi\) is in the kernel of \(\bar{\partial}_0\), and since \(w\) is non-zero, \(\xi\) and \(w\xi\) have the same number of zeros. This implies that the kernel of the evaluation map \(ev : \text{Ker}(\bar{\partial}) \rightarrow \times_i \Lambda_{x_i}\) is trivial, and the map \(ev\) is again an isomorphism.
There is a commutative diagram

\[
\begin{array}{ccc}
\text{Ker}(D) & \longrightarrow & l_{x_1} \times \ldots \times l_{x_k} \\
\cong \downarrow \varphi & & \cong \downarrow \varphi \\
\text{Ker}((\varphi^{-1})^*D) & \longrightarrow & \Lambda_{x_1} \times \ldots \times \Lambda_{x_k}
\end{array}
\]

where \( \varphi \) is an isomorphism of bundle pairs \( \varphi : (L, l) \rightarrow (\mathbb{C} \times D^2, \Lambda) \) and \( \varphi^*D \) is the induced complex linear operator on \( (\mathbb{C} \times D^2, \Lambda) \). Since three of the maps in the diagram are isomorphisms, so is the evaluation map \( \text{ev} : \text{Ker}(D) \rightarrow l_{x_1} \times \ldots \times l_{x_k} \).

**Proposition 2.2.6.** For every loop \( \gamma \) in \( \mathfrak{B}(\Sigma, b) \), the first Stiefel-Whitney class of \( \det(\overline{\partial}_{(E^1, F^1)}) \) restricted to \( \gamma \) satisfies

\[
w_1(\det(\overline{\partial}_{(E^1, F^1)}))_{|\gamma} = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha)
\]

where \( b_1 \) is the homology class that \( \partial \Sigma \) represents in \( H_1(L) \), and \( \alpha \) is a loop in \( L \) that a point on the boundary \( \partial \Sigma \) traces along \( \gamma \).

**Proof.** A loop \( \gamma \) in \( \mathfrak{B}(\Sigma, b) \) is a map \( \Phi : (\Sigma, \partial \Sigma) \times S^1 \rightarrow (M, L) \) and we are interested in the pull-back bundle pair \( \Phi^*(E^1, F^1) \).

Let \( U(\epsilon) = [0, \epsilon] \times \partial \Sigma \times I \) be an \( \epsilon \)-neighborhood of \( \partial \Sigma \times I \subset \Sigma \times I \). With this notation \( U(0) \) := \( \partial \Sigma \times I \). Let \( m \in \{0, 1\} \) be equal to 0 if \( w_1(\Phi^*F_{|\partial \Sigma}) = 0 \), and \( m = 1 \) if \( w_1(\Phi^*F_{|\partial \Sigma}) = 1 \). Then there is an isomorphism \( (E^1, F^1)_{|U(0)} \cong (\mathbb{C} \times U(0), \Lambda \times I) \), where the fiber of \( \Lambda \) at a point \( e^{i\theta} \times t \in \partial \Sigma \times I \) is given by \( e^{i\theta} \mathbb{R} \subset \mathbb{C} \). We can extend the trivialization \( \Phi^*E^1_{|U(0)} \cong \mathbb{C} \times U(0) \) to the neighborhood \( U(\epsilon) \).

At the two endpoints of the interval \( I \), glue the bundles \( \Lambda \times 1 \) and \( \Lambda \times 0 \) using a clutching map \( g \in \pi_1(O(1)) \), so that the bundle we obtain \( \Lambda \times_g I \rightarrow \partial \Sigma \times S^1 \) be isomorphic to \( \Phi^*F^1 \rightarrow \partial \Sigma \times S^1 \). The map \( g \) induces a clutching map \( \widetilde{g} \in U(1) \) for the complex bundles \( \mathbb{C} \times \partial \Sigma \times 1 \) and \( \mathbb{C} \times \partial \Sigma \times 0 \). Since the inclusion \( O(1) \rightarrow U(1) \) is nullhomotopic, the map \( \widetilde{g} \) can be extended to a trivial map in a neighborhood, thus
giving a trivialization $\Phi^* E^1_{|C_\epsilon \times S^1} \cong \mathbb{C} \times C_\epsilon \times S^1$, where $C_\epsilon = \epsilon \times \partial \Sigma \subset [0, \epsilon] \times \partial \Sigma \subset \Sigma$.

For every $t \in S^1$, we can pinch $\Sigma \times t$ along $C_\epsilon \times t$ to obtain a nodal curve $\Sigma^s$ with a normalization consisting of a disk $D^2$ and a closed Riemann surface $\Sigma^{cl}$, with special points $0 \in D^2$ and $p \in \Sigma^{cl}$. The bundle pair $\Phi^* (E^1, F^1)$ descends to a bundle pair over the family of nodal curves in the sense of Remark 2.1.1. By definition, a bundle pair over a nodal curve is given by bundles over its normalization with matching conditions at the special points. Therefore we have bundles $\hat{E}^1 \to \Sigma^{cl} \times S^1$ and $(\mathbb{C}, \Lambda) \times_{(\bar{g}, g)} I \to D^2 \times S^1$, with an isomorphism $\hat{E}^1_{|p \times S^1} \cong \mathbb{C} \times p \times t \to \mathbb{C} \times 0 \times t$ at the special points $(p, 0)$ for every $t \in S^1$.

We are interested in the first Stiefel-Whitney class of the family of real Cauchy-Riemann operators $\bar{\partial}_{\Phi^* (E^1, F^1)}$. Taking a family of complex linear Cauchy-Riemann operators $D^{\Sigma^{cl}}$ on $\hat{E}^1 \to \Sigma^{cl} \times S^1$ and gluing it to a family of complex linear Cauchy-Riemann operators $D^{D^2}$ on $(\mathbb{C}, \Lambda) \times_{(\bar{g}, g)} I \to D^2 \times S^1$ induces a family of Cauchy-Riemann operators $D^r$ on $\Phi^* (E^1, F^1) \to (\Sigma, \partial \Sigma) \times S^1$. Since the space of real Cauchy-Riemann operators is contractible, there is a canonical isomorphism $\det(\bar{\partial}_{\Phi^* (E^1, F^1)}) \cong \det(D^r)$.

By (2.1.2) we have

$$\det(D^r) \cong \det(D^{\Sigma^{cl}}) \otimes \det(D^{D^2}) \otimes \Lambda^{max}(\hat{E}^1_{|p \times S^1}),$$

and thus

$$w_1(\det(D^r)) = w_1(\det(D^{\Sigma^{cl}})) + w_1(\det(D^{D^2})) + w_1(\hat{E}^1_{|p \times S^1}).$$

The complex structure on the kernels and cokernels of the family of operators $D^{\Sigma^{cl}}$ induces a canonical orientation on $\det(D^{\Sigma^{cl}})$, and in particular the first Stiefel-Whitney class of $\det(D^{\Sigma^{cl}})$ is zero. Moreover, $\hat{E}^1_{|p \times S^1} \cong \mathbb{C} \times S^1$ also has a canonical orientation and first Stiefel-Whitney class equal to zero, and therefore the problem reduces to the family of operators on $(\mathbb{C}, \Lambda) \times_{(\bar{g}, g)} I \to D^2 \times S^1$. 
By Lemma 2.2.5, the family of complex linear operators $D^{D^2}$ is surjective and

$$\text{Ker}(D_t^{D^2}) \cong \bigotimes_{i=1}^{m+1} (\Lambda \times g \times I)_{x_i \times t} \cong \bigotimes_{i=1}^{m+1} (\Phi^* F^1)_{x_i \times t}$$

for $m + 1$ points $x_i$ on the boundary of $\partial \Sigma$. Therefore

$$w_1(\det(D^{D^2})) = w_1\left(\bigotimes_{i=1}^{m+1} (\Phi^* F^1)_{x_i \times t}\right) = \sum_{i=1}^{m+1} w_1(\Phi^* F^1_{|x_i \times S^1}) = (m + 1)w_1(F^1_{|\Phi(x_1 \times S^1)})$$

To summarize

$$w_1(\det(\bar{\partial}_{(E^1,F^1)})) \cdot \gamma = w_1(\det(\bar{\partial}_{(E^1,F^1)})) = w_1(\det(D^{\Sigma^i})) + w_1(\det(D^{D^2})) + w_1(E^1_{|y \times S^1}) = (m + 1)w_1(F^1_{|\Phi(x_1 \times S^1)})$$

Recall that $(m \mod 2)$ equals to $w_1(\Phi^* F^1_{|\partial \Sigma}) = w_1(F^1) \cdot b_1$ and let $\alpha := \Phi(x_1 \times S^1)$. Then the above formula becomes

$$w_1(\det(\bar{\partial}_{(E^1,F^1)})) \cdot \gamma = (w_1(F^1) \cdot b_1 + 1)(w_1(F^1) \cdot \alpha) = (w_1(TL) \cdot b_1 + 1)(w_1(TL) \cdot \alpha)$$

where the last equality is just $w_1(F^1) := w_1(\det(TL)) = w_1(TL)$.

When the Riemann surface $\Sigma$ has several boundary components, we pinch near each one of them. In this way we obtain a nodal surface formed by a closed Riemann surface and several disks attached at an interior point to the surface. We again consider the problem on $(E, F)$ and $(E^1, F^1)$ separately. And again the determinant line of $D$ is isomorphic to the tensor product of the determinant lines of families on each of the pieces, together with the top exterior powers of the fibers over the special points. Both the determinant line of a family of operators on the closed surface, and the fibers at the special points have canonical orientations coming from the complex
structures. Thus, the orientability question reduces to that on the disks. Applying Proposition 2.2.4 and Proposition 2.2.6 we obtain

\[ w_1(\det(D)_{\gamma}) = \sum_{i=1}^{h} (w_1(TL) \cdot b_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^{h} w_2(TL) \cdot \beta_i \quad (2.2.2) \]

where \( b_i \in H_1(L) \) is the class the \( i \)-th boundary component of \( \Sigma \) represents in \( L \), \( \alpha_i \) is a loop a point on the \( i \)-th boundary component traces in \( L \) along \( \gamma \), and \( \beta_i \) is the torus the \( i \)-th boundary component traces in \( L \) along \( \gamma \). This is Theorem 2.2.1 in the case of several boundary components.

### 2.3 The local system \( Z_{(w_1,w_2)} \) on \((L \times \mathcal{L}(L))^h\)

In this section we state the necessary definitions and theorems regarding local systems as discussed in [Ste] and we construct the system \( Z_{(w_1,w_2)} \) over the \( h \)-fold product of the Lagrangian \( L \) and its free loop space \( \mathcal{L}(L) \). In the following section we shall show its pull-back is canonically isomorphic to the local system twisted by the first Stiefel-Whitney class of \( \det(D) \).

**Definition 2.3.1.** We have a **system of local groups** in a path connected topological space \( L \) if

1. for every \( x \in L \) we are given a group \( G_x \)
2. for every class of paths \( \alpha_{xy} \) we are given a group isomorphism \( \alpha_{xy} : G_x \rightarrow G_y \)
3. the composition \( \beta_{yz} \circ \alpha_{xy} \) is the isomorphism corresponding to the path \( \alpha_{xyz} \)

**Theorem 2.3.1.** Let \( p_0 \in L \), \( G \) a group and \( \psi : \pi_1(L, p_0) \rightarrow \text{Aut}(G) \) a homomorphism. Then there is a unique system \( G_x \) of local groups in \( L \), such that \( G_0 \) is a copy of \( G \) and the operations of \( \pi_1(L, p_0) \) in \( G_0 \) are those determined by \( \psi \).

**Definition 2.3.2.** Let \( f : L_1 \rightarrow L_2 \) be a continuous map of connected topological spaces, with \( f(p_1) = p_2 \), and let \( \mathcal{G} = \{G_x\} \) be a local system on \( L_2 \), induced by \( \psi : \pi_1(L_2, p_2) \rightarrow \text{Aut}(G) \). The **pull-back system** \( \mathcal{G}' = f^*\mathcal{G} \) is induced by \( f_\# \circ \psi : \pi_1(L_1, p_1) \rightarrow \text{Aut}(G) \), where \( f_\# : \pi_1(L_1, p_1) \rightarrow \pi_1(L_2, p_2) \).

\(^1\)Recall that \( h \) is the number of boundary components of \( \Sigma \).
Definition 2.3.3. The local system of orientations is the system arising via the homomorphism \( \psi : \pi_1(L, p_0) \to \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2 \), assigning to \( \alpha \in \pi_1(L, p_0) \) the value of the first Stiefel-Whitney class of \( L \) evaluated on the class of \( \alpha \).

Remark 2.3.4. In [Ste] this system is called the system of twisted integer coefficients. It is the system twisted by the first Stiefel-Whitney class of \( L \). It is sometimes denoted \( \mathbb{Z}_{w_1}(L) \).

To construct the local system \( \mathbb{Z}_{(w_1, w_2)} \) on \( (L \times \mathcal{L}(L))^h \), we first construct a system over every component of \( L \times \mathcal{L}(L) \). This defines a system over \( L \times \mathcal{L}(L) \) and we pull-back \( h \) copies of it to the product \( (L \times \mathcal{L}(L))^h \) via the projection maps. The system \( \mathbb{Z}_{(w_1, w_2)} \) is defined as the tensor product of the pulled-back systems.

Let \( (p_i \times \gamma_j) \) be a basepoint for the component \( L_i \times \mathcal{L}(L)_j \subset L \times \mathcal{L}(L) \). Define a local system over \( L_i \times \mathcal{L}(L)_j \) using the homomorphism:

\[
\psi : \pi_1(L_i \times \mathcal{L}(L)_j, p_i \times \gamma_j) = \pi_1(L_i, p_i) \times \pi_1(\mathcal{L}(L)_j, \gamma_j) \to \text{Aut}(\mathbb{Z})
\]

\[
(\alpha, \beta) \mapsto \psi(\alpha, \beta)(t) = (-1)^{\epsilon_t}
\]

where \( \epsilon = (w_1(TL) \cdot \gamma_j + 1) \cdot (w_1(TL) \cdot \alpha) + w_2(TL) \cdot \beta \)

Remark 2.3.5. We can use twisted \( \mathbb{Q} \) or \( \mathbb{R} \) coefficients instead as well, if required by the method for achieving transversality.

To summarize, the system \( \mathbb{Z}_{(w_1, w_2)} \) over a component of \( (L \times \mathcal{L}(L))^h \) with a basepoint \( (\vec{p}, \vec{\gamma}) = (p_1, \gamma_1, .., p_h, \gamma_h) \) is given by the homomorphism

\[
\psi : \pi_1((L \times \mathcal{L}(L))^h, (\vec{p}, \vec{\gamma})) \to \text{Aut}(\mathbb{Z})
\]

\[
(\alpha_1, .., \beta_h) \mapsto \{ t \mapsto (-1)^{\epsilon_t} \}
\]

where \( \epsilon = \sum_{i=1}^{h} (w_1(TL) \cdot \gamma_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^{h} w_2(TL) \cdot \beta_i \)
2.4 Canonical isomorphism of $\mathcal{Z}_{w_1(\det(D))}$ with the pull-back of $\mathcal{Z}_{(w_1,w_2)}$

In this section we construct a canonical isomorphism between the local system on $\mathfrak{B}(\Sigma, b)$, twisted by the first Stiefel-Whitney class of $\det(D)$, and the pull-back of $\mathcal{Z}_{(w_1,w_2)}$. The former system is essentially the local system of orientations on the moduli space.

**Definition 2.4.1.** The local system $\mathcal{Z}_{w_1(\det(D))}$ over a component $\mathfrak{B}(\Sigma, b)_0 \subset \mathfrak{B}(\Sigma, b)$ with a basepoint $u_0$ is given by the homomorphism

$$\psi : \pi_1(\mathfrak{B}(\Sigma, b)_0, u_0) \to \text{Aut}(\mathbb{Z})$$

$$\gamma \mapsto \{t \mapsto (-1)^{\epsilon t}\}, \text{ where } \epsilon = w_1(\det(D)) \cdot \gamma$$

The group $\mathbb{Z}$ at the basepoint $u_0$ can be fixed by a choice of a trivialization of the fiber of $\det(D)$ over the basepoint.

There are canonical evaluation maps

$$ev^i : \mathfrak{B}(\Sigma, b) \to L, \text{ assigning to a map } u \text{ its evaluation at } x_{i,1} \in S^1 \cong (\partial \Sigma)_i,$$

$$ev^i_{L(L)} : \mathfrak{B}(\Sigma, b) \to L(L), \text{ assigning to a map } u \text{ the loop } u_{(\partial \Sigma)_i} : (\partial \Sigma)_i \to L.$$ 

Combining them yields a map $\overline{ev}_L \times \overline{ev}_{L(L)} = (ev^1 \times \ldots \times ev^h_{L(L)}) : \mathfrak{B}(\Sigma, b) \to (L \times L(L))^h$.

**Theorem 2.4.1.** The local system $\mathcal{Z}_{w_1(\det(D))}$ is isomorphic to the pulled-back system $(\overline{ev}_L \times \overline{ev}_{L(L)})^* \mathcal{Z}_{(w_1,w_2)}$.

**Proof.** We restrict ourselves to a particular component. Let $u_0 \in \mathfrak{B}(\Sigma, b)$ map under $(\overline{ev}_L \times \overline{ev}_{L(L)})$ to the basepoint $\overline{p}_0 \times \overline{\gamma}_0 \in (L \times L(L))^h$. It is enough to show that the action of $\pi_1(\mathfrak{B}(\Sigma, b), u_0)$ on the group $\mathbb{Z}_{u_0}$ induced by the system $\mathcal{Z}_{w_1(\det(D))}$ is the same as the one induced by the pulled-back system $(\overline{ev}_L \times \overline{ev}_{L(L)})^* \mathcal{Z}_{(w_1,w_2)}$. 

By definition the action induced by $Z_{w_1(\text{det}(D))}$ is $(-1)^\epsilon$, where $\epsilon = w_1(\text{det}(D)) \cdot \gamma$ for $\gamma \in \pi_1(\mathcal{B}(\Sigma, b), u_0)$. By Theorem 2.2.1 (and its extension (3.1) to several boundary components) we have

$$w_1(\text{det}(D)) \cdot \gamma = \sum_{i=1}^h (w_1(TL) \cdot \gamma_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^h w_2(TL) \cdot \beta_i$$

since $\gamma_i$ is a representative of the class $b_i$, and here we take $\alpha_i$ to be the loop in $L$ traced by the point $x_{1,i} \in S^1 \cong \partial \Sigma_i$ along $\gamma$.

The action of $\gamma \in \pi_1(\mathcal{B}(\Sigma, b), u_0)$ induced by the pull-back system is by definition the action of the image $\tilde{\gamma} \in \pi_1(((L \times \mathcal{L}(L))^h, (\bar{p}_0, \bar{\gamma}_0))$ of $\gamma$ under the evaluation maps. By (4.3) this is define to be $(-1)^\epsilon$, where

$$\epsilon = \sum_{i=1}^h (w_1(TL) \cdot \gamma_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^h w_2(TL) \cdot \beta_i$$

This shows the two actions are the same.

To make the isomorphism canonical, we need to choose a trivialization of the determinant line $\text{det}(D)$ over $u_0$. This will fix the group $\mathbb{Z}$ at $u_0$ and by Theorem 2.3.1 the two systems must be the same.

**Proposition 2.4.2.** A trivialization of $\text{det}(D)|_{u_0}$ is induced by trivializations of $\mathcal{F}$ and $4\mathcal{F}^1$ over the image of $u_0(\partial \Sigma)$ and a trivialization of $\mathcal{F}^1$ over $u_0(x_0)$ for some $x_0 \in \partial \Sigma$.

**Proof.** We have canonical isomorphisms $\text{det}(D) \cong \text{det}(\bar{\partial}_{(TM,TL)})$ and

$$\text{det}(\bar{\partial}_{(TM,TL)}) \otimes \text{det}(\bar{\partial}_{(4\mathcal{F}^1,4\mathcal{F}^1)}) \cong \text{det}(\bar{\partial}_{(\mathcal{E},\mathcal{F})}) \otimes \text{det}(\bar{\partial}_{(\mathcal{E}^1,\mathcal{F}^1)})$$

as discussed in Section 3. Therefore a choice of trivializations over $u_0$ of $\text{det}(\bar{\partial}_{(4\mathcal{F}^1,4\mathcal{F}^1)})$, $\text{det}(\bar{\partial}_{(\mathcal{E},\mathcal{F})})$ and $\text{det}(\bar{\partial}_{(\mathcal{E}^1,\mathcal{F}^1)})$ will induce one on $\text{det}(D)$. 

\[\Box\]
We know that \( \det(\bar{\partial}(E,F))|_{u_0} \) is canonically isomorphic to the determinant line of an operator over a disk, tensored with the determinant line over a closed surface and the top exterior power of a complex bundle. Both of the latter have canonical orientations coming from the complex structures, and thus we only need to choose a trivialization of the determinant over the disk. The bundle pair over the disk is trivial, and we can choose an isomorphism with the bundle pair \((\mathbb{C}^{n+3},\mathbb{R}^{n+3})\) and consider the standard Cauchy-Riemann operator. This gives an isomorphism of the determinant line with \( \det(\mathbb{R}^{n+3}) \) by evaluation at a boundary point. It implies that a choice of a trivialization of \( F \) over the image under \( u_0 \) of the boundary of \( \Sigma \), induces a trivialization of the determinant line \( \det(\bar{\partial}(E,F))|_{u_0} \). Likewise, a choice of a trivialization of \( 4F^1 \) over the image of the boundary under \( u_0 \), induces a trivialization of the determinant line \( \det(\bar{\partial}(4E^1,4F^1))|_{u_0} \).

Similarly, \( \det(\bar{\partial}(E^1,F^1))|_{u_0} \) is isomorphic to the determinant line of an operator over a disk, tensored with the determinant line over a closed surface and the top exterior power of a complex vector bundle. Both of the latter are canonically oriented, and again we only need to choose a trivialization of the determinant line over the disk. When \( w_1(F^1) \cdot b_1 = 0 \), the index is isomorphic to \( F^1|_{u_0(x_0)} \) and hence a choice of a trivialization of \( F^1|_{u_0(x_0)} \) induces a trivialization of the determinant line. When \( F^1 \) is not orientable along \( b_1 \), the index is isomorphic to the product of the fibers of \( F^1 \) over the images of two points \( x_0, x_1 \in \partial \Sigma \). We can use the orientation of the boundary of \( \Sigma \) to transport a choice of a trivialization of \( F^1|_{u_0(x_0)} \) to \( F^1|_{u_0(x_1)} \). In this way, a choice of a trivialization of \( F^1|_{u_0(x_0)} \) canonically determines an orientation of the index, and hence a trivialization of the determinant line of the operator on the disk. This induces a trivialization of the determinant line \( \det(\bar{\partial}(E^1,F^1))|_{u_0} \).

To summarize, a choice of trivializations of \( F \) and \( 4F^1 \) over the image under \( u_0 \) of the boundary of \( \Sigma \), and a choice of a trivialization of \( F^1 \) over the image under \( u_0 \) of a point on the boundary, induces a trivialization of \( \det(D)|_{u_0} \). Changing any of these trivializations, by homotopically non-trivial maps, changes the orientation of \( \det(D)|_{u_0} \), when \( w_1(L) \cdot b_1 = 0 \). If \( w_1(L) \cdot b_1 = 1 \), changing the orientation of \( F^1 \) over
the boundary point will not affect the orientation of $\det(D)$.

**Proposition 2.4.3.** The canonical isomorphism is independent of the choice of $u_0$, as long as the image $u_0(\partial D^2)$ is fixed.

**Proof.** Let us first describe the isomorphism $f : \mathcal{L}_{w_1(D^3)} \cong ev^* \mathcal{L}_{(w_1,w_2)}$. The trivializations of $F$ and $F^1$ over $u_0(\partial D^2)$ and $u_0(x_1)$ respectively, induce the isomorphism $\mathcal{L}_{w_1(D^3)|u_0} \cong \mathbb{Z}$. We have $ev^* \mathcal{L}_{(w_1,w_2)|u_0} = \mathbb{Z}$, and we define $f(1) = 1$. If $v \in \mathcal{L}_{w_1(D^3)|u_0}$, then we define $f(v) = \tilde{\phi}_{\gamma}^{-1} f(\phi_{\gamma} v)$, where $\gamma$ is a path from $u$ to $u_0$, $\phi_{\gamma}$ is the isomorphism induce by the path in the system $\mathcal{L}_{w_1(D^3)}$, and $\tilde{\phi}_{\gamma}$ is the isomorphism in $ev^* \mathcal{L}_{(w_1,w_2)}$. This is independent of the path $\gamma$.

Now if $u'_0$ is another point whose image of $\partial D^2$ is the same as the image $u_0(\partial D^2)$, we define an isomorphism $f' : \mathcal{L}_{w_1(D^3)} \cong ev^* \mathcal{L}_{(w_1,w_2)}$ in the same way. We shall show that $f(v) = f'(v)$. To do this it is enough to show it at the point $u'_0$. Take $v = 1 \in \mathcal{L}_{w_1(D^3)|u'_0} \cong \mathbb{Z}$. A path $\gamma$ induces an isomorphism $\mathbb{Z} = \mathcal{L}_{w_1(D^3)|u'_0} \cong \mathcal{L}_{w_1(D^3)|u_0} \cong \mathbb{Z}$ given by the evaluation of $w_2(TL)$ on the torus the image of the boundary traces in $L$, plus $(w_1(TL) \cdot (b_1) + 1) \cdot (w_1(TL) \cdot ev_{x_1})$. We denote this expression with $\epsilon$. The isomorphism $\tilde{\phi}_{\gamma}$ equals $\psi_{ev(\gamma)}$, where $\psi$ is the isomorphism in $\mathcal{L}_{(w_1,w_2)}$. Then $\psi_{ev(\gamma)}$ is equal to $\epsilon$ by definition. Therefore, $f(v) = \tilde{\phi}_{\gamma}^{-1} f(\phi_{\gamma}) = \psi_{ev(\gamma)} f(\epsilon \cdot 1) = \epsilon^2 f(1) = 1 = f'(v)$. \qed

### 2.5 The system of local orientations on the moduli space of simple maps

Here we give an isomorphism between the local system of orientations on the moduli space of simple maps, and the push-forward of the local system $(\mathcal{E}_D \times \mathcal{E}_{L(L)})^* \mathcal{L}_{(w_1,w_2)}$. At the end we discuss the relative sign between the boundary of the moduli space and the fiber product of the two components.

There is a forgetful map $f : \mathfrak{g}(\Sigma, b) \times (S^1)^k \times (\Sigma)^l \to \mathfrak{g}(\Sigma, b)$, and we can pull-back the system $\mathcal{L}_{w_1(D^3)}$ by $f$. Over the product $\mathfrak{g}(\Sigma, b) \times (S^1)^k \times (\Sigma)^l$, the system
\[^{f^*Z_{w_1}}_{w_1} \text{is canonically isomorphic to } f^*Z_{w_1} \otimes \det[(T^1)^k \times (T^1)^l] \text{ since } \Sigma \text{ is oriented. There is a projection map}\]

\[\pi: \mathcal{B}(\Sigma, b) \times (S^1)^k \times (\Sigma)^l \to (\mathcal{B}(\Sigma, b) \times (S^1)^k \times (\Sigma)^l)/\text{Aut}(\Sigma, \partial \Sigma; j)\]

When the maps are simple, the map \(\pi\) is a fibration with a fiber \(\text{Aut}(\Sigma, \partial \Sigma; j)\). The system \(Z_{w_1}^{\det(D)}\) is trivial along the fiber: the \(w_2\) term vanishes as the image of the torus is degenerated to a circle; the image that a point on the boundary traces along the loop is the boundary itself, and therefore the remaining term in the formula of Theorem 2.2.1 becomes \((w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot b_1) \equiv 0\). This together with the fact that the fiber is connected, implies that the system pushes-down to the factor space. The system \(\pi! \circ f^*Z_{w_1(\text{Ind}(D))}\) over \((\mathcal{B}(\Sigma, b) \times (S^1)^k \times (\Sigma)^l)/\text{Aut}(\Sigma, \partial \Sigma; j)\) at the image of the basepoint, is canonically determined by a choice of an orientation of \(\text{Aut}(\Sigma, \partial \Sigma; j)\). After a perturbation to achieve transversality, this system is the system of local orientations on the moduli space \(\mathcal{M}_{l,k}(\Sigma, b)\). In Section 2.4, we showed that after certain choices of trivializations, the system \(Z_{w_1}^{\det(D)}\) is canonically isomorphic to \((\overrightarrow{ev}_L \times \overrightarrow{ev}_{\text{L}(L)})^*Z_{(w_1, w_2)}\). This gives

**Theorem 2.5.1.** There is a local system \(Z_{(w_1, w_2)}\) on \((L \times \mathcal{L}(L))^b\), such that the local system of orientations on \(\mathcal{M}_{l,k}(\Sigma, b)\) is isomorphic to \(\pi! \circ f^* \circ (\overrightarrow{ev}_L \times \overrightarrow{ev}_{\text{L}(L)})^*Z_{(w_1, w_2)}\). The isomorphism is canonical once we choose a trivialization of \(TL\) over a basepoint in \(L\), and a trivialization of \(TL \oplus 3 \det(\text{TL})\) over those loops in \(L\) corresponding to a choice of a basepoint in each component of \(\mathcal{L}(L)\).

**The codimension one strata.** We know that given trivializations of \(F_{u(\partial \Sigma)}\) and \(F_{u(x_1)}\), we can induce an orientation on \(\det D_u\). We shall be interested what the sign of the isomorphism \(T_u \partial \mathcal{M}(b) \cong T_{u_1, u_2} \mathcal{M}(b') \times_{ev} \mathcal{M}(b'')\) depends on, provided we are given trivializations of \(F_{u(\partial \Sigma)}\) and \(F_{u(x_1)}\). Here \(u = (u_1, u_2)\). We know that

\[\det D^b \cong \det D^b' \otimes \det D^b'' \otimes \det TL\]

by 2.1.2. A trivialization of \(F\) over the image of the boundary \(u(\partial \Sigma)\), and of \(F^1\)
over \( u(x_1) \) for some \( x_1 \in \partial \Sigma \), induces an orientation of \( \det(D^b) \). Moreover these choices induce orientations on both \( \det(D^{b'}) \) and \( \det(D^{b''}) \), in the following way. Take the trivialization of \( F^1_{|u(x_1)} \) and transport it to the node following the orientation of \( \partial D^2 \). In this way we have a trivialization of \( F^1 \) over a point in the image of both \( u_1 \) and \( u_2 \), and since we also have a trivialization of \( F \) over the images of the boundary, this implies that we have an orientation on the determinant lines, as well as on \( F^1_{|\text{node}} \). In this way both sides of (2.5.1) are oriented. In the case when both Maslov indexes are even, the isomorphism (2.5.1) is orientation preserving as shown in [FOOO], [WW]. Therefore, in this case the sign of the isomorphism \( \partial \mathcal{M}_{t,k}(b) \cong \mathcal{M}_{t_1,k_1}(b') \times_{\text{ev}} \mathcal{M}_{t_2,k_2}(b'') \) depends only on the sign in the moduli space of domains, which depends only on the labeling of the points. Furthermore, since the orientations in (2.5.1) depend only on the order of the bubbles, the Maslov indexes, and on which bubble \( x_1 \) lies, the sign in the general case depends only this data as well.
Chapter 3

Open Gromov Witten Invariants

In this part we shall restrict ourselves to the case when the symplectic manifold \((M, \omega)\) has an anti-symplectic involution \(\tau\) with a non-empty fixed locus. The fixed locus of \(\tau\) is necessarily a Lagrangian submanifold \(L\). We shall construct a Gromov-Witten type invariants of the triple \((M, \omega, \tau)\) under certain topological restrictions.

Let us recall first the classical Gromov-Witten invariants. They are defined using the moduli space \(\overline{M}_l(\Sigma^{cl}, A)\) of unparametrized pseudo-holomorphic maps from a marked, closed Riemann surface \(\Sigma^{cl}\) to a symplectic manifold \(M\), which represent a fixed class \(A \in H_2(M)\). The heuristic idea is as follows. There are canonical maps \(ev_i : \overline{M}_l(\Sigma^{cl}, A) \to M\) given by the evaluation at the \(i\)-th marked point and \(f : \overline{M}_l(\Sigma^{cl}, A) \to \overline{M}_l(\Sigma^{cl})\) given by forgetting the target \(M\). Here \(\overline{M}_l(\Sigma^{cl})\) denotes the moduli space of marked closed Riemann surfaces. Since \(\overline{M}_l(\Sigma^{cl}, A)\) is orientable and all of its singular strata are of codimension greater than 1, it has a fundamental class, and the Gromov-Witten invariants are defined by evaluating cohomology classes of \(M^l \times \overline{M}_l(\Sigma^{cl})\) on the push-forward of the fundamental class in \(H_*(M^l \times \overline{M}_l(\Sigma^{cl}))\).

If we consider the moduli space of bordered Riemann surfaces we cannot in general proceed as above, since this space is not necessarily orientable and it has codimension 1 boundary strata. Several results have been obtained in this setting. Katz and Liu [KL], and Liu [Liu] work with an \(S^1\)-equivariant pair \((M, L)\). Fukaya [Fuk], and
Iacovino [Iac] define disk invariants for the Calabi-Yau threefold.

In the presence of an anti-symplectic involution $\tau : M \to M$ with $L = \text{fix}(\tau)$, disk invariants using point constraints have been defined by Cho [Cho], and Solomon [Sol] when the dimension of $L$ is less than 4, and the latter were computed for the quintic threefold in [PSW]. The idea is to consider all classes in the relative homotopy group, whose double with respect to $\tau$ is a fixed class $A \in \pi_2(M)$. Taking the moduli spaces of disk maps representing these classes, and identifying their boundaries will yield to a space without boundary. This space is related to the space of real sphere maps [Wel05, Wel08], and the construction is largely motivated by the fact that the latter space does not have boundary. Here we define disk invariants in higher dimensions and any type of constraints by adapting the above idea.

This part is organized as follows. First we define the moduli space of real sphere maps, and discuss the method for achieving transversality. We shall work with perturbed $J$-holomorphic maps, with certain restrictions on the allowed perturbations. Next, we construct the space $\tilde{M}(A)$ by identifying the boundaries of several moduli spaces of disk maps, and we show that the new moduli space is isomorphic to the moduli space of real sphere maps. We define and compute the relative sign of the gluing map on the boundary of the moduli spaces of disk maps. Using this, we express the first Stiefel-Whitney class of $\tilde{M}(A)$ by classes pulled-back from $L$ and the analogue of the Deligne-Mumford space $\bar{M}$, plus certain boundary divisors. Under some topological assumptions, this expression simplifies to the pull-back of the first Stiefel-Whitney class of $\bar{M}$. We then define the open Gromov-Witten disk invariants similarly to the classical case, by evaluating cohomology classes with coefficients twisted by $w_1(\tilde{M})$, on the push-forward of the moduli space $\tilde{M}(A)$.

### 3.1 Background

Here we give a brief description of the moduli space of real sphere maps. We use this space in the next sections as a model, which we reconstruct from several moduli
spaces of disk maps by gluing their boundaries. We also discuss the transversality results that are needed for the constructions later on.

**The moduli space of real sphere maps.** Let $(M, \omega)$ be a symplectic manifold, and let $\tau : M \to M$ be an anti-symplectic involution with a non-empty fixed locus $L \subset M$. For $J$ satisfying $\tau^* J = -J$, we consider the space of $J$-holomorphic maps $u : \mathbb{C}P^1 \to M$ in a fixed homology class $A \in \pi_2(M)$, which satisfy $u = \tau \circ u \circ c_{\mathbb{C}P^1}$, where $c_{\mathbb{C}P^1}$ is the standard conjugation on $\mathbb{C}P^1$. We call these maps real. The automorphisms of $\mathbb{C}P^1$, which leave invariant the fixed locus of the conjugation $c_{\mathbb{C}P^1}$, form a group $\text{Aut}_R(\mathbb{C}P^1)$. This group acts on the space of real maps, and the quotient space is the moduli space of real $J$-holomorphic sphere maps $\mathcal{M}(A)$. As usual, we can also consider the moduli space of pointed maps $\mathcal{M}_{l,k}(A)$, with $k$ marked points on the fixed locus of the conjugation $c_{\mathbb{C}P^1}$, which we call real, and $l$ pairs of ordered complex conjugate points $(z_i, \bar{z}_i)$. Although the Gromov compactification of this space has codimension one strata, this strata does not form a boundary. This fact is an important ingredient in showing that the image under the evaluation map is a pseudocycle. With the exception of certain cases [Wel08], however, we do not know when or whether this space is orientable. Hence, we do not know whether the target carries the correct local system to accommodate the pseudocycle. This prevents us to define a Gromov-Witten type invariant associated with this space as in the classical setting. On the other hand, the space of real sphere maps has a finite to one cover by moduli spaces of disk maps, the orientability of which we understand. We shall combine the properties of these spaces to construct invariants.

Another description of the moduli space of real spheres is as follows. Consider the space of $J$-holomorphic sphere maps, representing a fixed class $A \in \pi_2(M)$, with $k + 2l$ marked points. The automorphism group of $\mathbb{C}P^1$ acts on this space in the usual way, and it acts on the complex conjugation $c_{\mathbb{C}P^1}$ by conjugation with the element of the group. The quotient space is the moduli space of $J$-holomorphic sphere maps $\mathcal{M}_{k+2l}(A)$. There is an involution on this space, which sends a map $u$ to the map $\tau \circ u \circ c_{\mathbb{C}P^1}$, conjugates the first $k$ marked points $(z_i \mapsto \bar{z}_i)$, and conjugates and switches
pairwise the last $2l$ marked points $((z_{k+1}, z_{k+2}) \mapsto (\bar{z}_{k+2}, \bar{z}_{k+1}))$. The fixed locus of this involution is isomorphic to the moduli space of real sphere maps, and we have $\mathcal{R}M_{l,k}(A) \hookrightarrow \mathcal{M}_{k+2l}(A)$.

Remark 3.1.1. The space of real spheres described above is the same as the space considered by Welschinger in [Wel08], in the case when the permutation fixes the first $k$ marked points and is given by $(k+1, k+2) .. (k+2l-1, k+2l)$ on the last $2l$ marked points. See also [Sep], [Cey06] for the moduli space of real curves, and [Cey07] for its homology.

Transversality. We shall either restrict ourselves to a special class of symplectic manifolds called strongly semi-positive manifolds, or assume that all domains are stable. Both of these cases provide a favorable environment in which transversality can be achieved using the more classical approach of $(J, \nu)$-holomorphic maps. To go beyond them, we need to use more sophisticated methods to achieve transversality such as Kuranishi structures [FOOO, FO] or polyfolds [HWZ].

The moduli space of perturbed $J$-holomorphic maps was used by Ruan and Tian in [RT] to define the classical Gromov-Witten invariants for semi-positive symplectic manifolds. The semi-positive condition states that there are no spheres with positive symplectic area whose first Chern class is in the range $[3-n, 0)$. A map $u : \Sigma \rightarrow M$ is called $(J, \nu)$-holomorphic if it satisfy the perturbed equation $\bar{\partial}_J u = \nu$, where the perturbation $\nu$ is a section of the bundle $\text{Hom}(T\Sigma, TM)$ over $\Sigma \times M$. More precisely, one chooses an embedding of the universal family $\mathcal{U}$ of the Deligne-Mumford space into a large dimensional $\mathbb{C}P^{N}$, and considers a section of the bundle $\text{Hom}(T\mathbb{C}P^{N}, TM)$ over $\mathbb{C}P^{N} \times TM$. The pull-back of this section over $\mathcal{U} \times M$ defines the perturbation $\nu$. For a more detailed discussion see [Ion]. Using this perturbation, one achieves transversality for all maps with stable domains, which, in the semi-positive case, is enough to show that the image of the moduli space under the evaluation and forgetful maps is a pseudocycle in the product of several copies of $M$ and the Deligne-Mumford space. Note that if we can assume that all domains are stable (eg. by intersecting with a Donaldson divisor), there is no need for the semi-positive condition.
In the case of the moduli space of disk maps, one can proceed similarly [Sol], but has to work with strongly semi-positive rather than semi-positive manifolds. These are manifolds with no spheres with positive symplectic area whose first Chern class is in the range $[3 - n, 0]$. The necessity for this condition comes from considering the codimension one strata. If there is a multiply-covered unstable disk of Maslov index 0 in codimension one, we will not be able to achieve transversality, since the perturbation $\nu$ is zero on the unstable components. We also cannot argue that the image of the stratum will be contained in a lower dimensional space, by reducing the curve, since the Maslov index is 0. However, if we impose the strongly semi-positive condition, such disks will not appear. Then each stratum of the reduced moduli space is a smooth manifold with corners, with top and codimension one strata forming a manifold with boundary. Again, if all domains are stable, there is no need for the strongly semi-positive condition.

In the presence of an anti-symplectic involution $\tau$, in order to carry out the constructions in the next section we shall put some restrictions on the perturbation $\nu$. First, there is a map from the moduli space of disk domains to the moduli space of real sphere domains, which embeds into the moduli space of spheres. We can than use the embedding of the universal family in $\mathbb{C}P^N$ to pull-back a perturbation $\nu$ as before. We shall require this perturbation to satisfy $\tau \circ \nu \circ c_D^2 = \nu$ and $\tau \circ \nu \circ c_{\mathbb{C}P^1} = \nu$, where $c_D^2$ and $c_{\mathbb{C}P^1}$ are the conjugation on the disk and the sphere respectively. We shall also require that the almost complex structure $J$ satisfies $\tau^* J = -J$. The space of such almost complex structures is non-empty and contractible [Wel05]. Under these conditions, the restriction of a $(J, \nu)$-holomorphic sphere to its two hemispheres gives $(J, \nu)$-holomorphic disks, the reflection of a $(J, \nu)$-holomorphic disk is a $(J, \nu)$-holomorphic sphere, and if $u$ is a $(J, \nu)$-holomorphic disk, then so is $\tau \circ u \circ c_D^2$. We shall denote with $\mathcal{J}_R$ the space of pairs $(J, \nu)$ satisfying the above conditions, and being in the same connected component as $(J, 0)$.
3.2 Construction of $\tilde{\mathcal{M}}_{l,k}(A)$

In this section we construct a new moduli space $\tilde{\mathcal{M}}_{l,k}(A)$. We do this by gluing together several moduli spaces of disk maps, and we show that it is isomorphic to the moduli space of real spheres $\overline{\mathcal{M}}_{l,k}(A)$.

**Motivation.** Ideally, we would like to define a Gromov-Witten type invariant associated with the moduli space of disk maps $\overline{\mathcal{M}}(b)$. This space, however, has codimension one boundary strata, and heuristically speaking its fundamental class belongs to the relative top homology group $H_{\text{top}}(\overline{\mathcal{M}}(b), \partial \overline{\mathcal{M}}(b))$. The lack of a suitable pair in the target to map the pair $(\overline{\mathcal{M}}(b), \partial \overline{\mathcal{M}}(b))$ into, obstructs us to proceed as in the classical case. To overcome the problem arising from the existence of boundary, we shall glue together the boundaries of several moduli spaces of disk maps, to obtain a new space without boundary. Here we shall make use of the anti-symplectic involution on $M$, as the way we perform the gluing, will be guided by the way the codimension one strata of the moduli space of real sphere maps fit inside it.

Let us first consider the open part of the moduli space of real sphere maps. A choice of a preferred complex marked point $z_0$, determines a map to a disjoint union of moduli spaces of disk maps, mapping a sphere to the half disk containing $z_0$. 

\[ A \in H_2(M) \quad b \in H_2(M, L) \]

\[ \begin{array}{c}
\begin{array}{c}
\mathcal{M}(b) \times \mathbb{Z}_2 \\
\text{\textcircled{\text{decoration}}}
\end{array}
\end{array} \]
Moreover, the choice of $z_0$ divides the pairs of complex marked points into two groups - those whose first element lies in the same half sphere as $z_0$, and those whose first element is in the opposite half sphere. In order to construct a map back, we need a decoration of $+$ or $-$ on the interior marked points of the disks. This will determine whether the first element in the pair of complex conjugate marked points on the sphere is in the same or opposite half sphere as $z_0$.

Now we consider the boundary. In the moduli space of real sphere maps, we pass through a codimension one stratum by flipping one of the sphere components. The two sides of this stratum will be mapped, by the map discussed above, to two different moduli spaces of disk maps. We shall glue together these moduli spaces along the parts of their boundaries, which correspond to the stratum in the moduli space of real spheres.

The choice of $z_0$ also makes the choice of which bubble to be flipped canonical - we shall flip the one not containing $z_0$. This choice plays an important role when we show that the new moduli space is isomorphic to the moduli space of real sphere...
Construction. Let $\mathcal{M}_{l+1,k}(b)$ be the moduli space of $(J, \nu)$-holomorphic maps, which represent a fixed class $b \in \tilde{\pi}_2(M, L)$, and have $k$ boundary marked points and $l + 1$ interior marked points, one of which is preferred. Here the group $\tilde{\pi}_2(M, L)$ is the abelianization of $\pi_2(M, L)$. A decoration with $\pm$ on the interior marked points except on the preferred one, can be considered as an element of $\mathbb{Z}_{l+1}^2 = \{-1, +1\}$. Let $\tilde{\mathcal{M}}_{l+1,k}(b) = \mathcal{M}_{l+1,k}(b) \times \mathbb{Z}_{l+1}^2$ be the moduli space of $(J, \nu)$-holomorphic disk maps which represent the class $b \in \tilde{\pi}_2(M, L)$, and have a decoration of $+$ or $-$ on the interior marked points except $z_0$.

Let $o : \tilde{\pi}_2(M, L) \to \pi_2(M)$ be the correspondence obtained by choosing a lift to $\pi_2(M, L)$, and using the Schwarz reflection principle and $\tau$, to construct a real sphere map. Since $\pi_2(M, L)$ is abelian, the correspondence is independent of the lift. Let $\mathcal{M}_{l+1,k}(A) = \bigsqcup_{o(b) = A} \tilde{\mathcal{M}}_{l+1,k}(b)$ be the disjoint union of decorated moduli spaces of disk maps, representing a class $b$ whose double with respect to $\tau$ is a fixed class $A \in \pi_2(M)$.

Denote with $\bar{b}$ the class $-\tau_*(b)$. There is a map $g : \partial \mathcal{M}_{l+1,k}(A) \to \partial \mathcal{M}_{l+1,k}(A)$, which on a particular component of the boundary is given by

$$g : \tilde{\mathcal{M}}_{l_1+1,k_1}(b_1) \times_{ev} \tilde{\mathcal{M}}_{l_2,k_2}(b_2) \to \tilde{\mathcal{M}}_{l_1+1,k_1}(b_1) \times_{ev} \tilde{\mathcal{M}}_{l_2,k_2}(\bar{b}_2)$$

$$((u_1, \bar{x}_1, \bar{z}_1; \sigma_1), (u_2, \bar{x}_2, \bar{z}_2; \sigma_2)) \mapsto ((u_1, \bar{x}_1, \bar{z}_1; \sigma_1), (\tau \circ u_2 \circ c, c(\bar{x}_2), c(\bar{z}_2), -id_{Z_{l_2}^2}(\sigma_2))$$

Here $c$ is the conjugation on the disk. The subscript $l_1 + 1$ is used to emphasize that the preferred interior point $z_0$ belongs to this particular bubble - in other words we always flip the disk which does not contain $z_0$. More intuitively, the map $g$ is given by identity on the first factor and by flipping on the second.

Following [PSW], we define an equivalence $\sim_g$ on the singular strata as follows. On the codimension one strata, $u \sim v$ if $v = g(u)$. On the lower codimension strata, $u \sim v$ if there is a codimension one stratum containing $u$, and a codimension one stratum containing $v$, such that $v = g(u)$ when $u, v$ are considered as elements of
these strata.

**Theorem 3.2.1.** Let $\widetilde{\mathcal{M}}_{l+1,k}(A) = \mathcal{M}_{l+1,k}(A)/\sim_g$ be the space obtained by identifying the corners of the moduli spaces $\mathcal{M}_{l+1,k}(b)$ using the map $g$. The reduced space $\widetilde{\mathcal{M}}^r_{l+1,k}(A)$ has no boundary provided that there is at least one boundary marked point, or that the classes the boundary of the disk represent in $H_1(L)$ are not zero.

Moreover, there are continuous maps $ev_{x_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to L$ given by evaluation at the $i$-th boundary marked point, and $\tilde{ev}_{z_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to M$ given by $u(z_i)$ on $(u, z_i, +)$ and by $\tau \circ u(z_i)$ on $(u, z_i, -)$.

**Proof.** We first show that the map $g$ has no fixed points on the codimension one strata. Indeed, if the domain of the map $u$ is stable, it must have either at least 3 boundary marked points, on the order of which $g$ will act nontrivially, or a boundary and an interior marked points, and $g$ will act nontrivially on the decoration of the interior point. Otherwise, if $u = \tau \circ u \circ c$, then we can reduce the map [Sol], [Cho]. Since we are already working with the reduced moduli space, this case is excluded.

The evaluation maps at a boundary marked point $x_i$ fit together to form an evaluation map on $\widetilde{\mathcal{M}}_{l+1,k}(A)$, since $u(x_i) = \tau \circ u \circ c(c(x_i))$. This does not hold at an interior marked point, since $\tau$ does not fix the image. For this reason we shall use the decoration of the interior marked points to define a new evaluation map

$$\tilde{ev}_{z_i}(u) = \begin{cases} 
  u(z_i) & \text{if the decoration of } z_i \text{ is } +, \\
  \tau \circ u(z_i) & \text{if the decoration of } z_i \text{ is } - 
\end{cases}$$

Now, since $g$ changes the decoration, the maps $\tilde{ev}_{z_i}$ fit together and form an evaluation map at interior marked point $\tilde{ev}_{z_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to M$. □

**Remark 3.2.1.** There are two possible types of boundary - one having two components on which $g$ is defined and one having a single component on which $g$ will not be defined. The latter is avoided by requiring that either there is at least one boundary marked point, or that all classes which the boundary of the disk represent in $H_1(L)$ are not zero.
By taking the target manifold $M$ to be a point in the above construction, we obtain an analogue of the Deligne-Mumford moduli space which we denote with $\tilde{M}_{l+1,k}$. Naturally, there is a forgetful map $f : \tilde{M}_{l+1,k}(A) \to \tilde{M}_{l+1,k}$ given by forgetting the maps and stabilizing the domains.

**Remark 3.2.2.** If $k = 0$, the moduli space of disk domains has boundary on which $g$ is not defined, and thus $\tilde{M}_{l+1,0}$ has boundary as well.

We shall now show that the moduli space of real sphere maps $\mathbb{R}M_{l+1,k}(A)$ is isomorphic to the moduli space $\tilde{M}_{l+1,k}(A)$. First we set up a correspondence between decorated pointed $(J, \nu)$-holomorphic disk maps, and pointed real $(J, \nu)$-holomorphic sphere maps. Then we show that the Deligne-Mumford analogues $\tilde{M}_{l+1,k}$ and $\mathbb{R}M_{l+1,k}$ are isomorphic. Since we work with perturbed $J$-holomorphic maps, the only thing left is to give a correspondence on the unstable bubbles, which we do at the end.

We start with the correspondence between the maps. Fix an isomorphism $\mathbb{C}P^1 \cong D^2 \cup \bar{D}^2$. If $u : (D^2, \partial D^2) \to (M, L)$ is a $(J, \nu)$-holomorphic map in the class $b \in \tilde{\pi}_2(M, L)$, we define a map $\tilde{u} : \mathbb{C}P^1 \to M$ to be

$$
\tilde{u}(z) = \begin{cases} 
  u(z) & \text{for } z \in D^2 \subset \mathbb{C}P^1 \\
  \tau \circ u \circ c_{\mathbb{C}P^1}(z) & \text{for } z \in \bar{D}^2 \subset \mathbb{C}P^1 
\end{cases}
$$

The map $\tilde{u}$ is a $(J, \nu)$-holomorphic map in the class $o(b) = A \in \pi_2(M)$ which is real, in the sense that $\tilde{u} = \tau \circ \tilde{u} \circ c_{\mathbb{C}P^1}$.

Conversely, if $\tilde{u} : \mathbb{C}P^1 \to M$ is a real $(J, \nu)$-holomorphic map in the class $A \in \pi_2(M)$, then $u = \tilde{u}|_{D^2} : (D^2, \partial D^2) \to (M, L)$ is a $(J, \nu)$-holomorphic map from the disk to $(M, L)$, in a class $b$ with the property $o(b) = A$.

Suppose we have a collection of $k$ boundary and $l + 1$ decorated interior marked points on the disk, e.g. a tuple of the form $(x_1, \ldots, x_k, z_0, z_1^+, \ldots, z_l^-)$. To this we associate the collection of the $k$ marked points on $\partial D^2 \subset D^2 \subset \mathbb{C}P^1$, and $l$ pairs of ordered
conjugate points \((\phi(z_i), \overline{\phi(z_i)})\), where

\[
\phi(z_i) = \begin{cases} 
  z_i \in D^2 \subset \mathbb{CP}^1 & \text{if the decoration of } z_i \text{ is } + \\
  c_{\mathbb{CP}^1}(z_i) & \text{if the decoration of } z_i \text{ is } -
\end{cases}
\]

Note that we always assume the decoration of \(z_0\) is +. Conversely, suppose we have a collection of \(k\) real and \(l + 1\) pairs of ordered conjugate marked points on \(\mathbb{CP}^1\). Let \(z_0, \ldots, z_l\) be the first elements of each pair. Suppose \(z_0 \in D^2 \subset \mathbb{CP}^1\). To such a collection we associate the collection of the \(k\) marked points on the boundary of \(D^2\), and the decorated interior marked points \(\{\tilde{\phi}(z_i)\}\), where

\[
\tilde{\phi}(z_i) = \begin{cases} 
  (z_i, +) & \text{if } z_i \in D^2 \subset \mathbb{CP}^1 \\
  (c_{\mathbb{CP}^1}(z_i), -) & \text{if } z_i \in \overline{D^2} \subset \mathbb{CP}^1
\end{cases}
\]

It follows that the disjoint union of the spaces of decorated pointed \((J, \nu)\)-holomorphic disk maps in a class \(b\) with the property \(o(b) = A\), is isomorphic to the space of pointed real \((J, \nu)\)-holomorphic sphere maps in the class \(A\), having the first element of the 0-th marked pair lying in \(D^2 \subset \mathbb{CP}^1\).

Now we shall show that the two analogues of the Deligne-Mumford space are isomorphic. Denote with \(\text{Aut}_{\mathbb{R}^+}(\mathbb{CP}^1)\) the subgroup of the real automorphism group \(\text{Aut}_{\mathbb{R}}(\mathbb{CP}^1)\), which preserve the orientation of the real locus. We have \(\text{Aut}(D^2) \cong \text{Aut}_{\mathbb{R}^+}(\mathbb{CP}^1)\). Denote the space of marked spheres modulo the group \(\text{Aut}_{\mathbb{R}^+}(\mathbb{CP}^1)\) by \(\overline{\mathcal{M}}_{l+1,k}\), and the subset of \(\overline{\mathcal{M}}_{l+1,k}\) having the first element of the 0-th marked pair lying in \(D^2 \subset \mathbb{CP}^1\), by \(\overline{\mathcal{M}}^0_{l+1,k}\). The above discussion then gives an isomorphism between the moduli space of disks \(\mathcal{M}_{l+1,k}\) and \(\overline{\mathcal{M}}^0_{l+1,k}\).

**Theorem 3.2.2.** The map \(\tilde{\mathcal{M}}_{l+1,k} \to \overline{\mathcal{M}}^0_{l+1,k}\) given by the Schwarz reflection principle is an isomorphism.

**Proof.** We have that \(\mathcal{M}_{l+1,k} \cong \overline{\mathcal{M}}^0_{l+1,k}\) by the preceding discussion. We can use the equivalence \(\sim_g\) on both \(\mathcal{M}_{l+1,k}\) and \(\overline{\mathcal{M}}^0_{l+1,k}\). On the boundary of \(\overline{\mathcal{M}}_{l+1,k}\) the map \(g\) is given by \(\psi : \mathbb{CP}^1 \to \mathbb{CP}^1\) with \(\psi(z) = -z\). The quotient spaces are still isomorphic,
that is
\[ \tilde{M}_{l+1,k} \cong \mathbb{R} \tilde{M}_{l+1,k}^0 / \sim_g \]

On the other hand, \( \mathbb{R} \tilde{M}_{l+1,k}^0 / \sim_g \) is a double cover of \( \mathbb{R} \tilde{M}_{l+1,k} \). Over the open part this is because \( \text{Aut}_\mathbb{R}(\mathbb{C}P^1) \cong \text{Aut}_\mathbb{R}^+(\mathbb{C}P^1) \times \langle \psi \rangle \), and hence \( \mathbb{R} \tilde{M}_{l+1,k}^0 / \langle \psi \rangle = \mathbb{R} \tilde{M}_{l+1,k} \). Over the singular strata, \( \sim_g \) identifies a tuple of spheres with all possible flippings by \( \psi \) of the bubbles not containing \( z_0 \). The residual action of flipping the bubble containing \( z_0 \), is of order 2.

There is a section \( s : \tilde{M}_{l+1,k} \to \mathbb{R} \tilde{M}_{l+1,k}^0 / \sim_g \) given by choosing the sphere having \( z_0 \) in its upper half (the half of \( D^2 \subset \mathbb{C}P^1 \)). This means that the cover is two sheeted, and \( \mathbb{R} \tilde{M}_{l+1,k} \) is isomorphic to each of the sheets. Finally, by construction the image \( s(\mathbb{R} \tilde{M}_{l+1,k}) \) is equal to \( \mathbb{R} \tilde{M}_{l+1,k}^0 / \sim_g \cong \tilde{M}_{l+1,k} \).

We now consider having unstable bubbles. If the bubble is at an interior marked point, there is nothing to show. If the bubble is at a boundary marked point, then the two glued disk maps \( u \) and \( \tau \circ u \circ c \) give rise to the same sphere mod \( \text{Aut}_\mathbb{R}(\mathbb{C}P^1) \). Conversely, the two choices of a hemisphere of a sphere map, give the same disk map mod \( \sim_g \). Thus we have an isomorphism \( \tilde{M}_{l+1,k}(A) \cong \mathbb{R} \tilde{M}_{l+1,k}(A) \).

**Corollary 3.2.3.** There is a commutative diagram

\[
\begin{array}{ccc}
\tilde{M}_{l+1,k}(A) & \xrightarrow{\text{ev}_x \times \text{ev}_y \times \tau \text{ev}_y} & L \times M \times M \\
\cong & & \downarrow \text{id} \\
\mathbb{R} \tilde{M}_{l+1,k}(A) & \xrightarrow{\text{ev}_x \times \text{ev}_y \times \text{ev}_y} & L \times M \times M
\end{array}
\]

### 3.3 The (relative) sign of conjugation

To study the orientability of the new moduli space \( \tilde{M}(A) \), it will be useful to know what is the (relative) sign of the map \( g \) defined on the boundary of the moduli spaces \( \tilde{M}(b) \). Here we define and compute the relative sign of \( g \) both in the general case, and in the special case of a relatively spin Lagrangian.
First, we define what we mean by the relative sign of $g$. Suppose we have trivializations of $F = TL \oplus 3 \det TL$ over the image of $\partial D^2$, and of $F^1 = \det TL$ over the image of $x_1 \in \partial D^2$ under the map $u = (u_1, u_2) \in \tilde{\mathcal{M}}_{t+1,k}(b)$. This induces an orientation of $\det D_{u_1} \otimes \det D_{u_2} \otimes \det TL_{\text{node}}$, by transporting the trivialization of $F^1$ at $x_1$ to the nodal point (using the orientation of $\partial D^2$ to select the path). Moreover, since the unoriented image of $\partial D^2$ under $(u_1, u_2)$ and $(u_1, \tau \circ u_2 \circ c)$ is the same, the given trivializations induce orientations on $\det D_{u_1} \otimes \det D_{\tau \circ u_2 \circ c} \otimes F^1_{\text{node}}$. We call the sign of $g$ in this context the relative sign of $g$ at the point $u = (u_1, u_2)$. This sign is independent of the choice of trivializations that we begin with.

The map $g$ is defined on the fiber product $\tilde{\mathcal{M}}_{t+1,k+1}(b_1) \times_{ev} \tilde{\mathcal{M}}_{t,k+1}(b_2)$ by $id \times inv$, where

\[
inv : \tilde{\mathcal{M}}_{t,k}(b) \to \tilde{\mathcal{M}}_{t,k}(\bar{b}) \text{ is given by}
\]

\[
(u, \bar{x}, \bar{z}, \sigma) \mapsto (\tau \circ u \circ c, c(\bar{x}), c(\bar{z}), -Id_{\mathcal{Z}_2}(\sigma))
\]

Let us first find the sign of $inv$ at a point $v \in \tilde{\mathcal{M}}_{t,k}(b)$ given we have trivializations of $F^1_v(\partial D^2)$ and $F^1_{v(x_1)}$ inducing the orientations of both $\det D_v$ and $\det D_{\tau \circ u \circ c}$. The contribution to the sign of $inv$ coming from the boundary marked points is $-1$ for each boundary marked point, since the conjugation $c$ on $D^2$ reverses the orientation of its boundary. The orientation on the interior marked points depends on their decoration, and is chosen to be the one coming from the orientation of $D^2$, if the decoration is a plus, and the opposite one, if the decoration is a minus. Since the map $inv$ reverses both the orientation of $D^2$ and the decoration of the point, there is no contribution to the sign coming from the interior marked points. Additionally, the induced map on the automorphism group $\phi \mapsto \bar{\phi}(z)$ is orientation preserving as well.

Now we are left to consider the sign of $inv$ on the determinant line of $D_v$. First we recall how we induce the orientation on $\det(D_v)$. We do this by orienting $\det(\partial_{(E,F)})$ and $\det(\partial_{(E^1,F^1)})$, where $(E, F) = (TM \oplus 3 \det C TM, TL \oplus 3 \det TL)$.
and \((E^1, F^1) = (\det_C TM, \det TL)\). To orient \(\det(\bar{\partial}_{(E,F)})\) we push up the Maslov index and pinch the disk to obtain a sphere attached to a disk. The index over the disk is isomorphic to \(\text{Ind}(\bar{\partial}_{(\mathbb{C}^{n+1}, \mathbb{R}^{n+3})}) \cong \mathbb{R}^{n+3}\) and \(\tau\) acts as identity on it. The orientation on the determinant line on the sphere, comes from the canonical orientation on the kernel and cokernel, which in this case are complex vector spaces. The complex dimension of the index is \(c_1(E) \cdot \mathbb{C}P^1 + n + 3\), and since \(\tau^*J = -J\), the sign of \(\text{inv}\) on the sphere component is \((-1)^{c_1(E) \cdot \mathbb{C}P^1 + n + 3}\). On the other hand, \(c_1(E) \cdot \mathbb{C}P^1 = \frac{4\mu(b)}{2} = 2\mu(b)\), and therefore there is no contribution to the sign of \(\text{inv}\) coming from the Chern class. Finally, \(\tau\) acts on the canonical orientation of the incident condition with a sign \((-1)^{n+3}\), since the complex dimension of \(E\) is \(n + 3\). Now, let \(\tilde{w}_1(\partial b)\) denote the value of \(w_1(TL) \cdot (\partial b)\) as an integer, where \(\partial b \in H_1(L)\) is the class of the image of the boundary \(\partial D^2\). To orient \(\det(\bar{\partial}_{(E^1,F^1)})\) we push \(\mu(b) - \tilde{w}_1(\partial b)\) of the Maslov index to the sphere, and again pinch. The determinant line of the operator on the disk is isomorphic to \(ev_{x_1}^* \det(TL) \tilde{w}_1(\partial b)^{+1}\). This isomorphism depends on the orientation of \(\partial D^2\) in the case when \(w_1(TL) \cdot (\partial b) = 1\), and reversing the orientation of \(\partial D^2\) in this case, reverses the orientation of the determinant line. Since \(c\) reverses the orientation of \(\partial D^2\), and \(\tau\) acts as identity on \(\det(TL)\), the contribution to the sign of \(\text{inv}\) is \((-1)^{\tilde{w}_1(\partial b)}\). Again, the map \(\text{inv}\) will act on the incident condition by \((-1)\), and on the complex orientation of the index on the sphere with a sign equal to \((-1)\) to the dimension of the index, which is \(c_1(\det TM) \cdot (\mathbb{C}P^1) + 1 = \frac{\mu(b) - \tilde{w}_1(\partial b)}{2} + 1\). Thus, the overall sign of \(\text{inv}\) is \((-1)^{\frac{\mu(b) - \tilde{w}_1(\partial b)}{2} + \tilde{w}_1(\partial b) + k} = (-1)^{\frac{\mu(b) + \tilde{w}_1(\partial b) + k}{2}}\).

We shall now compute the relative sign of \(g\). Recall that the orientation on \(\det D_{u_1} \otimes \det D_{u_2} \otimes F^1\) is induced by transporting the trivialization of \(F^1_{u_1(x_1)}\) to the node. We have to consider two cases:

- \(x_1\) is on the bubble belonging to \(\tilde{\mathcal{M}}_{l_1+1,k_1+1}(b_1)\)
- \(x_1\) is on the bubble belonging to \(\tilde{\mathcal{M}}_{l_2,k_2+1}(b_2)\)

In the first case, the map \(g\) does not change the induced trivialization of \(F^1\) on the node, and therefore the sign of \(g\) is equal to the sign of \(\text{inv}\), that is \((-1)^{\frac{\mu(b_2) + \tilde{w}_1(\partial b_2) + k_2 + 1}{2}}\).
In the second case, however, the induced trivialization on the node changes when \( w_1(TL) \cdot (\partial b_2) = 1 \). This is because the path we transport the trivialization by, is chosen following the orientation of \( \partial D^2 \), and since \( c \) reverses this orientation, the induced trivialization of \( F^1 \) on the node will change when \( w_1(TL) \cdot (\partial b_2) = 1 \). Further, this will also change the orientation of \( \det D_{u_1} \) in the case when \( w_1(TL) \cdot (\partial b_1) = 0 \), since in this case the determinant line is isomorphic to a single copy of \( F_1 = \det TL \).

Finally, as before, we have a contribution of \((-1)^{\mu(b_2) + \tilde{w}_1(\partial b_2) + k_2 + 1 + \tilde{w}_1(\partial b_1) \tilde{w}_1(\partial b_2)} \)).

To combine the two cases let

\[
\epsilon = \begin{cases} 
0 & \text{if } x_1 \text{ is on the first bubble} \\
1 & \text{if } x_1 \text{ is on the second bubble}
\end{cases}
\]

Then the relative sign of \( g \) is given by

\[
rs(g) = \frac{\mu(b_2) + \tilde{w}_1(\partial b_2)}{2} + k_2 + 1 + \epsilon\tilde{w}_1(\partial b_1)\tilde{w}_1(\partial b_2) \mod 2
\]

We shall now consider the case when the Lagrangian \( L \) is relatively spin, that is orientable and \( w_2(L) \in Im(H^2(M) \to H^2(L)) \). In this case the moduli space \( \tilde{\mathcal{M}}_{l+1,k}(b) \) is orientable for any class \( b \). A choice of trivializations of \( F \) and \( F^1 \) as in Theorem 2.5.1, induces a canonical orientation of the determinant line of the linearized operator \( D \) for every class \( b \). Since all moduli spaces \( \tilde{\mathcal{M}}(b) \) are thus oriented, we can compute the sign of \( g \) on the boundary. Suppose \((u_1, u_2) \in \partial \tilde{\mathcal{M}}_{l+1,k}(b) \). Choose a path from the basepoint \( u_0 \) to \((u_1, u_2) \). This path transports the trivializations of \( F \) and \( F^1 \) to the images of \( \partial D^2 \) and \( x_1 \) under \((u_1, u_2) \), which induces the orientation of the moduli space at \((u_1, u_2) \). Similarly, choose a path from the basepoint of \( \tilde{\mathcal{M}}_{l+1,k}(b') \) to the point \((u_1, \tau \circ u_2 \circ c) = g((u_1, u_2)) \), where \( b' = b_1 + \bar{b}_2 \), and transport the respective trivializations of \( F \) and \( F^1 \) to induce the orientation. The sign of \( g \) then is equal to the relative sign of \( g \), plus the difference \( sp \) of the trivializations of \( F \) over the image of \( \partial D^2 \), and in the case when \( w_1(TL) \cdot (\partial b) = 0 \), the difference \( o \) of the trivializations.
of $F^1$ over the image of $x_1$. This gives

$$s(g) = rs(g) + (w_1(TL) \cdot (\partial b) + 1)o + sp$$

where $o \in \pi_0(O(1))$ and $sp \in \pi_1(SO(n+3))$. Note also that when $L$ is orientable, the relative sign of $g$ simplifies to $\frac{\mu(b_2)}{2} + k_2 + 1 \mod 2$. Since the moduli spaces $\tilde{M}_{l+1,k}(b)$ are orientable, this sign is independent of the choice of the point $(u_1, u_2)$, the paths from the basepoints to $(u_1, u_2)$ and the basepoints.

### 3.4 Orientability of $\tilde{M}_{l+1,k}(A)$

In this section we compute the first Stiefel-Whitney class of the new moduli space $\tilde{M}_{l+1,k}(A)$. When this class is a pull-back of a class $\kappa$ on the product $L \times \tilde{M}_{l+1,k}$, the image of $\tilde{M}_{l+1,k}(A)$ under $ev \times f$ will be a pseudocycle with coefficients in the local system twisted by $\kappa$. In Theorem 3.4.1 we give sufficient conditions for this to hold. In Theorem 3.4.2 we express the first Stiefel-Whitney class in terms of classes from $L$, $\tilde{M}_{l+1,k}$, and Poincare duals of boundary divisors in $\tilde{M}_{l+1,k}(A)$. The presence of the latter shows that in general the target will not carry the correct local system needed to accommodate the pseudocycle.

**Definition 3.4.1.** A tuple $(M, L, A, l, k)$ is called **admissible** if the following two conditions hold

1. All $b \in \tilde{\pi}_2^{-}(M, L)$ have Maslov index $\mu(b) \equiv 0 \mod 4$, where $\tilde{\pi}_2^{-}(M, L) = \{b \in \tilde{\pi}_2(M, L) | b = -\tau_r(b)\}$

2. The domains of all bubbles in codimension one of $\bigsqcup_{o(b)=A} \tilde{\mathcal{M}}_{l+1,k}(b)$ are stable if $k \neq 0$

**Example 3.4.2.** The pair $(M, L)$ satisfies the first condition in the following cases:

1. $H_1(L)$ has no $\mathbb{Z}_2$ torsion ($\Rightarrow \tilde{\pi}_2^{-}(M, L) = \emptyset$)

2. the Maslov index of any $b \in \pi_2(M, L)$ is a multiple of 4, e.g. $\mathbb{C}P^{4n-1}$, Calabi-Yau manifolds
Theorem 3.4.1. Suppose that the tuple \((M, L, A, l, k)\) is admissible, and that \(L\) is relatively spin. Then the first Stiefel-Whitney class of \(\widetilde{\mathcal{M}}_{l+1,k}(A)\) is the pull-back of the first Stiefel-Whitney class of \(\widetilde{\mathcal{M}}_{l+1,k}\).

Proof. The fact that \(L\) is relatively spin, implies that each of the moduli spaces \(\widetilde{\mathcal{M}}_{l+1,k}(b)\) is orientable. Therefore the orientability of \(\widetilde{\mathcal{M}}_{l+1,k}(A)\) depends only on the sign of \(g\) on the boundary of \(\widetilde{\mathcal{M}}_{l+1,k}(b)\). In the previous section we showed that the sign of \(g\) on a boundary component is given by

\[
s(g) = (-1)^{\frac{w(b)}{2} + k_2 + 1} + (w_1(TL) \cdot (\partial b) + 1) o + sp
\]

If this sign is \(-1\) on every boundary, then the moduli space \(\widetilde{\mathcal{M}}_{l+1,k}(A)\) is orientable. Otherwise, the first Stiefel-Whitney class of the moduli space, evaluated on a loop \(\gamma\), is equal to the sum mod 2 of the signs of \(g\) on the boundary components which the loop crosses, plus the number of boundary components crossed. We shall now examine this sum.

The sum of the elements involving \(o\) is equal to the evaluation of the first Stiefel-Whitney class \(w_1(TL)\) on the loop traced by the boundary marked point \(x_1\) inside \(L\), when \((w_1(TL) \cdot (\partial b) + 1) = 1\), and is zero otherwise. Since we assume that \(L\) is relatively spin, and in particular orientable, this evaluation is thus equal to zero.

The sum of the elements involving \(sp\) is equal to the evaluation of \(w_2(TL)\) on the \(\mathbb{Z}_2\)-homology class \(\tilde{T}_\gamma \in H_2(L; \mathbb{Z}_2)\), which the boundary \(\partial D^2\) traces in \(L\) along the loop \(\gamma\) in the moduli space \(\widetilde{\mathcal{M}}_{l+1,k}(A)\). This class bounds a class in \(H_3(M, L; \mathbb{Z}_2)\). Indeed, suppose the loop \(\gamma\) crosses \(s\) boundary divisors in \(\widetilde{\mathcal{M}}_{l+1,k}(A)\). Denote with \(\Gamma_\gamma\) the image of \(D^2\) along \(\gamma\). The boundary of \(\Gamma_\gamma\) consists of the image of \(\partial D^2\) along \(\gamma\), and the sum of the differences \(b_{i,2} - \bar{b}_{i,2}\), where \(b_{i,2}\) is the flipped bubble at the \(i\)-th crossing of a boundary divisor. Furthermore, the classes in \(\tilde{\pi}_2(M, L)\), which the maps represent, change each time a boundary divisor is crossed, by \(-b_{i,2} + \bar{b}_{i,2}\). Since \(\gamma\) is a loop, the sum of these changes must be zero. Therefore, the chain they represent in \(C_2(M, L)\) must bound a class \(B_\gamma \in C_3(M, L)\). Thus, the boundary of \(\tilde{\Gamma}_\gamma = \Gamma_\gamma - B_\gamma\)
is equal to the image of $\partial D^2$ which lies in $L$. Therefore, $\tilde{T}_\gamma$ is a cycle in the relative chain group $C_3(M, L)$, with $\partial \tilde{T}_\gamma = \text{Im}(\gamma|_{\partial D^2}) \in C_2(L)$. If we consider $\mathbb{Z}_2$ coefficients, then the image of $\partial D^2$ along $\gamma$ is a cycle, and we denote its class with $\tilde{T}_\gamma \in H_2(L; \mathbb{Z}_2)$. The map $\partial : H_3(M, L; \mathbb{Z}_2) \to H_2(L; \mathbb{Z}_2)$ then sends $\tilde{T}_\gamma$ to $\tilde{T}_\gamma$. Now, as in Corollary 2.2.2, the relatively spin condition implies that $w_2(L) \cdot \tilde{T}_\gamma = \delta(w_2(L)) \cdot \tilde{T}_\gamma = 0 \cdot \tilde{T}_\gamma = 0$.

The overall contribution from the Maslov indexes to the sum is equal to $\sum_{i=1}^s \frac{\mu(b_{i,2})}{2} \mod 2$, or equivalently $\sum_{i=1}^s \mu(b_{i,2}) \mod 4$. Under the admissibility condition, this contribution is zero. Indeed, we showed above that $\sum_{i=1}^s (b_{i,2} - \bar{b}_{i,2}) = 0$. Recall that $\bar{b} = -\tau_* b$. Then we have $\sum_{i=1}^s b_{i,2} = \sum_{i=1}^s -\tau_* b_{i,2} = -\tau_* \sum_{i=1}^s b_{i,2}$. Therefore $\sum_{i=1}^s b_{i,2} \in \tilde{\pi}^-(M, L)$, and thus its Maslov index is equal to zero mod 4.

Furthermore, if the flipped bubble is stable, $(-1)^{k_2+1}$ is precisely the sign of $g$ on the moduli space of domains $\tilde{\mathcal{M}}_{l+1,k}$, whose first Stiefel-Whitney class evaluated on a loop is similarly given as a sum. If all domains are stable, that sum is precisely equal to the sum giving the first Stiefel-Whitney class of $\tilde{\mathcal{M}}_{l+1,k}(A)$ (when $L$ is relatively spin and there is no contribution from the Maslov indexes). The only possible unstable domains are a bubble with no marked points and a bubble with one boundary marked point. The latter is excluded under the assumptions of the theorem. The former contributes to the sum, one for the node, and one for the boundary component, and hence can be disregarded. It follows that $w_1(\tilde{\mathcal{M}}_{l+1,k}(A)) = \mathfrak{f}^* w_1(\tilde{\mathcal{M}}_{l+1,k})$.

\[ w_1(\tilde{\mathcal{M}}_{l+1,k}(A)) \cdot \gamma = w_2(L) \cdot \tilde{T}_\gamma + (c_1(M) \cdot A + 1) \cdot (w_1(L) \cdot \text{ev}_{x_1}(\gamma)) + \left[ \mathfrak{f}^*(w_1(\tilde{\mathcal{M}}_{l+1,k})) + D_1^\gamma + D_2^\gamma + O_{x_1}^\gamma + U^\gamma \right] \cdot \gamma \]

where $\tilde{T}_\gamma$ is the $\mathbb{Z}_2$-homology class in $H_2(L, \mathbb{Z}_2)$ that the boundary $\partial D^2$ traces in $L$ along the loop $\gamma$, and $D_1^\gamma, O_{x_1}^\gamma$ and $U^\gamma$ are cohomology classes in $H^1(\tilde{\mathcal{M}}_{l+1,k}(A), \mathbb{Z}_2)$ whose Poincaré duals are codimension 1 strata in $\tilde{\mathcal{M}}_{l+1,k}(A)$, with $D_i$ having a second
bubble of Maslov index \( \mu(b_2) \equiv i \mod 4 \), \( O_{x_1} \) having \( x_1 \) on the second bubble and both Maslov indexes \( \mu(b_1) \) and \( \mu(b_2) \) odd, and \( U \) having a bubble with an unstable domain with one boundary marked point.

Proof. Recall that the formula for the first Stiefel-Whitney class of \( \tilde{\mathcal{M}}_{l+1,k}(b) \) evaluated on a loop \( \gamma \) is given by

\[
\begin{align*}
    w_2(L) \cdot \tilde{T}_\gamma + (w_1(L) \cdot (\partial b) + 1) \cdot ev_{x_1}(\gamma).
\end{align*}
\]

Recall also that the moduli space of domains \( \tilde{\mathcal{M}}_{l+1,k} \) is orientable. Hence, if the loop \( \gamma \) is inside one of the moduli spaces \( \tilde{\mathcal{M}}_{l+1,k}(b) \), we see that the formula in the theorem reduces to the formula for \( \tilde{\mathcal{M}}_{l+1,k}(A) \). Otherwise, we can think of the loop \( \gamma \) as several paths \( \alpha_1, \ldots, \alpha_r \) each in some moduli space \( \tilde{\mathcal{M}}(b_i) \), joint together at a boundary divisor. Choose a point \( u \) on \( \alpha_1 \) - this will separate the path \( \alpha_1 \) into two paths \( \beta_1 \) and \( \beta_2 \). Choose trivializations of \( F \) and \( F^1 \) over the images \( u(\partial D^2) \) and \( u(x_1) \) respectively. The path \( \beta_1 \) transports these trivializations to its other end, and hence to the beginning of \( \alpha_2 \), which again transports it. The first Stiefel-Whitney class evaluated on the loop \( \gamma \) is then given by the sum of the relative signs, plus the difference between the trivialization of \( F \) that we started with, and the one induced following the loop, as well as, the difference in the trivializations of \( F^1 \) in the case when \( w_1(L) \cdot (\partial b) = 0 \). The expression is given by

\[
\begin{align*}
    w_1(\tilde{\mathcal{M}}_{l+1,k}(A)) \cdot \gamma &= \sum rs(g) + w_2(L) \cdot \tilde{T}_\gamma + (w_1(L) \cdot \partial b + 1) \cdot (w_1(L) \cdot ev_{x_1}(\gamma)).
\end{align*}
\]

Note that \( w_1(L) \cdot \partial b \) is independent of the class \( b \) satisfying \( o(b) = A \). This is because if \( o(b) = A \), then \( \mu(b) = c_1(M) \cdot A \), and since \( \mu(b) \mod 2 \) is equal to \( w_1(L) \cdot \partial b \), we have that \( w_1(L) \cdot \partial b = c_1(M) \cdot A \mod 2 \).

The expressions for the relative signs of \( g \) are given by

\[
\begin{align*}
    rs(g) &= \frac{\mu(b_2) + \bar{w}_1(b_2)}{2} + \epsilon w_1(b_1)w_1(b_2) + k_2 + 1 \mod 2.
\end{align*}
\]

As discussed in the previous theorem, the contribution from the marked points is the same as their contribution in the moduli space of domains, except when the boundary component has a bubble with an unstable domain with one boundary marked.
point. This gives the expression $\mathcal{J}^*(w_1(\mathcal{M}_{l+1,k})) + U^\vee$ in the formula. The expression $\frac{\mu(b_2)+\tilde{w}_1(b_2)}{2}$ contributes to the sign when it equals to one mod 2, or equivalently $\mu(b_2)+\tilde{w}_1(b_2)$ equals to two mod 4, and equivalently $\mu(b_2)$ equals to one or two mod 4. This happens precisely when we cross the divisors $D_1$ or $D_2$. Finally, $\epsilon w_1(b_1)w_1(b_2)$ contributes to the sign, when the marked point $x_1$ lies on the bubble which is flipped, and the Maslov indexes of both $b_1$ and $b_2$ are odd, which happens when we cross the divisor $O_{x_1}$. The formula then follows.

3.5 Open Gromov-Witten disk invariants

The open Gromov-Witten disk invariant is defined using the homology class obtained by pushing forward the moduli space $\mathcal{M}_{l+1,k}(A)$ by the evaluation and forgetful maps $ev \times f : \mathcal{M}_{l+1,k}(A) \to M^{l+1} \times L^k \times \mathcal{M}_{l+1,k}$.

In this section we show that the absolute value of the pairing of this class with cohomology classes on the product is well-defined and is independent of a strongly semi-positive deformation of $\omega$.

Remark 3.5.1. When we work with non-orientable manifolds, to fix a fundamental class we need to choose an orientation of the orientation double cover of the manifold. In accordance with this, we shall choose trivializations as in Theorem 2.5.1, which in the relatively spin case determine an orientation on $det D_b$ for every class $b$. We then choose a basepoint $u_0 \in \mathcal{M}_{l+1,k}(A)$, whose image under the forgetful map is the basepoint of $\mathcal{M}_{l+1,k}$. Finally, we choose a lift $\tilde{u}_0$ of $u_0$ in the orientation double cover, and we pull-back the given orientation at $u_0$ induced by the orientations of $det D$ and $\mathcal{M}_{l+1,k}$. Here we assume that the moduli space has only one connected component. In case of several, we need to make the above choices for each component, and the invariant will be a sum over the connected components.

Theorem 3.5.1. Assume $(M,\omega)$ is a strongly semi-positive symplectic manifold with an anti-symplectic involution $\tau$. Assume also that the fixed locus of $\tau$ is relatively spin, and that the tuple $(M,L,A,l,k)$ is admissible. Choose a lift $\tilde{u}_0$ as in Remark
3.5. Then for a generic \((J, \nu) \in J_\mathbb{R}\), the image of \(\tilde{\mathcal{M}}_{l+1,k}(A)\) under the map \(ev \times f\) defines an element

\[
OGW_{\tilde{u}_0}^{\tilde{u}_0} \in H_*(M^{l+1} \times L^k \times \tilde{\mathcal{M}}_{l+1,k}; \mathbb{Z}_{w_1(\tilde{\mathcal{M}}_{l+1,k})})
\]

**Proof.** The regularity of the pair \((J, \nu)\) implies that we have transversality for all maps with stable domains. Furthermore, the image of \(\tilde{\mathcal{M}}_{l+1,k}(A)\) is contained in the image of the reduced space \(\tilde{\mathcal{M}}_{l+1,k}^r(A)\). By the strongly semi-positive condition, all strata in \(\tilde{\mathcal{M}}_{l+1,k}^r(A)\) is of expected or smaller dimension, and by Theorem 3.2.1, it has no boundary strata. As shown in Theorem 3.4.1, the orientation system on \(\tilde{\mathcal{M}}_{l+1,k}(A)\) is the pull-back of the orientation system on \(\tilde{\mathcal{M}}_{l+1,k}\), and therefore the image \(ev \times f[\tilde{\mathcal{M}}_{l+1,k}(A)]\) carries a homology class in \(H_*(M^{l+1} \times L^k \times \tilde{\mathcal{M}}_{l+1,k}, \mathbb{Z}_{w_1(\tilde{\mathcal{M}}_{l+1,k})})\).

**Remark 3.5.2.** If we assume that all domains are stable, the strongly semi-positive condition is not necessary. Note also that in this case, the second part of the admissibility condition is automatically satisfied.

**Proposition 3.5.2.** If \(\tilde{u}'_0\) is the lift of another point \(u'_0\), satisfying the conditions of Remark 3.5.1, and \(\tilde{\gamma}\) is a path between \(\tilde{u}_0\) and \(\tilde{u}'_0\), then

\[
OGW_{\tilde{u}_0}^{\tilde{u}'_0} = (sp + o(c_1(A) + 1) + \epsilon) \cdot OGW_{\tilde{u}_0}^{\tilde{u}_0}
\]

where \(sp \in \pi_1(SO(n+3)) = \mathbb{Z}_2\) is the difference between the two trivializations of \(F\), \(o \in O(1) = \mathbb{Z}_2\) is the difference between the two trivializations of \(F^1\), and \(\epsilon\) is equal to the value of the first Stiefel-Whitney class of \(\tilde{\mathcal{M}}_{l+1,k}\) evaluated on the image of \(\tilde{\gamma}\) in \(\tilde{\mathcal{M}}_{l+1,k}\).

**Proof.** Changing the trivializations of \(F\) and \(F^1\) over the images \(u'_0(\partial D^2)\) and \(u'_0(x_1)\), changes the orientation of \(\det D\) by \((sp + o(c_1(A) + 1))\), and hence the orientation of the orientation double cover changes by the same sign. Since both \(u_0\) and \(u'_0\) map to the base point in \(\tilde{\mathcal{M}}_{l+1,k}\), the change of the induced by pull-back orientation is equal to \(\epsilon\).

**Corollary 3.5.3.** Let \(h = (h_1^M, \ldots, h_{i+1}^M, h_t^1, \ldots, h_k^l, h^{DM}) \in H^*(M; \mathbb{Q}) \oplus H^*(L; \mathbb{Q}) \oplus H^*(\tilde{\mathcal{M}}_{l+1,k}; \mathbb{Q}_{w_1(\tilde{\mathcal{M}}_{l+1,k})})\) has degree equal to the dimension of \(\tilde{\mathcal{M}}_{l+1,k}(A)\). Then the
number
\[ OGW_{A,l+1,k}(h) = |h \cdot OGW^{\tilde{u}_0}_{A,l+1,k}| \]
is independent of \( \tilde{u}_0 \).

**Theorem 3.5.4.** The number \( OGW_{A,l+1,k}(h) \) is independent of the choice of a regular pair \( (J, \nu) \in J_R \) and of a strongly semi-positive deformation of \( \omega \).

**Proof.** This follows from a standard cobordism argument. If \( (J_1, \nu_1) \) and \( (J_2, \nu_2) \) are two regular pairs in \( J_R \), they can be connected by a path. For a generic such path, the universal moduli space will have top and smooth codimension one strata forming a manifold. The image of the singular and higher than one codimension strata will again be contained in a lower dimensional manifold. Then, up to a choice of \( \tilde{u}_1 \) and \( \tilde{u}_2 \), the two homology classes defined by \( (J_1, \nu_1) \) and \( (J_2, \nu_2) \) will be the same. Similar cobordism argument holds for a strongly semi-positive deformation of \( \omega \). \( \square \)

**Remark 3.5.3.** In the case when the moduli space \( \tilde{M}_{l+1,k} \) is orientable and \( L \) is spin, the above results hold without the absolute value in the definition of \( OGW_{A,l+1,k}(h) \) as well. The moduli space \( \tilde{M}_{l+1,k} \) is orientable if there are no boundary marked points, or when its dimension is zero or one.

**Example 3.5.4.** Let \( (M, L) = (\mathbb{C}P^{4s-1}, \mathbb{R}P^{4s-1}) \). This is a strongly semi-positive symplectic manifold, and the standard complex structure is regular. The tuple \( (M, L, A, l, 0) \) is admissible, since the Maslov index is a multiple of 4, and \( k = 0 \). Additionally, we consider classes \( A \) of odd degree to ensure that all \( \partial b \) are non-zero in \( \pi_1(L) \). The moduli space \( \tilde{M}_{l+1,0} \) is orientable, and hence we can use the dual of its fundamental class in the constraint \( h \). This, together with the fact that the moduli space \( \tilde{M}_{l+1,0}(A) \) is isomorphic to the moduli space of real spheres, allows us to interpret the number \( OGW_{A,l+1,0}(h) \) as the number of real spheres in the class \( A \) passing through the Poincare duals of \( (h_1^M, \ldots, h_{l+1}^M) \). In particular, if we take \( A \) to be the class of a line, \( l = 2 \), \( h = (pt^\vee, h^\vee) \), where \( h \) is the class of a hyperplane, then \( OGW_{A,2,0}(h) \) counts the number of real spheres passing through a complex point and intersecting a hyperplane. The real condition forces the curve to also pass through the complex conjugate of the point, and hence there is only one such curve. Thus, the number \( OGW_{ln,2,0}(pt^\vee, h^\vee) = 1 \).
Remark 3.5.5. Note that when the moduli space of domains $\widetilde{M}_{l+1,k}$ is orientable, we can use the dual of its fundamental class as a constraint, and thus obtain a count of the curves passing through constraints in $M$ and $L$. When $\widetilde{M}_{l+1,k}$ is not orientable, however, $H^0(\widetilde{M}_{l+1,k}, Q_{w_1(\widetilde{M}_{l+1,k})}) = 0$, and we have to use different constraints. This can be interpreted as a restriction on the domain of the map.

In these cases, one can define a count of curves without restricting the domains, if one can show that using a particular type of constraints in $M$ and $L$, in one parameter family the cut-down moduli space will not cross boundary divisors which contribute to the first Stiefel-Whitney class of $\widetilde{M}_{l+1,k}$. This is the approach of Solomon [Sol] and Cho [Cho]. Note that a boundary divisor contributes to $w_1(\widetilde{M}_{l+1,k})$ if the number of boundary marked points on the flipped bubble is odd.

To illustrate this approach, suppose $(M, L) = (\mathbb{C}P^3, \mathbb{R}P^3)$, $A$ has degree $d$, and we use only real point constraints at the boundary marked points, and hyperplane constraints at the interior points. From the dimension formula we see that the number of boundary marked points $k$ must equal $2d$. Suppose in one parameter family the cut down moduli space bubbles into classes $d_1$ and $d_2$ with $k_1$ and $k_2$ marked point respectively. This strata appears only if $4d_1 + 1 \geq 2k_1$ and $4d_2 + 1 \geq 2k_2$ by a dimension count. We add $2k_2$ to the first inequality to obtain

$$4d_1 + 1 + 2k_2 \geq 2k_1 + 2k_2 = 2k = 4d \Rightarrow 4d_2 - 1 \leq 2k_2$$

This together with the second inequality implies that $k_2$ must be even, and since $k$ was even, so should $k_1$ be. Thus, we never cross strata which contribute to the first Stiefel-Whitney class of $\widetilde{M}_{l+1,k}$. 
Bibliography


