A DUALITY THEOREM FOR DELIGNE-MUMFORD STACKS
WITH RESPECT TO MORAVA $K$-THEORY

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Man Chuen Cheng
August 2011
I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Soren Galatius, Primary Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Gunnar Carlsson

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Ralph Cohen

Approved for the Stanford University Committee on Graduate Studies.

Patricia J. Gumport, Vice Provost Graduate Education

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Abstract

In [7] Greenlees and Sadofsky used a transfer map to show that the classifying spaces of finite groups are self-dual with respect to Morava $K$-theory $K(n)$. By regarding these classifying spaces as the homotopy types of certain differentiable stacks, their construction can be viewed as a stack version of Spanier-Whitehead type construction. From this point of view, we will extend their results and prove a $K(n)$-version of Poincaré duality for Deligne-Mumford stacks. A few examples of stacks defined by finite groups and moduli stack of Riemann surfaces will be discussed at the end.
Acknowledgments

First and foremost, I would like to express my sincere thanks to my advisor Søren Galatius for his patient guidance and continuous encouragement throughout my graduate studies. He has been always generous with his time and idea to discuss mathematics with me. I am grateful to work in this interesting thesis project suggested by him.

I am thankful to the Stanford Mathematics Department. I would like to say thanks to Ralph Cohen, Gunnar Carlsson and Andrew Blumberg for teaching me algebraic topology, and to my friends Robin, Anssi, Eric, Tracy, Jonathan, Jeremy, Nathan, Martin, Jose, Jesse, Jeff, Daniel, Ken, Lan and my officemates for making my time at Stanford a very pleasant and enjoyable experience.

I would also like to express my gratitude to the Chinese University of Hong Kong where I had five great years for my undergraduate and master studies. I especially thank my master thesis advisor Luen Fai Tam. He is a good teacher and has a big influence on my mathematics career.

I also thank John Greenlees for sharing with me his viewpoint and idea on my thesis topic.

Last but not least, I am deeply indebted to my family for their love and support throughout the years. I dedicate this thesis to them.
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Chapter 1

Introduction and Statement of Results

Manifolds and classifying spaces of groups are two important classes of spaces. For a closed oriented manifold $M$, Poincaré duality gives us a simple relation between its homology and cohomology

$$\tilde H_*(M, \mathbb{Z}) \cong H^{m-\ast}(M, \mathbb{Z}).$$

Poincaré duality can be viewed as a consequence of the Thom isomorphism and the Spanier-Whitehead duality

$$\Sigma^\infty M_+ \simeq F(M^{-TM}, S).$$ \hspace{1cm} (1.1)

Classifying spaces of groups, on the other hand, usually have non-zero integral homology groups in infinitely many degrees and hence cannot satisfy a duality of the form $H^\ast(BG, \mathbb{Z}) \cong H_{m-\ast}(BG, \mathbb{Z})$. Nevertheless, they exhibit duality properties with respect to Morava $K$-theory.

For each fixed prime $p$, Morava $K$-theory is a sequence of homology theories $K(n), n \geq 0$, with coefficient ring $K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}]$ and $\deg v_n = 2(p^n - 1)$. In [16] Ravenel showed that for a finite group $G$, the $n$-th Morava $K$-theory cohomology ring $K(n)_*(BG)$ has finite rank over $K(n)_*$. Later, it was shown in [7] that
which can be regarded as a $K(n)$-version of Poincaré duality for $BG$. Strickland [18] showed that this duality can be obtained using a transfer map for a covering map version of the diagonal $BG \to BG \times BG$. The transfer map, constructed by equivariant stable homotopy theory, gives a map of spectra

$$\Sigma^\infty(BG \times BG)_+ \to \Sigma^\infty BG_+,$$

and composing with the collapse map $\Sigma^\infty BG_+ \to \mathbb{S}$ gives $\Sigma^\infty BG_+ \wedge \Sigma^\infty BG_+ \to \mathbb{S}$, with adjoint

$$\Sigma^\infty BG_+ \to F(\Sigma^\infty BG_+, \mathbb{S}).$$

The theorem of [11] is that this becomes an equivalence after $K(n)$-localization. In particular, taking $K(n)$-homology gives us the $K(n)$-self-duality of $BG$ in (1.2).

Despite the differences between $M$ and $BG$, the constructions of (1.1) and (1.4) underlying their dualities have common ingredients. To make the similarities more transparent, we will need the notion of stack, which is a generalization of spaces. Both $M$ and $BG$ are the homotopy types of certain simple stacks, and this setup puts the two spaces on an equal footing. Indeed, (1.4) can then be regarded as the stack version of the Spanier-Whitehead map.

From the viewpoint of stacks, it is natural to ask if the duality (1.2) can be generalized to stacks. For a topological stack $\mathfrak{X}$, let $\eta_\mathfrak{X} : \text{Ho}(\mathfrak{X}) \to \mathfrak{X}$ be its homotopy type. If $\mathfrak{X}$ is represented by a topological groupoid $\Gamma$, then $\text{Ho}(\mathfrak{X})$ can be taken to be the classifying space $B\Gamma$. If $E$ is a homology theory, then $E_n(\mathfrak{X})$ and $E^n(\mathfrak{X})$ are defined to be $E_n(\text{Ho}(\mathfrak{X}))$ and $E^n(\text{Ho}(\mathfrak{X}))$ respectively.

Suppose $\mathfrak{X}$ is a differentiable stack. It is said to be closed if it is compact and without boundary. There is a notion of tangent stack $T\mathfrak{X}$ and it induces a virtual vector bundle on $\text{Ho}(\mathfrak{X})$. We occasionally write $\mathfrak{X}^{-T\mathfrak{X}}$ for the corresponding Thom spectrum $\text{Ho}(\mathfrak{X})^{-T\mathfrak{X}}$. A differentiable stack is said to be Deligne-Mumford if it is locally of the form $[V/G]$ for a finite group $G$ and a $G$-representation $V$. 
The following is our main theorem concerning the duality of stacks with respect to Morava $K$-theory.

**Theorem 1.0.1.** Let $p$ be a prime and $K(n)$ be the corresponding $n$-th Morava $K$-theory. Let $\mathcal{X}$ be a differentiable local quotient stack (See definition 3.3.1). Then the following holds:

(a) There is a map of spectra

$$\psi : \Sigma^\infty \text{Ho}(\mathcal{X})_+ \longrightarrow F(\text{Ho}(\mathcal{X})^{-T\mathcal{X}}, S)$$

which reduces to (1.1) if $\mathcal{X}$ is a differentiable manifold and (1.4) if $\mathcal{X}$ is the quotient stack $[\ast / G]$ for a finite group $G$.

(b) Suppose $\mathcal{X}$ is a closed Deligne-Mumford stack. Then the map $\psi$ induces $K(n)$-duality

$$K(n)_*(\mathcal{X}) \cong K(n)^{-*}(\mathcal{X}^{-T\mathcal{X}}).$$

(c) Suppose in addition to the condition in (b), $p > 2$ and $\mathcal{X}$ is oriented. Then

$$K(n)_*(\mathcal{X}) \cong K(n)^{\dim \mathcal{X}-*}(\mathcal{X}).$$

**Remark 1.0.2.** (i) Theorem 1.0.1(c) is a consequence of 1.0.1(b) by the fact that $K(n)$-orientability is the same as ordinary orientability for $p > 2$ [17].

(ii) There is a version of $K(n)$-duality for compact Deligne-Mumford stacks with boundary by replacing $K(n)_*(\mathcal{X})$ by $\tilde{K}(n)_*(\mathcal{X}/\partial \mathcal{X})$ in (1.6).

As mentioned before, the map (1.5) in the main theorem can be regarded as a stack version of Spanier-Whitehead map. Analogous to the case of manifolds, the map is obtained from the Pontryagin-Thom construction of the diagonal map $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$. The general Pontryagin-Thom construction for local quotient stacks is due to Ebert and Giansiracusa [5].

For readers who are unfamiliar with the language of stacks, we will explain an important consequence of the above theorem in equivariant homotopy theory. Let $G$...
be a compact Lie group with adjoint representation $\mathfrak{g}$ and $M$ be a smooth $G$-manifold. This Lie group action defines a differentiable stack $[M/G]$. If the group action has finite stabilizer at each point of $M$, then $[M/G]$ is a Deligne-Mumford stack. Every differentiable stack has an associated homotopy type, which can be regarded as the best homotopy approximation of the stack by a space. In the case of quotient stack, the homotopy type $\operatorname{Ho}([M/G])$ of $[M/G]$ is given by the Borel construction $EG \times_G M$.

Consider the $G$-vector bundles $EG \times TM \to EG \times M$ and $EG \times M \times \mathfrak{g} \to EG \times M$ induced by $TM \to M$ and the projection. Their $G$-orbits are the non-equivariant vector bundles $EG \times G TM \to EG \times G M$ and $(EG \times M \times \mathfrak{g})/G \to EG \times G M$. For simplicity, we will denote them by $TM$ and $\mathfrak{g}$ respectively. The virtual bundle $-TM + \mathfrak{g}$ defines a Thom spectrum $(EG \times G M)^{-TM+\mathfrak{g}}$. If $X \simeq [M/G]$, $(EG \times G M)^{-TM+\mathfrak{g}}$ can be understood as $X^{-TX}$. Hence, in this special case of $X \simeq [M/G]$, theorem 1.0.1 can be interpreted as the following theorem of equivariant homotopy theory.

**Theorem 1.0.3.** Let $p$ be a prime and $K(n)$ be the corresponding $n$-th Morava $K$-theory. Let $G$ be a compact Lie group acting on an $m$-dimensional closed manifold $M$. Then the following holds:

(a) There is a map of spectra

$$
\Sigma^\infty(EG \times G M)_+ \longrightarrow F((EG \times G M)^{-TM+\mathfrak{g}}, S) \quad (1.7)
$$

which reduces to (1.1) if $G$ is trivial and (1.4) if $M$ is a point and $G$ is finite.

(b) Suppose $M$ is closed and the stabilizer subgroup $G_x$ is finite for all $x \in M$. Then the map (1.7) induces $K(n)$-duality

$$
K(n)_*(EG \times G M) \cong K(n)^{-*}(EG \times G M)^{-TM+\mathfrak{g}}.
$$

(c) Suppose in addition to the condition in (b), $p > 2$, $TM$ is orientable and the $G$-actions on both $TM$ and $\mathfrak{g}$ are orientation preserving. Then

$$
K(n)_*(EG \times G M) \cong K(n)^{m - \dim G - *}(EG \times G M). \quad (1.8)
$$
Remark 1.0.4. Similar to the case of stacks, there is also a version of $K(n)$-duality for compact manifolds with boundary

$$K(n)_*(EG \times_G M, EG \times_G \partial M) \cong K(n)^{n-\dim G-\bullet}(EG \times_G M).$$

As in the case of Poincaré duality, two consequences of the $K(n)$-duality of Deligne-Mumford stacks are the definitions of fundamental class and intersection product in homology.

**Definition 1.0.5.** Suppose $p > 2$ and $\mathcal{X}$ is a closed, oriented $q$-dimensional Deligne-Mumford stack. Let $\psi_* : K(n)_*(\mathcal{X}) \cong K(n)^{\dim \mathcal{X}-\bullet}$ be the isomorphism (1.6) in theorem 1.0.1.

(a) The $K(n)$-fundamental class of $\mathcal{X}$ is defined to be $[\mathcal{X}] = \psi_*^{-1}(1) \in K(n)_q(\mathcal{X})$, the pre-image of $1 \in K(n)^0(\mathcal{X})$ under $\psi_*$.

(b) The intersection product

$$\cap : K(n)_i(\mathcal{X}) \times K(n)_j(\mathcal{X}) \to K(n)_{i+j-q}(\mathcal{X})$$

in $K(n)_*(\mathcal{X})$ is defined to be the dual of cup product with respect to $\psi_*$. More precisely, $\alpha \cap \beta := \psi_*^{-1}(\psi_*(\alpha) \cup \psi_*(\beta))$.

Since $\psi_*([\mathcal{X}]) = 1$, the fundamental class $[\mathcal{X}]$ acts as identity in intersection product.

A key difference between manifolds and Deligne-Mumford stacks is that the later possess singularities of finite order. A Deligne-Mumford stack is pointwisely equivalent to $[*/G]$ for some finite group $G$. Therefore, in order to understand intersection product, it is interesting to begin with $K(n)_*([*/G]) = K(n)_*(BG)$.

For manifolds, the intersection product in ordinary homology admits a geometric interpretation for transverse submanifolds. We look for an analogous statement for stacks defined by finite groups. Submanifolds can be replaced by subgroups. For the transversality condition, we propose the following definition.
CHAPTER 1. INTRODUCTION AND STATEMENT OF RESULTS

Definition 1.0.6. Two subgroups $H$ and $K$ of a finite group $G$ is said to intersect transversely if $HK := \{hk | h \in H, k \in K\} = G$.

For any subgroup $i : H \hookrightarrow G$, write $[BH]$ for the image of the fundamental class of $[*/H]$ under the map $i_* : K(n)_0(BH) \to K(n)_0(BG)$. The following is our intersection formula for transverse subgroups of $G$.

Theorem 1.0.7. Suppose the $H, K$ are transverse subgroups of a finite group $G$. Then $[BH] \cap [BK] = [B(H \cap K)]$.

We are also particularly interested in the fundamental class of the moduli stack of Riemann surfaces. Let $\mathcal{M} = \mathcal{M}_{g,r}$ denote the moduli stack of Riemann surfaces of genus $g$ and $r$ marked points. It is the quotient stack represented by the action of the mapping class group $\Gamma = \Gamma_{g,r}$ on the $6g - 6 + 2n$-dimensional Teichmüller space $\mathcal{T} = \mathcal{T}_{g,r}$. Since $\mathcal{T}$ is contractible, $\text{Ho}(\mathcal{M}_{g,r}) \simeq B\Gamma_{g,r}$. Harvey [9] and Harer [8] constructed a Borel-Serre bordification of $\mathcal{T}$ for $n = 0$ and $n > 0$ respectively. It is a non-compact $(6g - 6 + 2r)$-dimension piecewise linear $\Gamma$-manifold $W$ with corners such that its interior is $\Gamma$-homeomorphic to $\mathcal{T}$. The bordification passes to the stack level and gives rise to a border $\partial \mathcal{M}$ for $\mathcal{M}$. We show that

Proposition 1.0.8. $K(n)_*(\mathcal{M}_{g,r}) \cong K(n)_*(B\Gamma_{g,r})$ has finite rank over $K(n)_*$. A $K(n)$-fundamental class $[\mathcal{M}_{g,r}]$ of $\mathcal{M}_{g,r}$ can be defined in $K(n)_{6g-6+2r}(\mathcal{M}, \partial \mathcal{M})$.

The organization of this thesis is as follow. Chapter 2 contains background materials for the paper. We recall several definitions in stable homotopy theory, discuss our general Pontryagin-Thom map and review the underlying construction of the Spanier-Whitehead duality and the transfer map. Some important properties of Morava $K$-theory and a brief recollection of equivariant stable homotopy theory will also be given. In chapter 3, we will give a quick introduction of differentiable stacks and describe how to do Pontryagin-Thom construction for morphisms of stacks. The proof of the main theorem will be presented in chapter 4. We will discuss examples of stacks defined by finite group and the moduli stacks of Riemann surfaces in chapter 5.
Chapter 2

Background and Preliminaries

2.1 Thom Spectra

There are several definitions of spectra in the literature. In this thesis, we will see two of them. The first one is by Adams [1] and adopts a more direct approach. In his definition, the underlying spaces of most examples of spectra are simple and many geometric constructions on spectra can be described explicitly. The second definition, as introduced by Lewis, May, Steinberger in [13], is geometrically more complicated but the spectra possess many nice categorical properties. For instance, their category of spectra has arbitrary limits and colimits. Many constructions and arguments can be done formally using machineries from category theory.

A brief introduction to the theory of [1] and [13] will be given in this section and section 2.6 respectively. To avoid confusion, whenever we mention spectra, it means spectra in the sense of Adams unless specified otherwise. We will also discuss the relation between these two notions of spectra at the end of section 2.6.

We now start with Adam’s definition of spectra. In this approach, a spectrum $D$ consists of a sequence of pointed spaces $\{D_n\}_{n=0}^{\infty}$ and structure maps $\sigma_n : \Sigma D_n \to D_{n+1}$, where $\Sigma D_n$ denotes the reduced suspension of $D_n$. Subspectra are defined in the obvious way, and a subspectrum $C$ of $D$ is said to be dense in $D$ if for any compact subset $X$ of $D_n$, the image of $\Sigma^k X$ under iterated $\sigma_i$’s is contained in $C_{n+k}$ for some $k \geq 0$. Given spectra $D$ and $D'$, define $P(D, D')$ to be the set of sequences
\( \{ f_n : D_n \to D'_n \} \) of based map which commute with the structure maps of \( D \) and \( D' \). Consider the set

\[ \coprod_C P(C, D') \]

where \( C \) runs through all the dense subspectra of \( D \). Two elements \( \{ f_n : C_n \to D'_n \}, \{ \tilde{f}_n : \tilde{C}_n \to D'_n \} \) of \( \coprod_C P(C, D') \) are said to be equivalent if \( f_n, \tilde{f}_n \) agree on a dense subspectrum of \( C \) and \( \tilde{C} \). A map \( D \to D' \) is then defined to be an equivalence class of this relation. We denote this category of spectra and maps by \( S \).

An important class of spectra is suspension spectra. Every pointed space \( X \) defines a suspension spectrum \( \Sigma \infty X \), whose \( n \)-th space is the \( n \)-fold suspension \( \Sigma^n X \) and \( \sigma_n \) is the obvious map \( \Sigma(\Sigma^n X) \cong \Sigma^{n+1} X \). The sphere spectrum \( S \) is defined to be \( \Sigma \infty S^0 \), where \( S^0 \) is the zero dimensional sphere with one of its two points the base point.

Another important class of spectra for us is Thom spectra of stable vector bundles. For a vector bundle \( V \to X \), its associated Thom space \( X^V \) is the fiberwise one-point compactification of \( V \) modulo the section of infinity. Alternately, if \( V \) is equipped with a metric, \( X^V \) can be described as the quotient \( D(V)/S(V) \), where \( D(V) := \{ v \in V : \|v\| \leq 1 \} \) and \( S(V) := \{ v \in V : \|v\| = 1 \} \) is the unit disc and sphere bundle of \( V \) respectively. If \( X \) is compact, \( X^V \) is the one-point compactification of \( V \).

Given vector bundles \( V \) and \( W \) over a paracompact space \( X \), \( V - W \) defines an element in the real K-theory \( KO(X) \) and an associated Thom spectrum \( X^{V-W} \). We will describe a construction of this spectrum. For the definition of the Thom spectrum of a negative bundle \( -W \) to make sense, one way is to embed \( W \) into an infinite dimensional trivial bundle and consider the complement of \( W \) in the trivial bundle. The following lemma provides the embedding.

**Lemma 2.1.1.** Suppose \( W \to X \) is a finite dimensional vector bundle over a paracompact space \( X \). Then there exists a vector bundle monomorphism \( \epsilon : W \to X \times \mathbb{R}^\infty \) such that for each \( x \in X \), there exists a neighborhood \( U \) and an integer \( n_U \) such that \( \epsilon(W|_U) \subset X \times \mathbb{R}^{n_U} \). The restrictions of any two such monomorphisms over a compact set of \( X \) are homotopic.

Let \( \epsilon : W \to X \) be a vector bundle monomorphism as described in the lemma. Since locally \( \epsilon \) factors through a finite subbundle of \( X \), there exists an open exhaustion
$X_1 \subset X_2 \subset \ldots$ of $X$ such that $\epsilon(W|X_n) \subset X_n \times \mathbb{R}^n$. Let $V_n := V|_{X_n}$, $W_n := W|_{X_n}$ and $W_n^\perp$ be the orthogonal complement of $\epsilon(W_n)$ in $X_n \times \mathbb{R}^n$. The $n$-th space of $X^{V-W}$ can be taken to be the Thom space $X_n^{V_n \oplus W_n^\perp}$. The structure map $\sigma_n$ is induced by the canonical vector bundle isomorphism

$$V_n \oplus W_n^\perp \oplus \mathbb{R} \xrightarrow{\cong} V_{n+1}|_{X_n} \oplus W_{n+1}^\perp|_{X_n},$$

where the summand $\mathbb{R}$ in the domain is resulted from taking orthogonal complement of $W_n$ in different ambient spaces $\mathbb{R}^n$ and $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$.

If $X'_1 \subset X'_2 \subset \ldots$ is another exhaustion of $X$ such that $\epsilon(W'|X'_n) \subset X'_n \times \mathbb{R}^n$, the spectra $(X^{V-W})'$ defined by the same procedure is equivalent to $X^{V-W}$. The inclusion maps

$$X_n \leftarrow X_n \cap X'_n \rightarrow X'_n$$

induce an equivalence between the two. Together with the last part of lemma 2.1.1, the homotopy type of $X^{V-W}$ is well-defined, independent of the choice of $\epsilon$ and exhaustion.

## 2.2 Pontryagin-Thom Construction

One of the main tools in the proof of the duality theorems in the introduction is the Pontryagin-Thom construction. In this chapter we will review the classical construction and discuss some of its variations.

Suppose $f : M \rightarrow N$ is a smooth map between compact manifolds. Let $\phi : M \rightarrow \mathbb{R}^n$ be an embedding of $M$ in a Euclidean space. Consider the embedding $e = (f, \phi) : M \rightarrow N \times \mathbb{R}^n$ and its normal bundle $E \rightarrow M$. By tubular neighborhood theorem, we may extend $e$ to an open embedding $\iota : E \rightarrow M \times \mathbb{R}^n$. 

\[
\begin{array}{ccc}
E & \xrightarrow{\iota} & N \times \mathbb{R}^n \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & N
\end{array}
\]
The data in the diagram determines a based map \( f^! : \Sigma^n N_+ \to M^E \), known as the Pontryagin-Thom collapse map, as follows: Consider \( \Sigma^n N_+ \) and \( M^E \) as \( \mathbb{N} \times \mathbb{R}^n \) and \( E \) with a basepoint infinity attached respectively. We define \( f^!(x) \) to be \( w \) if \( x = \iota(w) \in \mathbb{N} \times \mathbb{R}^n \subset \Sigma^n N_+ \) is in the image of \( \iota \) and to be the basepoint of \( M^E \) otherwise. Since the direct sum \( TM \oplus E \) is isomorphic to the vector bundle \( f^*TN \oplus e^n \) over \( M, M^E \) can be considered as the \( n \)-th level space of the spectrum \( Mf^*TN - TM \) and the map of based spaces \( \Sigma^n N_+ \to M^E \) induces a map of spectra

\[
\pi : \Sigma^\infty N_+ \to Mf^*TN - TM.
\]

While \( f^! \) depends on the embedding \( e \) and the tubular neighborhood \( \iota \), one can show that upon passing to stable homotopy type, \( f^! \) determines a well-defined homotopy class of maps of spectra, independent of the choices. For the case \( N \) is a point, this Pontryagin-Thom construction was used by René Thom to prove his celebrating Pontryagin-Thom theorem for the calculation of cobordism groups.

Our main goal in this section is to relax the assumptions in the classical case above to do Pontryagin-Thom construction in a more general setting. As seen from the classical case, two important ingredients in defining \( f^! \) are the embedding \( e : M \to \mathbb{N} \times \mathbb{R}^n \) and the open embedding \( \iota : E \to Y \times \mathbb{R}^n \). However, for non-compact spaces, the assumption that \( X \) can be embedded in \( Y \times \mathbb{R}^n \) is too restrictive. Instead, we embed \( e : X \to Y \times \mathbb{R}^\infty \), but require locally \( e \) factors through some finite dimensional subbundle \( Y \times \mathbb{R}^n \subset Y \times \mathbb{R}^\infty \). Also, we will need to have maps of spectra where the domain is the Thom spectrum of certain stable vector bundle but not of the form of a suspension spectrum.

This is our general Pontryagin-Thom construction.

**Proposition 2.2.1.** Let \( X \to Y \) be a continuous map between topological spaces and \( W_1, W_2 \) are vector bundles over \( Y \). Suppose we have the following data:

(a) a metric on \( W_1 \), an inclusion of vector bundles \( \epsilon : W_2 \to Y \times \mathbb{R}^\infty \) and an open exhaustion \( Y_1 \subset Y_2 \subset \ldots \) of \( Y \) such that the restrictions \( W_{i,n} := W_i|_{Y_n} \), \( i = 1, 2 \), satisfy \( \epsilon(W_{2,n}) \subset Y_n \times \mathbb{R}^n \). We will identify \( W_2 \) as a subbundle of \( Y \times \mathbb{R}^\infty \) through
\( \epsilon \) and denote by \( W_{2,n}^\perp \) the orthogonal complement of \( W_{2,n} \) in \( Y_n \times \mathbb{R}^n \). \( Q_n \to Y_n \) is taken to be the direct sum \( W_{1,n} \oplus W_{2,n}^\perp \);

(b) an open exhaustion \( X_1 \subset X_2 \subset \ldots \) of \( X \), vector bundles \( W'_n \to X_n \), equipped with metric, and isometric isomorphisms of vector bundles \( i_n : W'_n \oplus \mathbb{R} \cong W'_{n+1}|_{X_n} \) over \( X_n \);

(c) inclusions \( e : X \to W_1 \oplus W_2^\perp \) and \( \iota_n : W'_n \to W_1 \oplus W_2^\perp \) such that \( e \) lifts \( f \), the restriction of \( \iota_n \) to the zero section is \( e|_{X_n} \) and \( \iota_{n+1} \circ i_n|_{W'_n} = \iota_n \). Moreover, we require that \( \iota_n(D(W'_n)) \) to be a subset of \( \text{int}(D(W_1 \oplus W_2^\perp)) \) and \( \iota_m(D(W'_m)) \cap Q_n = \iota_n(D(W'_n)) \cap Q_n \) for all \( m \geq n \);

(d) The restriction \( \iota_n|_{\iota_n^{-1}(Q_n)} : \iota_n^{-1}(Q_n) \to Q_n \) is an open embedding and \( \iota_n(D(W'_n)) \cap Q_n \) is closed in \( Q_n \).

Then by Pontryagin-Thom construction, the above data determines a map of spectra

\[ f^! : Y^W \to X^{W'} \]

where \( W' \) is the stable vector bundle of \( X \) whose restriction to \( X_n \) is \( W'_n - \mathbb{R}^n \).

Remark 2.2.2. 1. Roughly speaking, the conditions in proposition 2.2.1 are listed according to their purposes. The first three conditions are used to define the spectra \( Y^W \), \( X^{W'} \) and the map \( f^! \) respectively. Condition (d) guarantees that each \( f^!_n \) is continuous.

2. Lemma 2.1.1 is useful for constructing the exhaustions satisfying the assumptions of the proposition 2.2.1 in the situation of paracompact spaces.

The following lemma about point set topology is needed in the proof of proposition 2.2.1.

Lemma 2.2.3. Let \( X \) be topological space. Suppose that \( A \) is an open subset of \( X \), \( B \) is a closed subset of \( X \) and \( A \subset B \). Then the inclusion \( B \hookrightarrow X \) induces a homeomorphism \( B/(B-A) \to X/(X-A) \).
Proof. Let \( q : B/(B - A) \to X/(X - A) \) be the map in the lemma. It is clear that \( q \) is a continuous bijection and so it suffices to show that \( q \) is a closed map. Let \( \pi_1 : B \to B/(B - A) \) and \( \pi_2 : X \to X/(X - A) \) be the quotient maps. Suppose \( Z \subset B/(B - A) \) is closed. Depending on whether \( Z \) contains \( [B - A] \) or not, \( \pi_1^{-1}(q(Z)) \) is either \( \pi_1^{-1}(Z) \cup (X - A) \) or \( \pi_1^{-1}(Z) \). Since \( \pi_1^{-1}(Z) \) is closed in \( B \) and both \( B, X - A \) are closed in \( X \), it shows \( \pi_2^{-1}(q(Z)) \), and hence \( q(Z) \), are closed. This proves the lemma.

We are ready to prove proposition 2.2.1.

Proof of proposition 2.2.1. First of all, we define the spectra in the map \( f' \). Our \( X^{W'} \) consists of the sequence of based spaces \( (X^{W'})_n = D(W'_n)/S(W'_n) \) and structure maps induced by \( i_n \). Similarly, the level spaces of \( Y^W \) are defined by \( (Y^W)_n = D(Q_n)/S(Q_n) \) with structure maps defined by the canonical isomorphisms

\[
Q_n \oplus \mathbb{R} \cong W_{1,n} \oplus W^\perp_{2,n} \oplus \mathbb{R} \cong W_{1,n+1} \mid x_n \oplus W^\perp_{2,n+1} \mid x_n \cong Q_{n+1} \mid x_n.
\]

Next, we define based maps \( (f')_n : Y_n^{Q_n} \to X_n^{W'_n} \) by Pontryagin-Thom construction. It is given by the composition

\[
Y_n^{Q_n} = D(Q_n)/S(Q_n) \\
\to \frac{D(Q_n)}{D(Q_n) - \iota_n(\text{int}(D(W'_n))) \cap Q_n} \\
\cong \frac{\iota_n(D(W'_n)) \cap Q_n - \iota_n(\text{int}(D(W'_n))) \cap Q_n}{\iota_n(D(W'_n)) \cap \iota_n^{-1}(Q_n)} \\
\cong \frac{D(W'_n) \cap \iota_n^{-1}(Q_n) - \text{int}(D(W'_n)) \cap \iota_n^{-1}(Q_n)}{S(W'_n) \cap \iota_n^{-1}(Q_n)} \\
= D(W'_n)/S(W'_n) \\
= X_n^{W'_n}
\]

where the first arrow is the quotient map, and the two homeomorphisms are given by lemma 2.2.3 and our assumption that \( \iota_n|_{\iota_n^{-1}(Q_n)} : \iota_n^{-1}(Q_n) \to Q_n \) is an open embedding.
The last arrow is induced by inclusions.

By \( i_{n+1} \circ i_n|_{W_n} = i_n \), the maps \((f^i)_n\) commute with the structure maps of \(Y^W\) and \(X^{W'}\). Hence, \(f^i\) is well-defined.

In the next two sections, we will see examples of Pontryagin-Thom constructions for diagonal maps of manifolds and for fiber bundles over paracompact spaces, which are the main ingredients for the constructions of the dualities (1.1) and (1.4).

### 2.3 Spanier-Whitehead Duality for Manifolds

Let \( M \) be a closed smooth manifold and \( TM \rightarrow M \) be its tangent bundle. The Spanier-Whitehead duality states that the map \( \Sigma^\infty M_+ \simeq F(M^{-TM}, S) \) in (1.1) is a weak equivalence. Here \( F(M^{-TM}, S) \) is the function spectrum from the Thom spectrum \( M^{-TM} \) to the sphere spectrum \( S \). This weak equivalence can be regarded as a spectrum version of Poincaré duality. When \( M \) is oriented, Poincaré duality follows by taking homology and applying Thom isomorphism. We will construct this map using the diagonal map \( \Delta : M \rightarrow M \times M \).

By Whitney embedding theorem, there is an embedding \( \phi : M \rightarrow \mathbb{R}^N \) of \( M \) into a Euclidean space and let \( E \rightarrow M \) be its normal bundle. Consider the composition

\[
M \xrightarrow{\Delta} M \times M \xrightarrow{Id \times s} M \times E
\]

where \( s : M \rightarrow E \) is the zero section. The normal bundle of this composition is the trivial \( N \)-dimensional vector bundle \( M \times \mathbb{R}^N \). Applying Pontryagin-Thom construction to the tubular neighborhood

\[
i : M \times \mathbb{R}^N \rightarrow M \times E
\]

(2.1)
gives a based map

\[
M_+ \wedge M^E \rightarrow (M \times \mathbb{R}^N)_+ = S^N \wedge M_+
\]

Since \( E \oplus TM \cong M \times \mathbb{R}^N \), the domain of this map is the \( N \)-th space of the spectrum
Σ^∞ M_+ ∧ M^{-TM}. Hence, we obtain a map of spectra

$$\Sigma^\infty M_+ \wedge M^{-TM} \to \Sigma^\infty M_+. \quad (2.2)$$

By further composing with the suspension of the collapse map of M to a point and taking adjoint, we obtain

$$\Sigma^\infty M_+ \wedge M^{-TM} \to \Sigma^\infty S^0 = S$$

and the equivalence (1.1).

Alternatively, We may obtain the map (2.2) by applying proposition 2.2.1 to \( f = \Delta : M \to M \times M \) with \( W_1 = 0 \) and \( W_2 = \pi_2^* TM \), where \( \pi_2 : M \times M \to M \) is the projection map of the second factor. We can take the required data in the assumptions of proposition 2.2.1 as follows:

The differential \( d\phi \) of the embedding \( \phi : M \to \mathbb{R}^N \) induces a map \( TM \to \phi^*(T\mathbb{R}^N) = M \times \mathbb{R}^N \) and \( \epsilon \) can be taken to be the product of \( \text{Id}_M \) with this induced map. Take \( e \) be the lifting of \( \Delta \) sending \( M \) to the zero section of the bundle \( \epsilon(\pi_2^* TM)^\perp \cong \pi_2^* E \) over \( M \times M \). As noted before, its normal bundle is the trivial \( \mathbb{R}^N \)-bundle and \( \iota \) in (2.1) defines a tubular neighborhood. This set of data can be summarized by the diagram

$$\begin{array}{c}
M \times \mathbb{R}^N \xrightarrow{\iota} \mathbb{R}^N \oplus \mathbb{R}^k \\
\downarrow s \quad \downarrow \quad \downarrow e \\
M \xrightarrow{\Delta} M \times M
\end{array}$$

(2.3)

Take \( X_{N+k} = M, Y_{N+k} = M \times M \) and \( W'_{N+k} = X \times \mathbb{R}^{N+k} = \mathbb{R}^N \oplus \mathbb{R}^k \) for \( k \geq 0 \) and empty if \( k < 0 \). The map \( i_{N+k} \) is the inclusions of the first \( N+k \) summands and \( i_{N+k} = \iota \oplus \text{Id}_{\mathbb{R}^k} \). Finally, it is easy to choose metrics on \( W' \) so that all the assumptions in proposition 2.2.1 are satisfied. Applying the proposition gives

$$\Delta^1 : \Sigma^\infty M_+ \wedge M^{-TM} \cong (M \times M)^{-\pi_2^*(TM)} \to M^{W'} = \Sigma^\infty M_+$$

which is the same as (2.2).
It is also possible to obtain Spanier-Whitehead duality for open manifolds or compact manifolds with boundary. For the first case, we need to consider the one-point compactification. Let $X^c$ denote the one point compactification of a Hausdorff topological space $X$. The continuous bijection $X_+ \to X^c$, which sends the disjoint basepoint to infinity and restricts to the identity on $X$, is a homeomorphism if and only if $X$ is compact.

Suppose $M$ be an open $n$-dimensional manifold. Note that Whitney embedding theorem also holds for non-compact manifolds: Any $n$-dimensional manifold can be embedded in $\mathbb{R}^{2n}$. To obtain a Pontryagin-Thom map for $\Delta$, we can perform the same construction as in the case of closed manifolds above. Proposition 2.2.1 gives us the map (2.2) again, but it does not lead to a duality after composing with the collapse map $\Sigma^\infty M_+ \to S$. Instead, one can show that, essentially because $M$ is locally compact, (2.2) factors through $\Sigma^\infty M^c \wedge M^{-TM}$

$$
\begin{array}{c}
\Sigma^\infty M_+ \wedge M^{-TM} \xrightarrow{\Delta^!} \Sigma^\infty M_+ \\
\downarrow \quad \exists \\
\Sigma^\infty M^c \wedge M^{-TM}
\end{array}
$$

The new map is also called $\Delta^!$

$$
\Delta^!: \Sigma^\infty M^c \wedge M^{-TM} \to \Sigma^\infty M_+
$$

Spanier-Whitehead duality for open manifolds states that the composition of $\Delta^!$ with the collapse map gives us a duality map for $M$, and its adjoint

$$
\Sigma^\infty M^c \simeq F(M^{-TM}, S)
$$

is an equivalence.

If $M$ is oriented, taking (co)homology and applying Thom isomorphism give the Poincaré duality for open manifolds

$$
\hat{H}^*(M^c; \mathbb{Z}) \cong H_{m-*}(M; \mathbb{Z})
$$
and
\[ H^*(M; \mathbb{Z}) \cong \tilde{H}_{m-*}(M^c; \mathbb{Z}). \]
The left hand side of the first isomorphism is canonical isomorphic to \( H^*_c(M, \mathbb{Z}) \), the cohomology of \( M \) with compact support.

Our final example is the Spanier-Whitehead duality for compact manifolds with boundary. We can construct a diagram analogous to (2.3)

\[
\begin{array}{ccc}
\text{int}(M) \times \mathbb{R}^N & \xrightarrow{\iota} & \pi_2^* E \\
\downarrow s & & \downarrow \\
\text{int}(M) & \xrightarrow{\Delta} & M \times \text{int}(M)
\end{array}
\]

with extra requirements that \( \epsilon(\text{int}(M) \times \mathbb{R}^N) \subset \text{int}(M) \times \text{int}(M) \) and \( \pi_2 \circ \iota \) is equal to the projection \( \text{int}(M) \times \mathbb{R}^N \rightarrow \text{int}(M) \).

From this set of data, we can do Pontryagin-Thom construction by a choice of exhaustions, vector bundles and maps as in the case of closed manifold to define a Pontryagin-Thom map of \( \Delta : \text{int}(M) \rightarrow M \times \text{int}(M) \). By our choice of the tubular neighborhood \( \iota \), the resulting map factors through \( \Sigma^\infty M_+ \wedge \text{int}(M)^{-TM} \rightarrow \Sigma^\infty (M/\partial M) \wedge \text{int}(M)^{-TM} \) to give

\[ \Delta^1 : \Sigma^\infty (M/\partial M) \wedge \text{int}(M)^{-TM} \rightarrow \Sigma^\infty \text{int}(M)_+, \]

By collar neighborhood theorem, we can perturb the identity map of \( M \) near the boundary to embed \( M \) to \( \text{int}(M) \) by a homotopy equivalence. Such an embedding allows us to replace \( \text{int}(M)_+ \) by \( M_+ \) in \( \Delta^1 \) above. After composing with collapse map and taking adjoint, we get

\[ \Sigma^\infty M/\partial M \simeq F(M^{-TM}, \mathbb{S}), \]

which is the Spanier-Whitehead duality for compact manifold with boundary. If \( M \) is oriented, Poincaré duality for manifolds with boundary follows by taking homology and applying Thom isomorphism.
2.4 Transfer map for Fiber Bundle

In this section we will consider transfer map for fiber bundle with a compact manifold as fiber. Let $G$ be a compact Lie group and $M$ be a compact manifold with a smooth $G$-action on it. Let $P \to Y$ be a principal $G$-bundle over a paracompact topological space $Y$. Then the the collapse map $M \to \ast$ induces a fiber bundle $f : X = P \times_G M \to Y = P \times_G \ast$ with fiber $M$ and structure group $G$.

Let $e' : M \to V'$ be an equivariant embedding of $M$ into some finite dimensional $G$-representation $V'$ with equivariant normal bundle $E' \to M$ and a tubular neighborhood $\iota' : E' \to V'$. By taking $P \times_G -$ to the fiberwise data above, we obtain vector bundles $E := P \times_G E' \to X = P \times_G M$, $V := P \times_G V' \to Y = P \times_G \ast$ and inclusions $e := P \times_G e'$ and $\iota := P \times_G \iota'$. The maps fit into the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\iota} & V \\
\uparrow{\text{zero}} & \nearrow{\epsilon} & \downarrow{f} \\
X & \xrightarrow{\iota} & Y
\end{array}
$$

By lemma 2.1.1, there exists a vector bundle monomorphism $\epsilon : V \to Y \times \mathbb{R}^\infty$ and an open exhaustion $Y_1 \subset Y_2 \subset \ldots$ of $Y$ such that $\epsilon(V|_{Y_n}) \subset Y_n \times \mathbb{R}^n$. We take $X_n = f^{-1}(Y_n)$ to give an open exhaustion of $X$.

Let $W'_n \to X_n$ be

$$E|_{X_n} \oplus f^*(V|_{Y_n}),$$

with the orthogonal complement taken in the ambient space $Y_n \times \mathbb{R}^n$. The inclusion of $V|_{Y_n}$ into the two ambient spaces $Y_n \times \mathbb{R}^n \subset Y_n \times \mathbb{R}^{n+1}$ induces an isomorphism $i_n : W'_n \oplus \mathbb{R} \to W'_{n+1}|_{X_n}$. These $W'_n$ define the virtual vector bundle $E - f^*(V)$ over $X$. The map $\iota_n : W'_n \to Y_n \times \mathbb{R}^n$, defined by the direct sum of $\iota|_{X_n}$ on the first summand and identity map on the second summand in the definition of $W'_n$, gives a tubular neighborhood of $\epsilon \circ e|_{X_n} : X_n \to V \to Y \times \mathbb{R}^n$. By choosing a suitable metric on $W'_n$, proposition 2.2.1 gives

$$f^! : \Sigma^\infty Y_+ \to X^E - f^*(V)$$
Note that since $E' \oplus TM$ is isomorphic to the trivial bundle $M \times V'$ over $M$, $E - f^*(V)$ is stably equivalent to $-P \times_G TM$, which is the negative of the vertical tangent bundle $T_{\text{vert}}X$ of $X$ by regarding $X$ as a fiber bundle over $Y$. Hence, $f^!$, which is called the dimension-shifted transfer map $\text{Tr}$ of the fiber bundle, can be written as

$$\text{Tr} : \Sigma^\infty Y_+ \to X^{-T_{\text{vert}}X}.$$  

For the case that $X$ is a finite $G$-set of cardinality $n$, $T_{\text{vert}}X$ is the zero bundle and the composition $f \circ \text{Tr} : \Sigma^\infty Y_+ \to \Sigma^\infty Y_+$ induces multiplication by $n$ on ordinary homology.

## 2.5 Morava $K$-theory

For a fixed prime $p$, Morava $K$-theory, denoted by $K(n)$, $n \geq 0$, is a sequence of multiplicative cohomology theories. The prime $p$ is implicit in the notation. $K(n)$ is derived from complex cobordism. We refer the readers to [12], [19] for the origin and construction of the theory. Here are some of its important properties.

(a) $K(0)^*(-) := H^*(-, \mathbb{Q})$, the rational cohomology theory;

(b) For $n \geq 1$, $K(n)$ is $2(p^n - 1)$-periodic with coefficient ring $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$, where the degree of $v_n$ is equal to $2(p^n - 1)$. $K(n)^*$ is a graded field in the sense that any graded module over it is free;

(c) $K(n)$ satisfies the Künneth formula, i.e. for spaces $X$ and $Y$,

$$K(n)^*(X \times Y) \cong K(n)^*(X) \otimes_{K(n)^*} K(n)^*(Y);$$

(d) $K(n)$ is complex oriented, i.e. every complex vector bundle has a Thom class.

Generalized Chern classes can be defined as a result. The $p$-series of the formal group law associated to $K(n)$ is given by $[p]x = v_n x^{p^n}$;

(e) For $p > 2$, as pointed out in the introduction, $K(n)$-orientability of a vector bundles is equivalent to ordinary orientability. It implies that if $V \to X$ is a
r-dimensional oriented vector bundle, then

\[ K(n)^*(X) \cong \tilde{K}(n)^{*+r}(X^V). \]

For \( p = 2 \), the question is more subtle. See [17] for details.

On one hand, these properties, together with techniques like spectral sequence, make \( K(n)^*(BG) \) rather computable for a finite group \( G \). For instance, by the simple \( p \)-series of \( K(n) \) and the Künneth formula, \( K(n)^*(BG) \) can be calculated easily for all finite abelian groups. On the other hand, by the results of [16] and [7], \( K(n)^*(BG) \) is a self-dual algebra of finite rank over \( K(n)^* \) and it gives extra structures to both these objects and morphisms between them. All these make Morava \( K \)-theory of finite groups interesting from both theoretical and computational points of view. It suggests us to study Deligne-Mumford stacks, which can be thought as manifolds with singularities of finite order, using Morava \( K \)-theory. The general theory of stacks will be introduced in the next chapter.

### 2.6 Equivariant Stable Homotopy Theory

We will recall basic definitions and theorems in equivariant stable homotopy theory from [13]. For simplicity and for our application, our focus will be on finite group cases. However, we also want to point out that all the definitions and results in this section can be extended to compact Lie groups with minor modifications.

Let \( G \) be a finite group. We say a real \( G \)-invariant inner product space \( U \) is a \( G \)-universe if it contains countably infinite copies of the trivial representation and any other irreducible \( G \)-representations it contains. \( U \) is called complete if it contains the regular representation (hence countably infinite copies of every irreducible \( G \)-representation). For finite dimensional representations \( V \subset W \) in \( U \), write \( W - V = V^\perp \cap W \).

From now on \( U \) will be assumed to be a complete \( G \)-universe. A \( G \)-spectrum \( D \) (indexed over \( U \)) consists of a collection of pointed \( G \)-spaces \( \{D(V)\}_V \) for finite dimensional representations \( V \in U \) and based \( G \)-maps \( \sigma_{VW} : \Sigma^{W-V}D(V) \to D(W) \).
for pairs $V \subset W$ of such representations such that the adjoints

$$\tilde{\sigma}_{VW} : D(V) \xrightarrow{\cong} \Omega^{W-V} D(W)$$

are homeomorphisms. Here the $G$-action on $\Sigma^{W-V} D(V)$ and $\Omega^{W-V} D(W)$ is given by diagonal action and conjugation respectively. The basepoint of each $D(V)$ is assumed to be $G$-fixed. The structure maps are required to satisfy $\sigma_{VV} = Id_{D(V)}$ and a compatibility condition for each triple $V \subset W \subset Z$ of finite dimensional representations in $U$.

A morphism $f : D \to E$ between $G$-spectra consists a collection of based maps $\{f_V : D(V) \to E(V)\}_V$ which commute with the structure maps $\sigma_{VW}$. The category of $G$-spectra is denote by $G\mathcal{S}U$.

We will also work with $G$-spectra indexed on the smaller universe $U^N$, the $N$-fixed points of $U$ for some normal subgroup $N \subset G$. The category of $G$-spectra indexed on $U^N$ is denoted by $G\mathcal{S}U^N$. Note that if $N = G$ then $U^G \cong \mathbb{R}^\infty$ is $G$-trivial. $G$-spectra indexed on this trivial universe $U^G$ are called naive $G$-spectra.

Let $i : U^N \to U$ be the inclusion. Evidently there is a functor $i^* : G\mathcal{S}U \to G\mathcal{S}U^N$ which forgets spaces indexed on representations $V$ not contained in $U^N$. It has a left adjoint $i_* : G\mathcal{S}U^N \to G\mathcal{S}U$. Also, there is a functor $\varepsilon^* : J\mathcal{S}U^N \to G\mathcal{S}U^N$, where $J = G/N$, which assigns $G$-action to $D \in J\mathcal{S}U^N$ by pullback of quotient map $\varepsilon : G \to J$. This functor has both left adjoint and right adjoint, namely taking $N$-orbit and $N$-fixed point respectively.

$$G\mathcal{S}U^N \xrightarrow{N\text{-orbit}} J\mathcal{S}U^N \xrightarrow{\varepsilon^*} G\mathcal{S}U^N \xrightarrow{i_*} G\mathcal{S}U$$

To simplify notations, we write $\varepsilon^\sharp = i_*\varepsilon^*$ and $D^N$ for $(i^*D)^N$ when $D \in G\mathcal{S}U$.

Analogous to $G$-CW complexes, $G$-CW spectra are spectra which are built from cell spectra $(G/H)_+ \wedge D^n$, where $H \subset G$ is a subgroup and $n \in \mathbb{Z}$. There is also a notion of homotopic maps for spectra, and the set of homotopy classes of maps from $D$ to $E$ is denoted by $[D, E]$. The following theorem [13, II, theorem 2.8(i)] relates homotopy classes of maps of spectra indexed on different universes.
**Theorem 2.6.1.** Suppose $N$ is a normal subgroup of $G$ and $i : U^N \to U$ is the inclusion. Let $D, E \in G \mathcal{S} U^N$ with $D$ a $N$-free $G$-CW spectrum. Then $i_*$ induces a bijection

$$[D, E]_{G \mathcal{S} U^N} \cong [i_* D, i_* E]_{G \mathcal{S} U}.$$  

Another important theorem for us is the following special case of [13, II, theorem 7.1].

**Theorem 2.6.2.** Let $D \in G \mathcal{S} U^G$ be $G$-free spectrum. Then there is a transfer map

$$\tau : \epsilon^b(D/G) \to i_* D$$

whose adjoint is an equivalence of nonequivariant spectra in $G \mathcal{S} U^G$

$$D/G \cong (i_* D)^G.$$  

We will not prove this theorem but will explain the construction of $\tau$ for later use. Although $\tau$ is a map of $G$-spectra, its construction involves $G^2$-spectra.

Let $U'$ be a complete $G^2$-universe. Then $U'^G \times 1$ can be regarded as a complete $G$-universe $U$ through the identification $G \cong 1 \times G \cong G^2 / (G \times 1)$ induced by the quotient map. Let $i : U^G \to U$ and $j : U = U'^G \times 1 \to U'$ be the inclusion of universe. The projections $\pi_0, \pi_1 : G^2 \to G$ of the first and second factor pull back $G$-spaces or spectra to ones with $G^2$-action.

Let $G^2$ acts on $G$ by $(g_1, g_2)g = g g_2 g_1^{-1}$. An $G^2$-embedding of $G$ into a $G^2$-representation $V$ induces a map $t : \mathbb{S} \to \Sigma^\infty G_+$ by Pontryagin-Thom construction and desuspension by $V$. Since $D$ is $G$-free, the spectrum $j_* i_* \pi_0^* D \in G^2 \mathcal{S} U'$ is $(G \times 1)$-free. By theorem 2.6.1, there is a bijection

$$[i_* \pi_0^* D, i_* \pi_0^* D \land G_+]_{G^2 \mathcal{S} U} \cong [j_* i_* \pi_0^* D, j_* i_* \pi_0^* D \land G_+]_{G^2 \mathcal{S} U'}.$$  

Using the canonical identifications

$$\epsilon^b(D/G) \cong (i_* \pi_0^* D) / (G \times 1)$$  

and

$$i_* D \cong (i_* \pi_0^* D \land G_+) / (G \times 1),$$
\( \tau \) is the \((G \times 1)\)-orbit of the preimage of

\[
1 \wedge t : j_* i_* \pi_0^* D = j_* i_* \pi_0^* D \wedge S \to j_* i_* \pi_0^* D \wedge G_+
\]

under the bijection above.

We would like to end this chapter by relating the two categories \( \mathcal{S} \) and \( \mathcal{S}_{\mathbb{R}\infty} \) of non-equivariant spectra defined in section 2.1 and this section. There is a functor \( L : \mathcal{S} \to \mathcal{S}_{\mathbb{R}\infty} \) such that for a spectrum \( D \in \mathcal{S} \),

\[
LD(\mathbb{R}^n) = \operatorname{colim}_{k \geq 0} \Omega^k D_{n+k},
\]

where the colimit is taken with respect to the adjoints \( D_i \to \Omega D_{i+1} \) of the structure maps of \( D \). As pointed out in [13], the functor \( L \) induces a bijection

\[
\mathcal{S}(D, D') \cong \mathcal{S}_{\mathbb{R}\infty}(LD, LD').
\]

Since all the spectra in \( \mathcal{S}_{\mathbb{R}\infty} \) which will appear in this thesis are of the form \( LD \), we are free to switch between the two definitions of non-equivariant spectra when considering questions of homotopy classes of maps.
Chapter 3

Differentiable Stacks

3.1 Definitions and Basics

In this section, we will give a brief introduction of the theory of stacks. A more detailed discussion of this subject can be found in [10].

The idea of stacks was first introduced by Deligne and Mumford to study moduli problem in algebraic geometry. Their motivation was to represent a moduli problem by an object in an appropriate category. For instance, for a CW-complex $X$, isomorphism classes of complex line bundle over $X$ are in one-one correspondence with homotopy classes of maps $[X, \mathbb{C}P^\infty]$. In other words, the moduli space of complex line bundles is represented by $\mathbb{C}P^\infty$ in the homotopy category of CW complexes.

Let $\textbf{Diff}$ be the category of smooth manifolds and smooth maps and $\textbf{Groupoid}$ be the category of groupoids and natural transformation. Roughly speaking, a stack is a contravariant pseudo-functor $\mathcal{X}: \textbf{Diff} \to \textbf{Groupoid}$ which satisfies sheaf-like properties so that we can glue compatible objects and morphisms. More precisely,

**Definition 3.1.1 (Stack).** A stack $\mathcal{X}$ on the site $\textbf{Diff}$ consists of

(a) a groupoid $\mathcal{X}(M)$ for each smooth manifold $M$;

(b) a functor $f^*: \mathcal{X}(M_2) \to \mathcal{X}(M_1)$ for each smooth map $f$;

(c) a natural transformation $\Phi_{f,g}: f^* \circ g^* \cong (g \circ f)^*$ for any smooth maps $f: M_1 \to M_2$ and $g: M_2 \to M_3$.
such that for an open cover $U_i$ of $M$,

(i) given objects $a_i \in \mathcal{X}(U_i)$ and isomorphisms $\psi_{ij} : a_i|_{U_i \cap U_j} \rightarrow a_j|_{U_i \cap U_j}$ which satisfy the cocycle condition $\psi_{jk} \circ \psi_{ij} = \psi_{ik}|_{U_i \cap U_j \cap U_k}$, there exist an object $a \in \mathcal{X}(M)$ and isomorphisms $\psi_i : a|_{U_i} \rightarrow a_i$ such that $\psi_{ij} = \psi_j \circ \psi_i^{-1}$;

(ii) given objects $a, b \in \mathcal{X}(M)$ and morphisms $\psi_i : a|_{U_i} \rightarrow b|_{U_i}$ which satisfy the compatibility conditions $\psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$ on intersection, there exists a unique morphism $\psi : a \rightarrow b$ such that $\psi|_{U_i} = \psi_i$.

Next is the definition of morphisms of stacks.

**Definition 3.1.2** (Morphisms of Stacks). (a) A morphism of stacks $u : \mathcal{X} \rightarrow \mathcal{Y}$ consists a collection of functors $u_M : \mathcal{X}(M) \rightarrow \mathcal{Y}(M)$ for each $M$ and natural transformations $u_f : u_M \circ f^* \Rightarrow f^* \circ u_N$ for each $f : M \rightarrow N$ such that $u_{gf} = u_f \circ u_g$.

(b) Given morphisms $u, v : \mathcal{X} \rightarrow \mathcal{Y}$, a 2-morphisms $\phi : u \rightarrow v$ is a collection of natural transformations $\phi_M : u_M \rightarrow v_M$ such that $v_f \circ \phi_M = \phi_N \circ u_f$.

From the definitions above, stacks on the site $\operatorname{Diff}$ form a 2-category. Note that all the 2-morphisms are invertible. Two stacks $\mathcal{X}, \mathcal{Y}$ are equivalent if there exist morphisms $u : \mathcal{X} \rightarrow \mathcal{Y}, v : \mathcal{Y} \rightarrow \mathcal{X}$ such that $v \circ u$ is 2-isomorphic to $1_{\mathcal{X}}$ and $u \circ v$ is 2-isomorphic to $1_{\mathcal{Y}}$.

By Yoneda embedding, a smooth manifold defines a stack.

**Example 3.1.3** (Manifold). Every smooth manifold $M$ defines a stack, which is also denoted by $M$, as follows. For any smooth manifold $N$, $M(N)$ is the set of smooth maps from $N$ to $M$. Hence, the only morphisms in the groupoid $M(N)$ are identity morphisms.

There is a version of Yoneda lemma in the 2-category of stacks.

**Lemma 3.1.4** (Yoneda Lemma). For any smooth manifold $M$ and stack $\mathcal{X}$, there is a canonical equivalence of categories between

$$\mathcal{X}(M) \simeq \operatorname{Mor}_{\operatorname{Stack}}(M, \mathcal{X}).$$
In particular, the category of stacks contains $\text{Diff}$ as a full subcategory.

**Example 3.1.5 (Quotient Stack).** Let $G$ be a Lie group and $M$ be a smooth $G$-manifold. The quotient stack $[M/G]$ is defined as follows:

(a) Given a smooth manifold $N$, $[M/G](N)$ is the groupoid whose objects are triples $(P,p,f)$, where $p : P \to N$ is a principal $G$-bundle and $f : P \to M$ is a $G$-map. We may represent such a triple by the diagram

$$
\begin{array}{ccc}
P & \overset{f}{\longrightarrow} & M \\
\downarrow{p} & & \\
N
\end{array}
$$

A morphism between objects $(P_i, p_i, f_i), i = 1, 2$ in the groupoid $[M/G](N)$ is given by a bundle map $q : P_1 \to P_2$ such that $f_2 \circ q = f_1$.

(b) Given a smooth map $g : N_1 \to N_2$, $g^* : [M/G](N_2) \to [M/G](N_1)$ sends an object $(P, p, f)$ of $[M/G](N_2)$ to $(g^*P, g^*p, f \circ p^*g)$ and a morphism $q : P_1 \to P_2$ to $g^*q : g^*P_1 \to g^*P_2$.

It is straightforward to check the data above satisfy the axioms of stacks.

**Remark 3.1.6.** If the $G$ action is free and properly discontinuous, $[M/G]$ is equivalent to the stack defined by the quotient manifold $M/G$.

**Example 3.1.7.** For $M = *$ is a point, $[*/G](N)$ is simply the groupoid of principal $G$-bundles over $M$ and bundle maps between them.

An equivariant map between $G$-manifolds induces an obvious morphism between the corresponding quotient stacks. More generally, given $G_i$-manifolds $M_i, i = 1, 2$ and a group homomorphism $\phi : G_1 \to G_2$, a $\phi$-map between $M_i$ is a smooth map $h : M_1 \to M_2$ such that for any $a \in G$ and $x \in M_i$, $h(ax) = \phi(a)h(x)$. It induces a morphism $h_* : [M_1/G_1] \to [M_2/G_2]$ which sends an object $(P, p, f)$ of $[M_1/G_1](N)$ to
the one in \([M_2/G_2](N)\) represented by the diagram

\[
\begin{array}{c}
G_2 \times_\phi P \\
\downarrow \\
N
\end{array} \xrightarrow{\phi \times f} \begin{array}{c}
G_2 \times_\phi M_1 \\
\downarrow \\
M_2
\end{array}.
\]

Here the second horizontal map sends \([b, x]\) to \(bh(x)\). \(h_\ast\) sends a morphism \(q : P_1 \to P_2\) between the objects \((P_j, p_j, f_j)\) in \([M_1/G_1](N)\) to the morphism \(G_2 \times_\phi q : G_2 \times_\phi P_1 \to G_2 \times_\phi P_2\) in \([M_2/G_2](N)\).

Next we define groupoid stacks. Recall that a Lie groupoid \(\Gamma = [\Gamma_1 \rightrightarrows \Gamma_0]\) is a groupoid where both the classes of objects \(\Gamma_0\) and morphisms \(\Gamma_1\) are smooth manifolds, source and target maps \(s, t : \Gamma_1 \to \Gamma_0\) are submersions and compositions of morphisms \(m : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1\), identity-assigning map \(e : \Gamma_0 \to \Gamma_1\) and inverse \(i : \Gamma_1 \to \Gamma_1\) are smooth. Here \(\Gamma_1 \times_{\Gamma_0} \Gamma_1 := \{(f, g) : t(f) = s(g)\}\) is a smooth manifold because \(s, t\) are submersions.

Given a Lie groupoid \(\Gamma\), a principal \(\Gamma\)-bundle \(P\) over a space \(X\) consists of maps \(p : P \to X\) and \(f : P \to \Gamma_0\) together with an action

\[
\mu : P \times_{\Gamma_0} \Gamma_1 = \{(y, g) : f(y) = s(g)\} \to P
\]

such that

(a) the action is compatible with \(p, f\) and \(m\), i.e. the diagrams

\[
\begin{array}{c}
P \times_{\Gamma_0} \Gamma_1 \\
\downarrow \\
X \times \Gamma_0
\end{array} \xrightarrow{\mu} \begin{array}{c}
P \\
\downarrow \\
X \times \Gamma_0
\end{array}
\]

and

\[
\begin{array}{c}
P \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Gamma_1 \\
\downarrow_{1 \times m} \\
P \times_{\Gamma_0} \Gamma_1
\end{array} \xrightarrow{\mu} \begin{array}{c}
P \times_{\Gamma_0} \Gamma_1 \\
\downarrow \\
P
\end{array}
\]
are commutative;

(b) for any \( x \in X \), there exists a neighborhood \( U \) of \( x \) and a pullback diagram

\[
\begin{array}{ccc}
P|_U & \xrightarrow{g} & \Gamma_1 \\
\downarrow \rho|_U & & \downarrow s \\
U & \longrightarrow & \Gamma_0
\end{array}
\]

such that \( f = t \circ g \) and \( m \circ (g \times 1) = g \circ \mu : P|_U \times_{\Gamma_0} \Gamma_1 \to \Gamma_1 \).

A morphism between principal \( \Gamma \)-bundles \( P_1, P_2 \) over \( X \) is an equivariant map \( P_1 \to P_2 \) which commutes with structure maps of \( P_1, P_2 \). It is clear that all such morphisms are invertible. The pullback of a principal \( \Gamma \)-bundle can also be defined in the obvious way. These properties allow the following definition.

**Definition 3.1.8 (Groupoid Stack).** A Lie groupoid \( \Gamma \) defines a stack \([\Gamma_0/\Gamma_1]\) by associating to any smooth manifold \( M \) the groupoid \([\Gamma_0/\Gamma_1](M)\) of principal \( \Gamma \)-bundles over \( X \). For smooth maps \( f : M_1 \to M_2, g : M_2 \to M_3 \), the functor \( f^* \) is defined by pullback and the natural transformation \( \Phi_{f,g} : f^* \circ g^* \cong (g \circ f)^* \) is defined by the canonical principal \( \Gamma \)-bundle isomorphism between \( f^*(g^*P) \) and \( (g \circ f)^*(P) \). The local nature of \( \Gamma \)-bundles ensures that \([\Gamma_0/\Gamma_1]\) satisfies the sheaf-like axioms in the definition of stacks.

Similar to quotient stacks, a morphism of groupoid \( \Gamma \to \Gamma' \) induces a morphisms between \([\Gamma_0/\Gamma_1] \to [\Gamma'_0/\Gamma'_1]\). Indeed, it is clear that the association \( \Gamma \mapsto [\Gamma_0/\Gamma_1] \) defines a 2-functor between the categories of Lie groupoids and stacks.

**Example 3.1.9.** For a Lie groupoid \( \Gamma \), its source and target maps define a principal \( \Gamma \)-bundle \( \Gamma_1 \to \Gamma_0 \) by \( p = s \) and \( f = t \). The identity-assigning map \( e : \Gamma_0 \to \Gamma_1 \) gives a canonical section to this bundle. Any principal \( \Gamma \)-bundle is locally a pullback of \( s : \Gamma_1 \to \Gamma_0 \).

**Example 3.1.10.** Suppose that \( G \) is a Lie group acting on a smooth manifold \( M \). Let \( \Gamma \) be the Lie groupoid defined by \( \Gamma_0 = M, \Gamma_1 = G \times M \) and \( s, t : \Gamma_1 \to \Gamma_0 \) be the projection and action map respectively. Then a principal \( \Gamma \)-bundle over \( X \) is a
principal $G$-bundle $P \to X$ together with a $G$-map $P \to M$ as described in example 3.1.5. An important related example is the $\Gamma$-bundle $EG \times M \to EG \times_G M$ with $f : EG \times M \to M$ be the projection. This is called the universal principal $\Gamma$-bundle.

While the definition of stacks is rather abstract and algebraic, Lie groupoids are much more concrete geometric objects. One may ask if the geometric structures on Lie groupoids can be passed to the stacks they define. It is indeed possible and leads to the extension of many definitions and constructions in differential topology to this subcategory of stacks defined by Lie groupoids. In order to give a proper definition for this interesting class of stacks, we should start with the notion of 2-fiber product in the 2-category of stacks.

**Definition 3.1.11.** Given a diagram of morphisms of stacks

\[
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\
\downarrow & & \downarrow ^g \\
\mathfrak{Z} & \to & \mathfrak{Y}
\end{array}
\]

the fiber product $\mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}$ is defined as follows. For a smooth manifold $M$,

\[
\mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}(M) := \{(p, q, \phi) \mid p : M \to \mathfrak{X}, q : M \to \mathfrak{Y}, \phi : f \circ p \Rightarrow g \circ q\}.
\]

A morphism $(p_1, q_1, \phi_1) \to (p_2, q_2, \phi_2)$ is given by a pair of morphisms

\[
(\psi_{p_1,p_2} : p_1 \to p_2, \psi_{q_1,q_2} : q_1 \to q_2)
\]

such that

\[
\phi_2 \circ f(\psi_{p_1,p_2}) = g(\psi_{q_1,q_2}) \circ \phi_1.
\]

**Remark 3.1.12.** We want to remind the readers that as 2-fiber product, the diagram

\[
\begin{array}{ccc}
\mathfrak{Y} \times_\mathfrak{X} \mathfrak{Z} & \to & \mathfrak{Y} \\
\downarrow & & \downarrow ^f \\
\mathfrak{Z} & \to & \mathfrak{X}
\end{array}
\]
is 2-commutative but not necessarily strictly commutative. The two compositions from $\mathfrak{g} \times \mathfrak{x} \mathfrak{z}$ to $\mathfrak{x}$ may not be equal but there is a 2-isomorphism between them. For simplicity, we will write fiber products for 2-fiber products and 2-isomorphisms are implicit in 2-commutative diagrams.

**Example 3.1.13.** Let $G$ be a compact Lie group and $H$ be a subgroup. The inclusion map $H \to G$ induces a morphism of stacks $i : [*/H] \to [*/G]$ which sends a principal $H$-bundle $P \to N$ to the principal $G$-bundle $G \times_H P \to N$. It is easy to check the following is a fiber square.

$$
\begin{array}{ccc}
G/H & \to & * \\
\downarrow & & \downarrow \\
[*/H] & \to & [*/G]
\end{array}
$$

where the left vertical map is associated to the principal $H$-bundle $G \to G/H$.

**Lemma 3.1.14.** Let $G$ be a compact Lie group and $H, K$ be subgroups. The inclusions of subgroups induce a morphisms of stacks $i : [*/H] \to [*/G]$ and $j : [*/K] \to [*/G]$ as in example 3.1.13. Then

$$
[*/H] \times_{[*/G]} [*/K] \simeq [[[G/H] \times (G/K)]/G].
$$

**Proof.** Note that an object in $([*/H] \times_{[*/G]} [*/K])(N)$ is represented by a triple $(P_1, P_2, \phi)$, where $p_1 : P_1 \to N$ is a principal $H$-bundle, $p_2 : P_2 \to N$ is a principal $K$-bundle and $\phi : G \times_H P_1 \cong G \times_K P_2$ is an isomorphism of principal $G$-bundles. The principle $G$-bundle $G \times_H P_1 \to N$ and the $G$-map

$$
(G \times_H \varepsilon_1, (G \times_K \varepsilon_2) \circ \phi) : G \times_H P_1 \to G/H \times G/K
$$
determines an object in $[((G/H) \times (G/K))/G](N)$. Here $\varepsilon_i, i = 1, 2,$ denotes the collapse map $P_i \to *$.

Conversely, given the data of a principal $G$-bundle $P \to N$ and a $G$-map $f : P \to G/H \times G/K$, we get a principal $H$-bundle $P_1 = f^{-1}([H] \times G/K)$, a principal $K$-bundle $P_2 = f^{-1}(G/H \times [K])$ and an canonical isomorphism of principal $G$-bundles
over $N$ given by

$$\phi = G \times_H P_1 \cong P \cong G \times_K P_2.$$  

This determines an object in $([*/H] \times_{[*/G]} [*/K])(N)$. It is straightforward to check the two correspondences above give an equivalence of the stacks $[*/H] \times_{[*/G]} [*/K]$ and $[((G/H) \times (G/K))/G]$.

\[\square\]

Remark 3.1.15. Another way of seeing the equivalence is by considering the following diagram, where each of its faces is a pullback diagram.

\[\begin{array}{ccc}
G/H \times G/K & \longrightarrow & G/K \\
\downarrow & & \downarrow \\
[*/H] \times_{[*/G]} [*/K] & \longrightarrow & [*/K] \\
\downarrow & & \downarrow \\
[*/H] & \longrightarrow & [*/G]
\end{array}\]

The two pullback squares with $* \rightarrow [*/G]$ follow from example 3.1.13. All the vertical maps are principal $G$-bundles.

Lemma 3.1.16. Let $\Gamma$ be a Lie groupoid and $P \rightarrow M$ be a principal $\Gamma$-bundle. It defines an object in $[\Gamma_0/\Gamma_1](M)$ and hence a map $p_1 : M \rightarrow [\Gamma_0/\Gamma_1]$ by Yoneda lemma. Similarly, let $p_2 : \Gamma_0 \rightarrow [\Gamma_0/\Gamma_1]$ be the map associated to the principal $\Gamma$-bundle $\Gamma_1 \rightarrow \Gamma_0$ in example 3.1.9. Then there is a fiber product diagram

\[\begin{array}{ccc}
P & \longrightarrow & \Gamma_0 \\
\downarrow & & \downarrow \\
M & \longrightarrow & [\Gamma_0/\Gamma_1]
\end{array}\]
Proof. Suppose \( N \) is a smooth manifold. Then

\[
(M \times_{[\Gamma_0/\Gamma_1]} \Gamma_0)(N) \cong \langle (f : N \to M, g : N \to \Gamma_0, \phi : p_1 \circ f \to p_2 \circ g) \rangle \\
\cong \langle (f, g, \phi : f^* P \cong g^* \Gamma_1) \rangle \\
\cong \{ (f, f' : N \to P | p_1 \circ f' = f) \} \\
\cong \{ f' : N \to P \}
\]

\( = P(N) \).

The third equivalence follows from the fact that given \( f : N \to M \), the data of an isomorphism class of \( g \) and \( \phi \) uniquely determines a section of \( P \to N \) by pulling back the section \( e : \Gamma_0 \to \Gamma_1 \). Hence, \( M \times_{[\Gamma_0/\Gamma_1]} \Gamma_0 \cong P \).

The following definitions are due to [5].

**Definition 3.1.17.**

1. A morphism \( f : \mathcal{X} \to \mathcal{Y} \) of stacks on the site \( \text{Diff} \) is said to be a representable submersion if for any manifold \( M \) and any morphism \( M \to \mathcal{Y} \), the fiber product \( \mathcal{X} \times_{\mathcal{Y}} M \) is a smooth manifold and the pullback \( \mathcal{X} \times_{\mathcal{Y}} M \to M \) is a submersion.

2. A morphism \( f : \mathcal{X} \to \mathcal{Y} \) of stacks on the site \( \text{Diff} \) is representable if for any representable submersion \( M \to \mathcal{Y} \) from a manifold \( M \), the fiber product \( \mathcal{X} \times_{\mathcal{Y}} M \) is a smooth manifold and the pullback \( \mathcal{X} \times_{\mathcal{Y}} M \to M \) is smooth.

**Remark 3.1.18.** This definition of representable morphisms is different from the one in [10]. Any smooth map between smooth manifolds are representable according to our definition, but it is not case in [10].

**Example 3.1.19.** Suppose \( \mathcal{Y} = [Y/G] \) is a quotient stack and \( f : \mathcal{X} \to \mathcal{Y} \) is representable. Then \( X := \mathcal{X} \times_{\mathcal{Y}} Y \) is naturally a \( G \)-space and the pullback of \( f \) under \( Y \to \mathcal{Y} \) is a \( G \)-map \( X \to Y \).

**Definition 3.1.20** (Differentiable Stacks). A stack \( \mathcal{X} \) is called a differentiable stack if there exists a smooth manifold \( M \) and a representable morphism \( M \to \mathcal{X} \) such that for any \( N \to \mathcal{X} \), the pullback \( M \times_{\mathcal{X}} N \to N \) is a surjective submersion. The map \( M \to \mathcal{X} \) is said to be an atlas of \( \mathcal{X} \).
Example 3.1.21 (Atlases for Quotient and Groupoid Stacks). Let $G$ be a compact Lie group acting on $M$. Then the morphism $M \to [M/G]$ associated to the diagram

$$
\begin{array}{ccc}
G \times M & \xrightarrow{\mu} & M \\
\downarrow^{\pi_2} & & \downarrow M
\end{array}
$$

is an atlas of $[M/G]$. More generally, the morphism $p : \Gamma_0 \to [\Gamma_0/\Gamma_1]$ associated to the trivial $\Gamma$-bundle $\Gamma_1 \to \Gamma_0$ in example 3.1.9 is an atlas of $[\Gamma_0/\Gamma_1]$ by lemma 3.1.16.

As seen from example 3.1.21, every Lie groupoid gives rise to a differentiable stack. There is a construction in the opposite direction. Suppose $X$ is a differentiable stack and $M \to X$ is an atlas of $X$. Let $M_0 = M$, $M_1 = M \times_X M$ and $M_2 = M \times_X M \times_X M$ with projection maps $p_1, p_2 : M_1 \to M_0$ and $p_{12}, p_{13}, p_{23} : M_2 \to M_1$ where the indices indicate the factors of projection. A Lie groupoid $[M_1 \rr M_0]$ can be defined by taking $s = p_0$ and $t = p_1$. Identity-assigning map $e : M_0 \to M_1$ and the inverse map $M_1 \to M_1$ is induced by the diagonal map $M \to M \times M$ and interchange of factors respectively. By the canonical identification

$$M_1 \times_{M_0} M_1 = (M \times_X M) \times_M (M \times_X M) \cong M \times_X M \times_X M = M_2,$$

composition of morphisms is given by $p_{13}$. The following proposition shows that the two constructions between Lie groupoids and differentiable stacks are inverses up to equivalence.

**Proposition 3.1.22.** Let $\Gamma$ be a Lie groupoid and $\Gamma_0 \to [\Gamma_0/\Gamma_1]$ be the atlas associated to the trivial $\Gamma$-bundle. Then the groupoid $[\Gamma_0 \times [\Gamma_0/\Gamma_1] \Gamma_0 \rr \Gamma_0]$ associated to this atlas is canonically isomorphic to $\Gamma$. Conversely, given a differentiable stack $\mathfrak{X}$ and an atlas $M \to \mathfrak{X}$, the associated Lie groupoid $[M_1 \rr M_0]$ defines a differentiable stack $[M_0/M_1]$ which is equivalent to $\mathfrak{X}$.

This proposition states the close relationship between differentiable stacks and Lie groupoids. If $\Gamma$ is a Lie groupoid with boundary, in other words, both $\Gamma_0, \Gamma_1$ are manifolds with boundary, then it defines a differentiable stack $\mathfrak{X}$ with boundary in...
the same way in terms of $\Gamma$-bundle. The boundary $\partial X$ of $X$ is defined by the Lie groupoid $\partial \Gamma$ and the inclusion $\partial \Gamma \to \Gamma$ induces a morphism $\partial X \to X$.

The existence of atlas allows us to consider differentiable stacks geometrically. Certain geometric definitions become possible for differentiable stacks in terms of their atlases.

**Definition 3.1.23.** Let $P$ be a property of maps between smooth manifolds which is preserved under pullback by submersion. Then a representable morphism $f : X \to Y$ between differentiable stacks has property $P$ if for one (equivalently any) atlas $M \to Y$, the pullback $X \times_Y M \to M$ has property $P$.

Some important examples of properties preserved under pullback by submersion include smoothness, properness, open embedding, closed embedding, local diffeomorphism and immersion. In particular, the notions of open substacks and closed substacks make sense. Compactness of differentiable stacks can then be defined in terms of open cover as in the case of topological spaces.

Another example of definition made possible by atlas is vector bundle. In differential topology, a vector bundle over a smooth manifold $M$ with an open cover $\{U_i\}$ can be defined by a collection of vector bundles on $U_i$ of $M$ and transition functions on $U_i \cap U_j$ which satisfy a cocycle condition on $U_i \cap U_j \cap U_k$. This same construction allows us to define vector bundles over differentiable stacks. More explicitly,

**Definition 3.1.24.** A vector bundle over a differentiable stack $X$ is given by an atlas $M \to X$, a vector bundle $V \to M$ together with a vector bundle isomorphism $\phi : p_1^* V \cong p_2^* V$ such that $p_{12}^* \phi \circ p_{23}^* \phi = p_{13}^* \phi$.

Indeed, the data in the above definition gives rise to a Lie groupoid $[V_1 \rightrightarrows V_0]$ where $V_0 = V, V_1 = p_i^* V$, with source and target maps $s : p_i^* V \to V$ and $t : p_i^* V \phi \to p_2^* V \to V$ respectively. The cocycle condition on $\phi$ ensures that composition of morphisms in $[V_1 \rightrightarrows V_0]$ is well-defined. The obvious morphism of Lie groupoids induces a representable morphism of stacks $\mathfrak{V} := [V_0/V_1] \to X = [M_0/M_1]$. For any $f : Y \to X$, the induced map $f^* \mathfrak{V} = Y \times_X \mathfrak{V} \to Y$ has a natural vector bundle structure. For simplicity, we also denote $f^* \mathfrak{V}$ by $\mathfrak{V}$. 

Definition 3.1.26 (Tangent stack). Let $\mathcal{X}$ be a differentiable stack and $M \rightarrow \mathcal{X}$ be an atlas. The structure maps of the associated groupoid $[M \times_{\mathcal{X}} M \rightrightarrows M]$ induce maps of tangent bundles

$$T(M \times_{\mathcal{X}} M \times_{\mathcal{X}} M) \rightrightarrows T(M \times_{\mathcal{X}} M) \rightrightarrows TM$$

which define a Lie groupoid $[T(M \times_{\mathcal{X}} M) \rightrightarrows TM]$. The tangent stack $T\mathcal{X} \rightarrow \mathcal{X}$ of $\mathcal{X}$ is defined to be $[TM/T(M \times_{\mathcal{X}} M)] \rightarrow [M/(M \times_{\mathcal{X}} M)]$.

Remark 3.1.27. 1. There are canonical equivalences between $T\mathcal{X} \rightarrow \mathcal{X}$ defined using different atlases. Hence tangent stack is a well-defined notion.

2. Unlike smooth manifolds, tangent stack is not always a vector bundle over $\mathcal{X}$. Indeed, $T\mathcal{X} \rightarrow \mathcal{X}$ may not be representable (for example if $\mathcal{X} = *[S^1]$). Nevertheless, there is a stable vector bundle over $\text{Ho}(\mathcal{X})$ associated to $T\mathcal{X}$. We will discuss this in the next section.

We end this section with the definition of Deligne-Mumford stack. It can be considered as the stack version of orbifold, or manifold with singularities of finite order.

Definition 3.1.28 (Deligne-Mumford Stack). A differentiable stack $\mathcal{X}$ is called a Deligne-Mumford stack if the diagonal map $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is proper and there exists an atlas $M \rightarrow \mathcal{X}$ which is a local diffeomorphism.

3.2 Homotopy theory for Topological Stacks

We are interested in doing homotopy theory on differentiable stacks. One way of doing this is by associating a topological space to each differentiable stack to represent its homotopy type. For instance, since $[*G]$ classifies principal $G$-bundles, the classifying space $BG$ is the most reasonable choice to represent the homotopy type of $[*G]$. 

CHAPTER 3. DIFFERENTIABLE STACKS

As seen from this example, a differentiable stack may not have the homotopy type of a manifold. It is necessary for us to include both general topological spaces and differentiable stacks in a certain category. Hence, we have to consider another category of stacks.

Let $\text{Top}$ be the category of topological spaces and continuous maps. A stack on the site $\text{Top}$ is defined to be a contravariant pseudo-functor $\mathcal{X} : \text{Top} \rightarrow \text{Groupoid}$ satisfying sheaf-like properties as analogous to definition 3.1.1. Most of the previous definitions for stacks still make sense after replacing $\text{Diff}$ by $\text{Top}$. Two exceptions are the definitions of representable morphisms and topological stacks given below.

**Definition 3.2.1.**
1. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks on the site $\text{Top}$ is said to be a representable if for any space $Y$ and any morphism $Y \rightarrow \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y$ is equivalent to a space.

2. A stack $\mathcal{X}$ on the site $\text{Top}$ is called a topological stack if there exists a space $X$ and a representable morphism $X \rightarrow \mathcal{X}$ such that for any $Y \rightarrow \mathcal{X}$, the pullback $M \times_{\mathcal{X}} N \rightarrow N$ admits local section. The map $X \rightarrow \mathcal{X}$ is said to be an atlas of $\mathcal{X}$.

The category of stacks on the site $\text{Top}$ forms a 2-category. Every topological space and topological groupoid defines a topological stack analogously. By the forgetful functor from Lie groupoids to topological groupoids, one can associate to a differentiable stack $[\Gamma_0/\Gamma_1]$ a topological stack defined by the underlying topological groupoid of $\Gamma$. We will denote a differentiable stack and its underlying topological stack by the same symbol.

**Definition 3.2.2.**
1. Let $\mathcal{X} \rightarrow \mathcal{Y}$ be topological stacks. A representable morphism $\mathcal{X} \rightarrow \mathcal{Y}$ is said to be a weak equivalence if for any space $Y$ and any morphism $Y \rightarrow \mathcal{Y}$, the pullback $Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow Y$ is a weak equivalence.

2. A homotopy type of a topological stack $\mathcal{X}$ is a space $\text{Ho}(\mathcal{X})$ together with a weak equivalence of stacks $\eta : \text{Ho}(\mathcal{X}) \rightarrow \mathcal{X}$.

Note that a representable morphism $\mathcal{X} \rightarrow \mathcal{Y}$ between topological stacks pulls back a homotopy type $\text{Ho}(\mathcal{Y}) \rightarrow \mathcal{Y}$ of $\mathcal{Y}$ to a homotopy type $\mathcal{X} \times_{\mathcal{Y}} \text{Ho}(\mathcal{Y}) \rightarrow \mathcal{X}$ of $\mathcal{X}$. The
next two results show the existence and uniqueness of homotopy type for topological
stacks.

**Lemma 3.2.3.** Suppose \( \mathfrak{X} \) is a topological stack and \( \eta_i : X_i \to \mathfrak{X}, i = 1, 2 \) are weak
equivalences of stacks from topological spaces. Then there is a canonical chain of weak
equivalences of spaces between \( X_1 \) and \( X_2 \).

Proof. By the definition of weak equivalence, the fiber product \( X_1 \times_\mathfrak{X} X_2 \) is a space
and the two induced maps

\[
X_1 \leftarrow \bar{X}_1 \times_\mathfrak{X} \bar{X}_2 \rightarrow \bar{X}_2
\]

give the desired chain of weak equivalences.

**Proposition 3.2.4.** [15] If \( X \to \mathfrak{X} \) is an atlas of a topological stack \( \mathfrak{X} \), then there is
a weak equivalence \( B\Gamma \to \mathfrak{X} \) from the classifying space \( B\Gamma \) of the topological groupoid
\( \Gamma = [X \times_\mathfrak{X} X \rightrightarrows X] \).

For the case of quotient stacks, the space \( \text{Ho}([X/G]) \) can be taken to be the
classifying space of the action groupoid \( [G \times X \rightrightarrows X] \), which is given by the Borel
construction \( EG \times_G X \). The morphism \( EG \times_G X \to [X/G] \) is induced by the principal
\( G \)-bundle \( EG \times X \to EG \times_G X \) and the projection map \( EG \times X \to X \). If \( E \) is a
homology theory, we write \( E_*(\mathfrak{X}) = E_*(\text{Ho}(\mathfrak{X})) \). By lemma 3.2.3, it is well-defined
and functorial with respect to \( \mathfrak{X} \). We would also like to point out that homotopy type
can also be constructed functorially, as shown in [4] and [15].

Finally, we want to explain the notion of normal bundle of a morphism of stacks
and its associated Thom spectrum. Suppose \( f : \mathfrak{X} \to \mathfrak{Y} \) is a proper representable
morphism of differentiable stacks. Let \( N \to \mathfrak{Y} \) be an atlas of \( \mathfrak{Y} \) and \( M := \mathfrak{X} \times_{\mathfrak{Y}} Y \to \mathfrak{X} \)
be the induced atlas of \( \mathfrak{X} \). Then \( f \) is represented by a map \( \tilde{f} : M \to N \). The vector
bundles \( (\tilde{f})^*TN, TM \) over \( M \) satisfy the cocycle condition on triple intersections
and descend to vector bundles over \( \mathfrak{X} \), which will be denoted by \( f^*TN \) and \( TM \)
respectively. Now, suppose \( \eta : \text{Ho}(\mathfrak{X}) \to X \) is a homotopy type of \( \mathfrak{X} \). The formal
difference \( \eta^*(f^*TN) - \eta^*TM \) defines a Thom spectrum \( \text{Ho}(\mathfrak{X})^\nu(f) \), which is well-
defined up to homotopy type, independent of \( N \to \mathfrak{Y} \) and \( \eta : \text{Ho}(\mathfrak{X}) \to \mathfrak{X} \).
A particularly important case is the normal bundle of diagonal map. Let \( M \to \mathcal{X} \) be an atlas of a differentiable stack \( \mathcal{X} \). Consider the diagonal map \( \Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X} \) and its pullback under the atlas \( M \times M \to \mathcal{X} \times \mathcal{X} \). The fiber product \((M \times M) \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}\) is canonically diffeomorphic to \( M \times \mathcal{X} \) and there is a fiber product diagram

\[
\begin{array}{ccc}
M \times \mathcal{X} & \xrightarrow{\tilde{\Delta}} & M \times M \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}
\end{array}
\] (3.1)

Note that \( \tilde{\Delta} \) is the map \((s,t)\) of the Lie groupoid \([M \times \mathcal{X} M \rightrightarrows M]\) associated to the atlas \( M \to \mathcal{X} \). As the pullback of \( \Delta \), its normal bundle descends to give a stable vector bundle which represents \( T\mathcal{X} \). The dimension of this normal bundle, \( \dim(M \times M) - \dim(M \times \mathcal{X} M) \), is defined to be the \( \dim(T\mathcal{X}) = \dim(\mathcal{X}) \).

**Example 3.2.5.** Let \( G \) be a compact Lie group with adjoint representation \( \mathfrak{g} \) and \( M \) be a smooth \( G \)-manifold. Consider the quotient groupoid \( \mathcal{X} = [M/G] \). The purpose of this example is to verify that the stable vector bundle associated to \( T\mathcal{X} \) over \( \text{Ho}(\mathcal{X}) \) is the one described in the introduction.

Using the atlas \( M \to [M/G] \), the top horizontal map of (3.1) becomes

\[
\tilde{\Delta} : G \times M \xrightarrow{(\pi_2,\mu)} M \times M.
\]

It is a map of principal \( G^2 \)-bundles over \( \Delta \) with \( G^2 \) acts on \( G \times M \) and \( M \times M \) by \((g_1,g_2)(g,x) = (g_2g_1^{-1},g_1x)\) and \((g_1,g_2)(x_1,x_2) = (g_1x_1,g_2x_2)\) respectively. Note that the tangent space of \( G \) is equivariantly isomorphic to \( G \times \mathfrak{g} \) with \( G^2 \) acts on \( \mathfrak{g} \) through the first factor. Hence, the normal bundle of \( \tilde{\Delta} \) is

\[
\tilde{\Delta}^*(TM \times TM) - (\mathfrak{g} \oplus TM) \cong TM - \mathfrak{g},
\]

with \( G^2 \) acts on \( TM \) and \( \mathfrak{g} \) through the first factor. These two vector bundles on \( G \times M \) descend to \( \mathcal{X} \) and pull back to \( EG^2 \times_{G^2} (G \times TM) \) and \( EG^2 \times_{G^2} (G \times M \times \mathfrak{g}) \) over \( \text{Ho}(\mathcal{X}) = EG \times_{G^2} (G \times M) \). Note that \( EG^2 \times_{1 \times G} G \) is a \( G \)-free contractible space.
so is also a model for $EG$. From the diagram

\[
\begin{array}{ccc}
EG \times G^2 (G \times TM) & \longrightarrow & (EG \times 1 \times G) \times G TM \longrightarrow EG \times G TM \\
\downarrow & & \downarrow \\
EG \times G^2 (G \times M) & \longrightarrow & (EG \times 1 \times G) \times G M \longrightarrow EG \times G M
\end{array}
\]

and a similar one for $EG^2 \times G^2 (G \times M \times g)$, it follows that the stable vector bundle associated to $T\mathcal{X}$ over $\text{Ho}(\mathcal{X})$ is the one described in the introduction.

### 3.3 Crude Vector Bundles and Pontryagin-Thom Construction for Local Quotient Stacks

In order to prove our duality theorem for differentiable stacks, we will need to be able to apply Pontryagin-Thom construction to maps between differentiable stacks. Ebert and Giansiracusa showed the construction for the case of local quotient stacks [5].

**Definition 3.3.1** (Local Quotient Stack). A differentiable stack $\mathcal{X}$ is said to be a differentiable local quotient stack if the diagonal map $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is proper and there exists a countable cover of open substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is equivalent to $[M_i/G_i]$, where $G_i$ is compact Lie group acting on a differentiable manifold $M_i$.

**Remark 3.3.2.** It is easy to check that a Deligne-Mumford stack is a local quotient.

To fit our purpose, we will need the following version of Pontryagin-Thom construction for differentiable stacks. It is a slight generalization of [5] but the construction is essentially the same.

**Proposition 3.3.3.** Suppose $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a proper representable morphism of differentiable stacks with $\mathcal{Y}$ a local quotient stack. (This would imply that $\mathcal{X}$ is also a local quotient). Let $W_1$ and $W_2$ be vector bundles over $\mathcal{Y}$. Then there exists a Pontryagin-Thom map

\[\text{Ho}(\mathcal{Y})^W \longrightarrow \text{Ho}(\mathcal{X})^{f'(W)+\nu(f)} \quad (3.2)\]
where $W$ is the stable vector bundle $W_1 - W_2$ over $\text{Ho}(\mathcal{Y})$ and $\nu(f)$ is the stable normal bundle of $f$.

The rest of this section will be devoted to the construction of the map (3.2). We will first look the situation for global quotient stacks.

Suppose $G$ is a compact Lie group and $N$ is a smooth manifold with smooth $G$-action. Let $\mathcal{Y}$ be the quotient stack $[N/G]$. Suppose $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a representable morphism of stacks and $M$ be the manifold $\mathcal{X} \times_{\mathcal{Y}} N$. Then $M$ is naturally equipped with a smooth $G$-action with quotient $\mathcal{X}$ and $f$ is represented by a $G$-equivariant smooth map $\tilde{f} : M \rightarrow N$.

We can embed $M$ equivariantly into some finite dimensional $G$-representation $V$ and hence also into the product $N \times V$ over $\tilde{f}$. Let $\mathcal{V} := [(N \times V)/G] \rightarrow \mathcal{Y} = [N/G]$ be the vector bundle induced by projection. Let $E \rightarrow M$ be the normal bundle of the embedding $\tilde{e} : M \rightarrow N \times V$. It is a $G$-vector bundle, and there is a $G$-open embedding $\tilde{i} : E \rightarrow N \times V$ extending $\tilde{e}$. The stack quotient of all this data can be summarized in the following diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\epsilon} & \mathcal{V} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

By taking homotopy type, it gives

\[
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{V} \\
\downarrow & & \downarrow \\
\text{Ho}(\mathcal{X}) & \longrightarrow & \text{Ho}(\mathcal{Y}).
\end{array}
\]

Given such a diagram, one can choose a vector bundle monomorphism $\epsilon : \mathcal{V} \rightarrow \text{Ho}(\mathcal{Y}) \times \mathbb{R}^\infty$ as in lemma 2.1.1) to construct the data in the assumptions of proposition 2.2.1 to give a map of spectra. It is similar to the example of transfer map construction in section 2.4 but we need to choose the exhaustions of $\text{Ho}(\mathcal{X})$ and $\text{Ho}(\mathcal{Y})$ more carefully in this case.

However, if $\mathcal{Y}$ is a local quotient but not a global one, such a finite dimensional vector bundle $\mathcal{V}$ over $\mathcal{Y}$ and embedding $e : \mathcal{X} \rightarrow \mathcal{Y}$ over $f$ do not exist in general.
Nevertheless, it is possible to do the construction above locally in a compatible fashion to produce a Pontryagin-Thom map. The correct substitutes for vector bundles and embedding in this case are crude vector bundles and crude embedding. Briefly speaking, a crude vector bundle is a family of vector spaces parametrized nicely over a base space or stack, and the fiber dimension is allowed to vary over the base space, as opposite to the case of vector bundle where the fiber dimension is locally constant.

**Definition 3.3.4** (Crude vector bundle). Let $X$ be a space or a topological stack. A crude vector bundle $V \to X$ consists of the following data:

(a) a countable, locally finite cover $\{X_S\}_{S \in J}$ of $X$ indexed on a partially ordered set $J$ such that $X_T \subset X_S$ if $S \leq T$;

(b) a vector bundle $V_S \to X_S$ for each $S \in J$ and a monomorphism of vector bundles $b_{S,T} : V_S|_{X_T} \hookrightarrow V_T$ for each $S \leq T$ such that $b_{S,S} = Id_{V_S}$ and $b_{S,U} = b_{T,U} \circ b_{S,T}|_{X_U}$ whenever $S \leq T \leq U$.

We define the total space of $V$ to be

$$\bigcup V_S = \coprod_{S \in J} \{S\} \times V_S / \sim$$

where the equivalence relation is generated by $(S,v) \sim (T,w)$ if $S \leq T$ and $b_{S,T}(v) = w$. There is an obvious projection map $\pi : \bigcup V_S \longrightarrow X$. The fiber $\pi^{-1}(x)$ of each $x \in X$ has a natural vector space structure. The dimension of fiber defines an lower semi-continuous function on $X$.

Of course, if the open cover in the data above consists of only one open set $X$ or if all the $b_{S,T}$ are isomorphism, then $V$ is isomorphic to a vector bundle.

Direct sum, pullback, morphism and metric of crude vector bundles can be defined in the obvious ways. For instance, given crude vector bundles $V, V'$, defined by the collections $\{V_S \to X_S\}_{S \in J}$ and $\{V'_S' \to X'_S'\}_{S' \in J'}$ respectively, the direct sum $V \oplus V'$ is a crude vector bundle defined by the collection $\{V_S|_{X_S \cap X'_S'} \oplus V'_S'|_{X_S \cap X'_S'} \to X_S \cap X'_S\}_{(S,S') \in J \times J'}$.

For our applications, we will be mainly interested in crude vector bundles of the following form.
Example 3.3.5. Suppose \( \{ Y_\alpha \}_{\alpha \in I} \) is a countable locally finite cover of \( Y \) and \( \{ V_\alpha \to Y_\alpha \}_{\alpha \in I} \) is a collection of vector bundles. Let \( J \) be the set of all non-empty finite subsets of \( I \), partially ordered by inclusion, and \( Y_S = \bigcap_{\alpha \in S} Y_\alpha \) for each \( S \in J \). Then \( \{ Y_S \}_{S \in J}, V_S = \bigoplus_{\alpha \in S} V_\alpha |_{Y_S} \) and the obvious monomorphisms \( b_{S,T} \) induced by inclusion of summands define a crude vector bundle on \( Y \).

For \( S \in J \), let \( Y'_S = Y_S \setminus (\bigcup_{\alpha \in S} Y_\alpha) \) and \( V'_S = V_S |_{Y'_S} \). Then the collection \( \{ Y'_S \}_{S \in J} \) form a partition of \( Y \) and \( (\bigcup V_U) |_{Y'_S} = V'_S \) is a vector bundle.

The substitute for embedding is crude embedding.

Definition 3.3.6. Given a crude vector bundle \( V \to Y \) and a continuous map \( f : X \to Y \), a crude embedding over \( f \) is defined to be an embedding \( e : X \to \bigcup V_S \) such that \( \pi \circ e = f \). The map is sometimes denoted by \( e : X \to V \) for simplicity.

Since in our applications all crude embeddings are into crude vector bundles of the form in example 3.3.5, we will only give the definition of crude tubular neighborhood in this special case, which admits the following simpler version compared to the general situation.

Definition 3.3.7. Suppose \( V \to Y \) is a crude vector bundle as in example 3.3.5 and \( e : X \to V \) is a crude embedding over \( f : X \to Y \). A crude tubular neighborhood of \( e \) consists of the following data:

(a) a crude vector bundle \( E \to X \) defined by the collection \( E_S \to X_S := f^{-1}(Y_S) \) and \( c_{S,T} : E_S |_{X_T} \to E_T \);

(b) inclusions \( t_S : E_S \to \bigcup V_U \), whose restriction to the zero section is \( e |_{X_S} \), such that \( t_T \circ c_{S,T} = t_S |_{E_S |_{X_T}} \) for \( S \leq T \) and

\[
 t_S |_{i_S^{-1}(V'_S)} : i_S^{-1}(V'_S) \to V'_S
\]

is an open embedding.

After introducing all these crude objects, we can return to the construction of Pontryagin-Thom map (3.2) for local quotient stacks. This can be done in three main steps.
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Step 1. Construction of crude embedding.

Pick locally finite covers \( \{ \mathcal{U}_a \}_{a \in I} \) and \( \{ \mathcal{V}_a \}_{a \in I} \) of \( \mathcal{Y} \) such that \( \mathcal{Y}_a = [N_a/G_a] \subset \mathcal{U}_a = [U_a/G_a] \) with \( N_a \) relatively compact in \( U_a \). Then \( \mathfrak{X}_a := f^{-1}(\mathcal{Y}_a) \) is equivalent to \( [M_a/G_a] \) for some \( G_a \)-manifold \( M_a \) and \( f|\mathfrak{X}_a : \mathfrak{X}_a \to \mathcal{Y}_a \) is represented by a \( G_a \)-map \( M_a \to N_a \). As in the global quotient case, there exists a \( G_a \)-representation \( V_a \) and a \( G_a \)-map \( M_a \to N_a \times V_a \) whose \( G_a \)-quotient \( \tilde{e}_a : \mathfrak{X}_a \to \mathcal{V}_a := (N_a \times V_a)/G_a \) lifts \( f|\mathfrak{X}_a \). The vector bundles \( \mathcal{V}_a \to \mathcal{Y}_a \) define a crude vector bundle as described in example 3.3.5.

Let \( (\lambda_a) \) be a partition of unity subordinate to the cover \( \mathcal{Y}_a \) and \( (\mu_a) \) be a family of bump functions so that \( \text{supp}(\mu_a) \subset \mathcal{Y}_a \) and \( \mu_a \lambda_a = \lambda_a \). Define \( e : \mathfrak{X} \to \bigcup \mathcal{V}_U \) by \( e = \sum \lambda_a \tilde{e}_a \). This gives a crude embedding over \( f \).

Step 2. Construction of crude tubular neighborhood.

Next we will construct a crude tubular neighborhood of the crude embedding \( e \). For each \( S \in J \), \( S = \{ \alpha_1, \alpha_2, \ldots, \alpha_k \} \), let \( \mathcal{E}_S \to \mathcal{X}_S \) be the normal bundle of \( \sum_{i=1}^k \tilde{e}_{\alpha_i} : \mathcal{X}_S \to \mathcal{V}_S \). The collection of all \( \mathcal{E}_S \) form a crude vector bundle \( \mathcal{E} \to \mathfrak{X} \).

Let \( \Delta_S \subset \mathbb{R}^S \) denote the \((|S| - 1)\)-dimensional simplex spanned by \( S \). In other words, any point \( p \in \Delta_S \) is represented by a formal linear combination \( \sum_{i=1}^k a_i \{ \alpha_i \} \) with \( a_i \geq 0 \) and \( \sum_{i=1}^k a_i = 1 \). Let \( \varepsilon_p = \sum_{i=1}^k a_i \tilde{e}_{\alpha_i} : X_S \to V_S \). It is an embedding isotopic to \( \sum_{i=1}^k \tilde{e}_{\alpha_i} \) and so its normal bundle is isomorphic to \( \mathcal{E}_S \). Let \( \mathcal{F}_p \) denote the subspace of tubular neighborhoods \( \iota : \mathcal{E}_S \to \mathcal{Y}_S \) of \( e_p \) such that if \( z \in \mathcal{E}_S \) is a fiber of a point in \( \mathfrak{X}_T \), then \( \iota(z) \) is a fiber of a point in \( \mathcal{Y}_T \). Let \( \mathcal{F}_S = \bigcup_{p \in \Delta_S} \mathcal{F}_p \) and give it the subspace topology of \( \text{Map}(\mathcal{E}_S, \mathcal{Y}_S) \).

For \( S \subset T \), let \( j : \Delta_S \to \Delta_T \) be the inclusion map. Note that \( e_p|_{\mathfrak{X}_T} = e_{j(p)} \) for \( p \in \Delta_S \). Let \( r_{S,T} : \mathcal{F}_S \to \mathcal{F}_T \) denote the map defined by sending a tubular neighborhood \( \iota : \mathcal{E}_S \to \mathcal{Y}_S \) of \( e_p \) to the tubular neighborhood

\[
\iota \oplus \text{Id} : \mathcal{E}_S|_{\mathfrak{X}_T} \oplus \mathcal{V}_{T \setminus S}|_{\mathcal{Y}_T} \to \mathcal{Y}_{S|_{\mathcal{Y}_T}} \oplus \mathcal{V}_{T \setminus S}|_{\mathcal{Y}_T} \cong \mathcal{Y}_T
\]

of \( e_{j(p)} \).
By induction on $|S|$, it is possible to choose maps $h_S : Δ_S \to F_S$ with $h_S(p) \in F_p$, such that $h_T|_{Δ_S} = r_{S,T} \circ h_S$ whenever $S \subseteq T$. Let $q : E \to X$ be the projection. For $x \in X$, let $S(x) := \{α ∈ I | f(x) ∈ Y_α\}$. For $e ∈ E$, define

$$ι(e) := h_{S(q(e))} \left( \sum_{α ∈ S(q(e))} λ_α(f(q(e)))\{α\} \right) (e) \in \bigcup_{S ∈ J} V_S.$$

The restriction of $ι$ to the zero section is $e$. By a reparametrization if necessary, $ι$ is injective and form a crude tubular neighborhood of $e$. We can choose metrics on $E$ and $V$ such that $ι(\text{int} D(E)) \subset D(V)$ and $ι|_{D(E)}$ is proper. We will need this properness assumption in step 3 to construct compact exhaustions of $\text{Ho}(X)$ and $\text{Ho}(Y)$ for Pontryagin-Thom construction.

**Step 3.** Pontryagin-Thom Construction on homotopy type.

In this last step, we will pass our construction from stacks to their homotopy types. We then apply proposition 2.2.1 to get the desired map (3.2) in proposition 3.3.3.

So far we have constructed the following data in the category of stacks:

$$\begin{array}{ccc}
E & \xrightarrow{ι} & V \\
\downarrow{e} & & \downarrow{f} \\
X & \xrightarrow{f} & Y
\end{array}$$

Consider the weak equivalence $η_Y : \text{Ho}(Y) \to Y$. Its pullback under $f$ is a weak equivalence $\text{Ho}(Y) \times_Y X \to X$ from a topological space. We denote this map by $η_X : \text{Ho}(X) \to X$. By our convention $η^*_X E, η^*_Y V$ are denoted by $E, V$. The crude vector bundles in the diagram above pull back under $η_Y$ to give a commutative diagram

$$\begin{array}{ccc}
E & \xrightarrow{\text{Ho}(ι)} & \text{Ho}(V) \\
\downarrow{\text{Ho}(e)} & & \downarrow{\text{Ho}(f)} \\
\text{Ho}(X) & \xrightarrow{\text{Ho}(f)} & \text{Ho}(Y)
\end{array}$$
of crude embedding and crude tubular neighborhood of topological spaces. For simplicity, the spaces \( E = \eta^*_X \mathfrak{E}, V = \eta^*_Y \mathfrak{V}, \text{Ho}(\mathfrak{X}), \text{Ho}(\mathfrak{Y}) \) will be denoted by \( E, V, X, Y \) respectively for the remaining of this section.

We also need an analogue of lemma 2.1.1 for crude vector bundles, which is proved in [5].

**Lemma 3.3.8.** Let \( Y \) be a paracompact space with \((Y_i)_{i \in I}\) a countable locally finite cover. For each \( i \), let \( \pi_i : V_i \to Y_i \) be a finite-dimensional vector bundle, and let \((V_S \to Y_S)_{S \in J}\) be the crude vector bundle constructed as in example 3.3.5 and \( V = \bigcup_S V_S \).

Assume that there exist open neighborhoods \( \tilde{Y}_i \) of \( Y_i \) and vector bundles \( \tilde{V}_i \to \tilde{Y}_i \) with \( \tilde{V}_i|_{Y_i} = V_i \), such that \((\tilde{Y}_i)_{i \in I}\) is locally a finite cover of \( Y \). Then there exists a monomorphism \( \epsilon : V \to Y \times \mathbb{R}^\infty \) such that any \( y \in Y \) has an open neighborhood \( U \subset Y \) such that \( \epsilon(V|_U) \subset U \times \mathbb{R}^{n_U} \) for some integer \( n_U \).

We now construct the data in the assumptions of proposition 2.2.1.

First of all, pick metrics on \( W_1, W_2 \). By choosing an isometric embedding \( \epsilon : V \oplus W_2 \to Y \times \mathbb{R}^\infty \) as in lemma 3.3.8, we may identify \( V \oplus W_2 \), and hence also \( V \) and \( W_2 \), as subbundles of \( Y \times \mathbb{R}^\infty \). Note that by the definition of metric of direct sum, \( V \) and \( W_2 \) are orthogonal. As a result, \( \text{Ho}(\epsilon) : X \to V \) and \( \text{Ho}(\iota) : E \to V \) can be considered as maps to \( W_2^\perp \subset W_1 \oplus W_2^\perp \).

Take compact exhaustion \( X_1 \subset X_2 \subset \ldots \) of \( X \) so that each \( \epsilon((V \oplus W_2)|_{\text{Ho}(f(X_n))}) \subset Y \times \mathbb{R}^n \). The properness of \( \iota|_{D(E)} \), by our construction in step 2, implies that the pullback \( \text{Ho}(\iota)|_{D(E)} \) is also proper. Hence for any compact set \( K \) in \( Y \), the preimage \( \text{Ho}(\iota)|^{-1}_{D(E)}(D(V|_K)) \) is contained in some \( D(E|_{X_n}) \). As a result, we can choose an open exhaustion \( Y_1 \subset Y_2 \subset \ldots \) of \( Y \) such that

\[
\epsilon(W_2|_{Y_n}) \subset \epsilon((V \oplus W_2)|_{Y_n}) \subset Y \times \mathbb{R}^n
\]

and

\[
\text{Ho}(\iota)(D(E|_{X_m})) \cap V|_{Y_n} = \text{Ho}(\iota)(D(E|_{X_n})) \cap V|_{Y_n}
\]

for all \( m \geq n \).
For each \( n \), we define the vector bundle \( Q_n \to Y_n \), using the notations in proposition 2.2.1, to be \( W_{1,n} + W_{2,n}^\perp \). Our \( W'_n \to X_n \) is defined to be the direct sum

\[
\text{Ho}(f)^*(W_{1,n}) \oplus \text{Ho}(f)^*((V_n \oplus W_{2,n})^\perp) \oplus E_n
\]

(3.5)

where the orthogonal complement is taken with respect to the inclusion \( V_n \oplus W_{2,n} \hookrightarrow \text{Ho}(f)(X_n) \times \mathbb{R}^n \).

The map \( i_n : W'_n \oplus \mathbb{R} \to W'_{n+1}|_{X_n} \) is given by the direct sum of the canonical isomorphisms on the first and third summands of (3.5) and the isomorphism

\[
\text{Ho}(f)^*((V_n \oplus W_{2,n})^\perp) \oplus \mathbb{R} \cong \text{Ho}(f)^*((V_{n+1} \oplus W_{2,n+1})^\perp)
\]

induced by the inclusion of ambient spaces \( \mathbb{R}^n \to \mathbb{R}^{n+1} \).

To define \( \iota_n : W'_n \to W_1 \oplus W_2^\perp \), note that

\[
W_1 \oplus W_2^\perp = W_1 \oplus (V \oplus W_2)^\perp \oplus V.
\]

(3.6)

The map \( \iota_n \) is given by the inclusion of the first two summands from (3.5) to (3.6) and \( \text{Ho}(\iota)|_{E_n} \) on the third summand.

The vector bundles and maps we just constructed satisfy all the assumptions of proposition 2.2.1 except that the image of the disc bundle \( D(W'_n) \) under \( \iota_n \) lies in the interior of the disc bundle of \( W_1 \oplus W_2^\perp \) of radius two but not the unit one. This can be fixed by post-composing \( e \) and \( \iota \) with \( \text{Id}_Y \times \frac{1}{2} : Y \times \mathbb{R}^\infty \to Y \times \mathbb{R}^\infty \).

This finishes the construction of Pontryagin-Thom map (3.2) and proves proposition 3.3.3.
Chapter 4

Proof of the Main Theorem

The goal of this chapter is to prove our main theorem 1.0.1. To do this, we will first recall the proof of the $K(n)$-duality of classifying spaces of finite groups in [7] in section 4.1. It is the special case of our $K(n)$-duality of stacks when $\mathfrak{X} \simeq \ast / G$. In section 4.2, we will construct the map (1.5) in the main theorem using the Pontryagin-Thom construction of stacks described in section 3.3. The $K(n)$-duality of the special cases $\mathfrak{X} \simeq [M/G]$ for finite $G$ will be established in section 4.3 and 4.4. The proof of theorem 1.0.1 will be completed in section 4.5.

4.1 Result for Finite Group cases

As mentioned in the introduction, Greenlees and Sadofsky proved that the $K(n)$-homology of the classifying space of a finite group is self dual. Their proof uses equivariant stable homotopy theory of [13]. We will review their results and proof below.

Let $G$ be a finite group and $U$ be a complete $G$-universe. We will primarily work in three different categories of spectra, namely $G\mathbb{S}U$, $G\mathbb{S}U^G$ and $\mathbb{S}U^G$.

Consider the cofiber sequence of pointed spaces

$$EG_+ \to S^0 \to \widetilde{EG}.$$  \hfill (4.1)
The first map collapses $EG$ to the non-base point of $S^0$. The cofiber $\tilde{E}G$ is homotopy equivalent to the unreduced suspension of $EG$.

Consider the $n$-th Morava $K$-theory spectrum. It is a non-equivariant spectrum, but can also be regarded as a naive $G$-spectrum with trivial $G$-action. We use the same notation $K(n)$ to represent it in both $G\Sigma$ and $G\Sigma G$. Consider the map

$$\tau : i_*K(n) \simeq F(S^0, i_*K(n)) \to F(EG_+, i_*K(n))$$

induced by the collapse map $EG_+ \to S^0$. Here $F(X, -)$ denotes the function $G$-spectrum of maps from a $G$-space or spectrum $X$. It is right adjoint to taking smash product with $X$.

The smash product of (4.1) and $\tau$ gives the following commutative diagram in $G\Sigma U$

$$
\begin{array}{ccc}
EG_+ \wedge i_*K(n) & \rightarrow & i_*K(n) \\
\downarrow & & \downarrow \\
EG_+ \wedge F(EG_+, i_*K(n)) & \rightarrow & F(EG_+, i_*K(n)) \\
\end{array}
$$

Both rows are cofiber sequences. Define the last term

$$t_G(i_*K(n)) = \tilde{E}G \wedge F(EG_+, i_*K(n))$$

to be the Tate spectrum of $i_*K(n)$.

Using the complex orientability of Morava $K$-theory and the result of Ravenel that $K(n)^*(BG)$ has finite rank [16], Greenlees and Sadofsky proved

**Theorem 4.1.1.** $t_G(i_*K(n)) \simeq \ast$.

The contractibility of the cofiber $t_G(i_*K(n))$ implies the $G$-equivalence

$$EG_+ \wedge F(EG_+, i_*K(n)) \simeq F(EG_+, i_*K(n)).$$

Also, since $\tau$ is a non-equivariant equivalence, the left vertical map $EG_+ \wedge \tau$ in diagram (4.2) is a $G$-equivalence. Hence
Corollary 4.1.2. The composite

\[ EG_+ \wedge i_* K(n) \to i_* K(n) \to F(EG_+, i_* K(n)) \]  \hspace{1cm} (4.3)

is a \( G \)-equivalence.

By the \( G \)-freeness of \( EG \) and theorem 2.6.2,

\[ BG_+ \wedge K(n) \simeq (EG_+ \wedge i_* K(n))^G \]

and

\[ F(EG_+, i_* K(n))^G \simeq F(BG_+, K(n)) \]

are equivalences in \( SU^G \).

Combining these two with the \( G \)-fixed point of (4.3), we get

\[ BG_+ \wedge K(n) \simeq F(BG_+, K(n)) \]  \hspace{1cm} (4.4)

which gives the self \( K(n) \)-duality of \( BG \) by taking homotopy groups.

4.2 Construction of Duality maps of Stacks

In this section we will construct the map \( \psi \) in (1.5) which induces the \( K(n) \)-duality of Deligne-Mumford stacks in theorem 1.0.1. As pointed out in the introduction, an essential step is the Pontryagin-Thom construction of diagonal map of stack. We will do this basically by applying proposition 3.3.3, but with specific choices of crude vector bundles and crude tubular neighborhood which are needed for the proof of theorem 1.0.1.

Suppose \( \mathfrak{X} \) is a local quotient stack and \( \Delta : \mathfrak{X} \to \mathfrak{Y} := \mathfrak{X} \times \mathfrak{X} \) is its diagonal map. Let \( \{ \mathfrak{X}_\alpha \}_{\alpha \in I'} \), where \( \mathfrak{X}_\alpha = [M_\alpha/G_\alpha] \), be a countable locally finite open cover of \( \mathfrak{X} \). Assume \( 0 \notin I' \) and \( I = I' \cup \{0\} \). Let \( \mathfrak{Y}_0 := \mathfrak{Y} - \Delta(\mathfrak{X}) \) and \( \mathfrak{Y}_\alpha := \mathfrak{X}_\alpha \times \mathfrak{X}_\alpha, \alpha \in I' \). Then \( \{ \mathfrak{Y}_\alpha \}_{\alpha \in I} \) is a locally finite cover of \( \mathfrak{Y} \).

For each \( \alpha \in I' \), consider the restriction \( \Delta|_{\mathfrak{X}_\alpha} : \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha \times \mathfrak{X}_\alpha \). It pulls back the
CHAPTER 4. PROOF OF THE MAIN THEOREM

atlas $M_\alpha \times M_\alpha \rightarrow \mathcal{X}_\alpha \times \mathcal{X}_\alpha$ to an atlas of $\mathcal{X}_\alpha$:

\[
\begin{array}{ccc}
G_\alpha \times M_\alpha & \rightarrow & M_\alpha \times M_\alpha \\
\downarrow & & \downarrow \\
\mathcal{X}_\alpha & \rightarrow & \mathcal{X}_\alpha \times \mathcal{X}_\alpha
\end{array}
\] (4.5)

Here the top horizontal map is $(g, x) \mapsto (x, gx)$. The stacks can be expressed as quotients $\mathcal{X}_\alpha \simeq [(G_\alpha \times M_\alpha)/G^2_\alpha]$ and $\mathcal{X}_\alpha \times \mathcal{X}_\alpha \simeq [(M_\alpha \times M_\alpha)/G^2_\alpha]$, where $G^2_\alpha$ acts on $G_\alpha$, $G_\alpha \times M_\alpha$ and $M_\alpha \times M_\alpha$ by $(g_1, g_2)g = g_2 gg^{-1}_1$, $(g_1, g_2)(g, x) = (g_2 gg^{-1}_1, g_1 x)$ and $(g_1, g_2)(x_1, x_2) = (g_1 x_1, g_2 x_2)$ respectively.

Let $\tilde{e}_\alpha : G_\alpha \hookrightarrow V_\alpha$ be a $G^2_\alpha$-embedding of $G_\alpha$ into a $G^2_\alpha$-representation $V_\alpha$. This induces an $G^2_\alpha$-embedding

\[
G_\alpha \times M_\alpha \hookrightarrow M_\alpha \times M_\alpha \times V_\alpha \\
(g, x) \mapsto (x, gx, \tilde{e}_\alpha(g))
\]

over $G_\alpha \times M_\alpha \rightarrow M_\alpha \times M_\alpha$. By taking $G^2_\alpha$-stack quotient, this becomes an embedding $e_\alpha : \mathcal{X}_\alpha \rightarrow \mathcal{Y}_\alpha := [(M_\alpha \times M_\alpha \times V_\alpha)/G^2]$ over $\Delta |_{\mathcal{X}_\alpha} : \mathcal{X}_\alpha \rightarrow \mathcal{X}_\alpha \times \mathcal{X}_\alpha = \mathcal{Y}_\alpha$.

We will apply the construction of Pontryagin-Thom map (3.2) described in section 3.3 to produce a Pontryagin-Thom map

\[
\Delta' : \Sigma^\infty \text{Ho}(\mathcal{X})_+ \wedge \text{Ho}(\mathcal{X})^{-T\mathcal{X}} \rightarrow \Sigma^\infty \text{Ho}(\mathcal{X})_+
\] (4.6)

for $\Delta$ with $W_1 = 0, W_2 = \pi^*_2 T\mathcal{X}$. In doing so, the following choices are specified in the construction:

(a) use the collection of vector bundles $\mathcal{V}_\alpha \rightarrow \mathcal{Y}_\alpha$ and embeddings $e_\alpha$ constructed in the last paragraph to produce a crude embedding over $\Delta$;

(b) in the step of constructing crude tubular neighborhood, replace $\mathcal{F}_S$ by the subset $\mathcal{F}_S'$ of $\iota_S$ satisfying the extra condition that

\[
\mathcal{E}_S \xrightarrow{\iota_S} \mathcal{V}_S \rightarrow \mathcal{Y}_S = \mathcal{X}_S \times \mathcal{X}_S \xrightarrow{\pi^*_2} \mathcal{X}_S \text{ is equal to } \mathcal{E}_S \rightarrow \mathcal{X}_S.
\] (4.7)
The consequence of the second condition is that given any open \( U \subset \mathcal{X} \), the restriction map \( \iota|_U : \iota|_{e|_U} \) factors through \( \mathcal{V}|_{\mathcal{X} \times U} \). This property is needed when we define a relative version of \( \psi \).

We are ready to define the map (1.5) in theorem 1.0.1. The composition of (4.6) with the collapse map \( \Sigma^\infty \text{Ho}(\mathcal{X})_+ \to \mathbb{S} \) gives

\[
\Sigma^\infty \text{Ho}(\mathcal{X})_+ \wedge \text{Ho}(\mathcal{X})^{-TX} \to \mathbb{S}.
\]

The map \( \psi \) in (1.5) is defined to be the adjoint of this one.

### 4.3 Special case of Global Quotient Stacks by Finite Group Action

In this section we will prove the \( K(n) \)-duality of differentiable stacks defined by finite group action on manifolds. Since Deligne-Mumford stacks are locally of this form, this result can be regarded as a local version of theorem 1.0.1.

Suppose \( G \) is a finite group and \( M \) is a \( m \)-dimensional \( G \)-manifold without boundary. Let \( W \) be a finite dimensional \( G \)-representation and \( \phi : M \to W \) be a \( G \)-embedding. Let \( E \to M \) be its equivariant normal bundle. Then the Thom spectrum \( M^{-TM} \in G\mathbb{S}U \) can be defined to be \( \Sigma^{-W}M^E \), the desuspension of \( M^E \) by \( W \). It is a \( G \)-spectrum. The equivariant version of Spanier-Whitehead theorem states that for closed manifold \( M \),

\[
\Sigma^\infty M_+ \simeq F(M^{-TM}, \mathbb{S}) \tag{4.8}
\]

is a \( G \)-equivalence in \( G\mathbb{S}U \). To obtain this map, apply Pontryagin-Thom construction to the composition \( M \to M \times M \to M \times E \) and take desuspension \( \Sigma^{-W} \) to get

\[
\Sigma^\infty M_+ \wedge M^{-TM} \to \Sigma^\infty M_+.
\]

The map (4.8) is the adjoint of the composition of this map with the collapse map.
\[ \Sigma^\infty M_+ \rightarrow \mathbb{S}. \]

For non-compact \( M \), (4.8) is not a \( G \)-equivalence, but becomes one after factoring through \( \Sigma^\infty M_+ \rightarrow \Sigma^\infty M^c \):

\[ \Sigma^\infty M^c \simeq F(M^{-TM}, \mathbb{S}) \] (4.9)

by the equivariant version of Spanier-Whitehead theorem for open manifolds.

By using the equivariant version of Spanier-Whitehead duality and the result of Greenlees-Sadofsky, we will show that

**Proposition 4.3.1.** Let \( G \) be a finite group and \( M \) is a smooth manifold without boundary. Then

\[ \tilde{K}(n)_*(EG_+ \wedge_G M^c) \cong K(n)^-((EG \times_G M)^{-TM}). \]

In particular, if \( M \) is compact, then

\[ K(n)_*(EG \times_G M) \cong K(n)^-((EG \times_G M)^{-TM}). \]

**Proof.** By smashing the \( G \)-equivalences (4.3) and (4.9) and taking \( G \)-fixed point, we obtain an equivalence in \( \mathfrak{S}U^G \).

\[ (EG_+ \wedge M^c \wedge i_*K(n))^G \rightarrow (F(EG_+, i_*K(n)) \wedge F(M^{-TM}, \mathbb{S}))^G \]
\[ \simeq F(EG_+ \wedge M^{-TM}, i_*K(n))^G \]
\[ \simeq F(EG_+ \wedge_G M^{-TM}, K(n)) \] (4.10)

By (2.6.2), since \( EG_+ \wedge M^c \wedge i_*K(n) \) is \( G \)-free, the adjoint of the equivariant transfer map gives

\[ (EG_+ \wedge_G M^c) \wedge K(n) \simeq (EG_+ \wedge M^{-TM} \wedge K(n))/G \]
\[ \simeq (EG_+ \wedge M^{-TM} \wedge i_*K(n))^G. \]
Combining this with (4.10), we get the equivalence

$$\kappa : EG_+ \wedge_G M^c \wedge K(n) \simeq F(EG_+ \wedge_G M^{-TM}, K(n))$$  \hspace{1cm} (4.11)

in $\mathcal{S}U^G$. The proposition follows from taking homotopy of this equivalence.

In order to interpret the result of proposition 4.3.1 as a local version of our main theorem 1.0.1, we have to relate the maps $\kappa$ in (4.11) and $\psi$ in (1.5). Suppose $G$ is a finite group and $M$ is a smooth $G$-manifold without boundary as in proposition 4.3.1. Let $\mathfrak{X} \simeq [M/G]$ and fix its homotopy type $\mathrm{Ho}(\mathfrak{X}) = EG \times_G M \to \mathfrak{X}$. Then $\psi$, as constructed in section 4.2, induces a map of spectra

$$\psi_{[M/G]} : \Sigma^\infty(EG \times_G M)_+ \longrightarrow F((EG \times_G M)^{-TM}, S),$$

which factors though $\Sigma^\infty(EG \times_G M)_+ \to \Sigma^\infty EG_+ \wedge_G M^c$ to give

$$\psi^c_{[M/G]} : \Sigma^\infty EG_+ \wedge_G M^c \longrightarrow F((EG \times_G M)^{-TM}, S)$$  \hspace{1cm} (4.12)

The following proposition relates the maps $\psi^c_{[M/G]}$ and $\kappa$. Its proof, which involves comparing the underlying constructions in the definitions of the two maps, will be given in the next section.

**Proposition 4.3.2.** $\psi^c_{[M/G]} \wedge K(n)$ is homotopic to $\kappa$ in (4.11).

Combining this result with proposition 4.3.1,

**Corollary 4.3.3.** The map $\psi^c_{[M/G]}$ induces $K(n)$-equivalence for global quotients by finite group.

### 4.4 Comparison of Two Duality Maps

The purpose of this section is to prove proposition 4.3.2, which states that the map $\kappa$ obtained by equivariant stable homotopy theory is homotopic to $\psi^c_{[M/G]} \wedge K(n)$.
constructed by our Pontryagin-Thom construction of stacks. To compare the two maps, it is necessary for us to review the construction of them.

Let \( U' \) be a complete \( G^2 \)-universe. Its \( (G \times 1) \)-fixed point \( U := U'^{G \times 1} \) can be regarded as a complete \( G \)-universe. We will need to work in three different categories, namely the category of \( G^2 \)-spectra \( G^2 \mathcal{G} U' \), naive \( G^2 \)-spectra \( G^2 \mathcal{G} U'^{G^2} \) and non-equivariant spectra \( \mathcal{G} U'^{G^2} = \mathcal{G} U^{G} \). The sphere spectrum will be denoted by \( S' \) in the first one and \( S \) in the last two.

Two main ingredients in the definition of \( \kappa \) are the transfer map

\[
\epsilon^\ast (EG_+ \wedge_G M^c \wedge K(n)) \to i_\ast (EG_+ \wedge M^c \wedge K(n))
\]

and the Spanier-Whitehead duality (4.9) for \( G \)-manifolds. The constructions of them start with the following data of spaces and maps.

(a) Regard \( G \) as a \( G^2 \)-space with action \((g_1, g_2)g = g_2gg_1^{-1}\) and let \( \tilde{e} : G \to V \) be a \( G^2 \)-embedding of \( G \) into a \( G^2 \)-representation \( V \). The normal bundle of \( \tilde{e} \) is isomorphic to \( G \times V \) and let \( \iota_1 : G \times V \to V \) be a tubular neighborhood;

(b) An \( G \)-embedding of \( M \) into a \( G \)-representation \( W \) with normal bundle \( E \to M \) and a tubular neighborhood \( \iota_2 : M \times W \to M \times E \) over \( M \triangle M \times M \xrightarrow{\text{zero}} M \times E \).

We will construct three Pontryagin-Thom maps \( \zeta_0, \zeta_1 \) and \( \zeta_2 \), which are related to the definition of \( \kappa \), by the data above. Let \( \pi_0, \pi_1 : G^2 \to G \) be the projection of the first and second factor respectively. They pull back a \( G \)-space or \( G \)-spectrum \( X \) to \( \pi_1^* X \) which is equipped with a \( G^2 \)-action.

**Construction of \( \zeta_0 \).**

Using \( \iota_1, \iota_2 \), a tubular neighborhood of the embedding

\[
e : G \times \pi_0^* M \to \pi_0^* M \times \pi_1^* E \times V,
\]

where \( e(g, x) = (x, gx, \tilde{e}(g)) \), is given by

\[
\nu : G \times V \times \pi_0^*(M \times W) \to \pi_0^* M \times \pi_1^* E \times V
\]
which sends \((g, v, x, w)\) to \((x, gt_2(x, w), t_1(g))\). The maps fit into the commutative diagram

\[
\begin{array}{ccc}
G \times V \times \pi_0^* (M \times W) & \xrightarrow{\iota} & \pi_0^* M \times \pi_1^* E \times V \\
\uparrow \text{zero} & & \downarrow \epsilon \\
G \times \pi_0^* M & \xrightarrow{\iota} & \pi_0^* M \times \pi_1^* M
\end{array}
\] (4.13)

Pontryagin-Thom construction produces a \(G^2\)-based map

\[
S^V \wedge \pi_0^* M_+ \wedge \pi_1^* M^E \rightarrow \Sigma^V G_+ \wedge \pi_0^*(\Sigma^W M_+).
\]

By taking desuspension \(\Sigma^{-V-\pi_1^*W}\), smashing with \(EG_+\) and factoring through \(M_+ \rightarrow M^c\), it gives a map of \(G^2\)-spectra

\[
\zeta_0 : EG_+^2 \wedge \pi_0^* M^c \wedge \pi_1^* M^{-T^M} \rightarrow S' \wedge EG_+^2 \wedge G_+ \wedge \pi_0^* M_+.
\]

**Construction of \(\zeta_1\).**

Consider the right vertical map of diagram (4.13). It induces a vector bundle

\[
EG^2 \times_{G^2} (\pi_0^* M \times \pi_1^* E \times V) \rightarrow (EG \times_G M)^2,
\]

which is a subbundle of

\[
EG^2 \times_{G^2} (\pi_0^* M \times \pi_1^* (M \times W) \times V) \rightarrow (EG \times_G M)^2.
\]

Take an embedding

\[
\epsilon : EG^2 \times_{G^2} (\pi_0^* M \times \pi_1^* (M \times W) \times V) \rightarrow (EG \times_G M)^2 \times \mathbb{R}^\infty
\]

over \((EG \times_G M)^2\) by lemma 3.3.8. Since \(TM \oplus E \cong M \times W\), the Thom spaces of the restrictions of certain subbundles of

\[
\epsilon(EG^2 \times_{G^2} (\pi_0^* M \times \pi_1^* (M \times W) \times V)) \oplus \epsilon(EG^2 \times_{G^2} (\pi_0^* M \times \pi_1^* E \times V))
\]
over an exhaustion of \((EG \times_G M)^2\) can be taken to define the non-equivariant spectrum \((EG \times_G M)_+ \land (EG \times_G M)^{-TM} \in \mathcal{S}U^{G^2}\).

The pullback of \(\epsilon\) along the quotient map

\[
EG^2 \times \pi_0^* M \times \pi_1^* M \times \mathbb{R}^\infty \to (EG \times_G M)^2 \times \mathbb{R}^\infty
\]
is an embedding

\[
\epsilon' : EG^2 \times \pi_0^* M \times \pi_1^* (M \times W) \times V \to EG^2 \times \pi_0^* M \times \pi_1^* M \times \mathbb{R}^\infty
\]
over \(EG^2 \times \pi_0^* M \times \pi_1^* M\). Similar to the definition of \((EG \times_G M)_+ \land (EG \times_G M)^{-TM}\) above, a naive \(G^2\)-spectrum \(\pi_0^*(EG \times M)_+ \land \pi_1^*(EG \times M)^{-TM} \in G^2\mathcal{S}U^{G^2}\) can be defined using \(\epsilon'\) and an exhaustion of \(EG^2 \times \pi_0^* M \times \pi_1^* M\).

The direct product of the diagram (4.13) with \(EG^2\)

\[
\begin{array}{ccc}
EG^2 \times G \times V \times \pi_0^* (M \times W) & \longrightarrow & EG^2 \times \pi_0^* M \times \pi_1^* E \times V \\
\uparrow & & \downarrow \\
EG^2 \times G \times \pi_0^* M & \longrightarrow & EG^2 \times \pi_0^* M \times \pi_1^* M
\end{array}
\] (4.14)
together with \(\epsilon'\) induce a map of naive \(G^2\)-spectra

\[
\zeta_1 : EG^2_+ \land \pi_0^* M^c \land \pi_1^* M^{-TM} \to \Sigma^\infty (EG^2 \times G \times \pi_0^* M)_+.
\]
by Pontryagin-Thom construction.

**Construction of \(\zeta_2\).**

The construction of \(\zeta_2\) is similar to that of \(\zeta_1\). The \(G^2\)-quotient of diagram 4.14 above and \(\epsilon\) yield the desired map of non-equivariant spectra

\[
\zeta_2 : (EG_+ \land_G M^c) \land (EG \times_G M)^{-TM} \to \Sigma^\infty (EG^2 \times G^2 (G \times \pi_0^* M))_+.
\]

After all these set up, we can prove proposition 4.3.2.
Proof of Proposition 4.3.2. Let $\delta$ denote the collapse map of any suspension spectra in the categories $S^G_\ast G$, $G^2S^G_\ast$ and $G^2S^G_\ast'$. By examining the definitions of

$$\varepsilon^\ast(EG_+ \wedge_G M_+ \wedge K(n)) \to i_\ast(EG_+ \wedge M_+ \wedge K(n))$$

and the Spanier-Whitehead map, one can check that $\kappa$ is obtained from smashing $K(n)$ with a map

$$b : EG_+ \wedge_G M^c \to F(EG_+ \wedge_G M^{-TM}, S),$$

whose adjoint is the preimage of $\delta \circ \zeta_0$ under the bijections

$$[(EG_+ \wedge G M^c) \times (EG \times G M)^{-TM}, S]_{S^G_\ast G}$$

$$\cong [\pi_0^\ast(EG_+ \wedge M^c) \wedge \pi_1^\ast(EG \times M)^{-TM}, S]_{G^2S^G_\ast}$$

$$\cong [EG_+^G \wedge \pi_0^\ast M^c \wedge \pi_1^\ast M^{-TM}, S']_{G^2S^G_\ast'}. \quad (4.15)$$

Here the first bijection is the adjunction of adjoint pair of taking orbit and assigning trivial action, and the second one is by the change of universe functor (Theorem 2.6.1). On one hand, by constructions, $j_\ast i_\ast \zeta_1$ and $\zeta_0$ are homotopic. It is also clear that $\delta \circ \zeta_2$ is the adjoint of $\delta \circ \zeta_1$. Hence, $b$ is homotopic to $\delta \circ \zeta_2$ and $\kappa$ is the smash product of $K(n)$ with the adjoint of $\delta \circ \zeta_2$.

On the other hand, the $G^2$-orbit of diagram 4.14 shows that $\zeta_2$ is the Pontryagin-Thom map of the diagonal map of $[M/G]$. Hence, the adjoint of $\delta \circ \zeta_2$ is $\psi^c_{[M/G]}$. It completes the proof of proposition 4.3.2.

4.5 Proof of the General case

Finally, we will prove that the map $\psi$ in (1.5) in Theorem 1.0.1 is a $K(n)$-equivalence when $X$ is a closed Deligne-Mumford Stack. To do so, we will define a relative version
of $\psi$ and show that it is a $K(n)$-equivalence by the fact that $X$ admits an open cover consisting of quotient stacks by finite group. Then the global $K(n)$-equivalence of $\psi$ follows from this local $K(n)$-equivalence and standard five lemma argument.

Suppose that $X$ is a Deligne-Mumford Stack. Throughout the proof, we fix a space $\text{Ho}(\mathcal{X} \times \mathcal{X})$ and a weak equivalence $\eta_{\mathcal{X} \times \mathcal{X}} : \text{Ho}(\mathcal{X} \times \mathcal{X}) \to \mathcal{X} \times \mathcal{X}$ to represent the homotopy type of $\mathcal{X} \times \mathcal{X}$. The pullback of $\eta_{\mathcal{X} \times \mathcal{X}}$ under $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ determines a weak equivalence $\eta_{\mathcal{X}} : \text{Ho}(\mathcal{X}) = \text{Ho}(\mathcal{X} \times \mathcal{X}) \to \mathcal{X}$.

Let

$$\psi : \Sigma^\infty \text{Ho}(\mathcal{X})_+ \to F(\text{Ho}(\mathcal{X})^{-T(\mathcal{X})}, \mathbb{S})$$

be the map (1.5) constructed in section 4.2. In defining $\psi$, which involves the Pontryagin-Thom map of the diagonal map $\Delta$ of $\mathcal{X}$, we constructed a crude embedding $e$ over $\Delta$ and a crude tubular neighborhood $\iota$ which fit into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\iota} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xleftarrow{\Delta} & \mathcal{X} \times \mathcal{X}
\end{array}
$$

In addition, we chose metrics $h_{\mathcal{E}}, h_{\mathcal{Y}}$ on the crude vector bundles $\mathcal{E}$ and $\mathcal{Y}$. This data constructed at stack level was passed to homotopy type by $\eta_{\mathcal{X} \times \mathcal{X}}$. The map $\iota$, vector bundles $W'_n \to \text{Ho}(\mathcal{X})_n, Q_n \to \text{Ho}(\mathcal{X} \times \mathcal{X})_n$ over exhaustions of $\text{Ho}(\mathcal{X}), \text{Ho}(\mathcal{X} \times \mathcal{X})$ and $\iota_n$ were defined and they determined a Pontryagin Thom map $\Delta'$ by proposition 2.2.1.

Suppose $\mathcal{U} \subset \mathcal{X}$ is a open substack. Let $\eta_{\mathcal{U}} : \text{Ho}(\mathcal{U}) = \mathcal{U} \times \mathcal{X} \text{Ho}(\mathcal{X}) \to \mathcal{U}$ be the pullback of $\eta_{\mathcal{X}}$ under $\mathcal{U} \hookrightarrow \mathcal{X}$ and identify $\text{Ho}(\mathcal{U})$ as a open subset of $\text{Ho}(\mathcal{X})$. Similarly, we define $\eta_{\mathcal{U}'}$ and $\text{Ho}(\mathcal{U}') \subset \text{Ho}(\mathcal{X} \times \mathcal{X})$ for any open substack $\mathcal{U}' \subset \mathcal{X} \times \mathcal{X}$.

Recall that the extra condition (4.7) was imposed in the construction of crude tubular neighborhood. The upshot is that the restrictions $\iota|_{\mathcal{U}} := \iota|_{\mathcal{E}|_{\mathcal{U}}}$ factors through
\[ \mathcal{Y}|_{\mathcal{X} \times \mathcal{U}} \] Hence we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}|_{\mathcal{U}} & \xrightarrow{\iota_{\mathcal{U}}} & \mathcal{Y}|_{\mathcal{X} \times \mathcal{U}} \\
\downarrow & & \downarrow \\
\mathcal{U} & \xrightarrow{\Delta_{\mathcal{U}}} & \mathcal{X} \times \mathcal{U}
\end{array}
\] (4.16)

For each \( \mathcal{U} \), pick a continuous function \( \rho_{\mathcal{U}} : \mathcal{U} \to [1, \infty) \) so that the image of \( D_\mathcal{U}(\mathcal{E}|_{\mathcal{U}}) \), the unit disk bundle of \( \mathcal{E}|_{\mathcal{U}} \) with respect to the metric \( \rho_{\mathcal{U}} \), under \( \iota \) is contained in \( \mathcal{Y}|_{\mathcal{U} \times \mathcal{U}} \subset \mathcal{Y}|_{\mathcal{X} \times \mathcal{U}} \). We require that \( \rho_{\mathcal{X}} \equiv 1 \).

We pass the data of the diagram (4.16) and the metric \( \rho_{\mathcal{U}} \) at stack level to homotopy type by \( \eta_{\mathcal{X} \times \mathcal{U}} \). The pullback of \( \Delta_{\mathcal{U}} \) under \( \eta_{\mathcal{X} \times \mathcal{U}} \) is a map \( \text{Ho}(\Delta_{\mathcal{U}}) : \text{Ho}(\mathcal{U}) \to \text{Ho}(\mathcal{X} \times \mathcal{U}) \). The exhaustions \( \{ \text{Ho}(\mathcal{U}) \cap \text{Ho}(\mathcal{X})_n \}_n \), \( \{ \text{Ho}(\mathcal{X} \times \mathcal{U}) \cap \text{Ho}(\mathcal{X} \times \mathcal{X})_n \}_n \) of \( \text{Ho}(\mathcal{U}) \) and \( \text{Ho}(\mathcal{X} \times \mathcal{U}) \) together with the corresponding restrictions of \( \epsilon, W'_n, Q_n, \iota_n \) determine a Pontryagin Thom map

\[ \Delta^!_{\mathcal{U}} : \Sigma^\infty \text{Ho}(\mathcal{X})_+ \wedge \text{Ho}(\mathcal{U})^{-T\mathcal{X}} \to \Sigma^\infty \text{Ho}(\mathcal{U})_+ \]

by proposition 2.2.1. The usual composition with collapse map and taking adjoint gives

\[ \Sigma^\infty \text{Ho}(\mathcal{X})_+ \to F(\text{Ho}(\mathcal{U})^{-T(\mathcal{X})}, \mathcal{S}). \]

By our choice of metric \( \rho_{\mathcal{U}} \) on \( \mathcal{E}|_{\mathcal{U}} \), this map factors through \( \Sigma^\infty \text{Ho}(\mathcal{X})_+ \to \Sigma^\infty \text{Ho}(\mathcal{X})/(\text{Ho}(\mathcal{X}) - \text{Ho}(\mathcal{U})) \) and so we get a map

\[ \psi_{\mathcal{U}} : \Sigma^\infty \text{Ho}(\mathcal{X})/(\text{Ho}(\mathcal{X}) - \text{Ho}(\mathcal{U})) \to F(\text{Ho}(\mathcal{U})^{-T(\mathcal{X})}, \mathcal{S}). \]

Since we require \( \rho_{\mathcal{X}} \equiv 1 \), \( \psi_{\mathcal{X}} = \psi \).

We want to prove that these relative versions of \( \psi \) are homotopy compatible with respect to inclusion of open substacks.

**Lemma 4.5.1.** Suppose \( \mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{X} \) are open substacks of \( \mathcal{X} \). Then the following
diagram
\[
\begin{array}{c}
\Sigma^\infty \text{Ho}(\mathfrak{X})/(\text{Ho}(\mathfrak{X}) - \text{Ho}(\mathfrak{U}_2)) \xrightarrow{\psi_{\mathfrak{U}_2}} F(\text{Ho}(\mathfrak{U}_2)^{-T(\mathfrak{X})}, \mathbb{S}) \\
\downarrow \\
\Sigma^\infty \text{Ho}(\mathfrak{X})/(\text{Ho}(\mathfrak{X}) - \text{Ho}(\mathfrak{U}_1)) \xrightarrow{\psi_{\mathfrak{U}_1}} F(\text{Ho}(\mathfrak{U}_1)^{-T(\mathfrak{X})}, \mathbb{S})
\end{array}
\]

commutes up to homotopy. Here the vertical maps are induced by the obvious quotient and inclusion.

Proof. To prove the lemma, we need to compare the constructions of the two compositions. Both of them are constructed from the following diagram

and the same metric \( h_{\mathfrak{U}} \). The only difference is that for the crude vector bundle \( E|_{\mathfrak{U}_1} \), we use the metric \( \rho_{\mathfrak{U}_2} h_E \) for the composition through top right corner and \( \rho_{\mathfrak{U}_1} h_E \) for the composition through bottom left one. Note that data \( \{ \text{Ho}(\mathfrak{U}_1) \cap \text{Ho}(\mathfrak{X}) \}_n \), \( \{ \text{Ho}(\mathfrak{X} \times \mathfrak{U}_1) \cap \text{Ho}(\mathfrak{X} \times \mathfrak{X}) \}_n \) and the corresponding restrictions of \( \epsilon, W'_n, Q_n, t_n \) are independent of the choices of the metric on \( h_E|_{\mathfrak{U}_1} \).

Since the space of continuous functions not less than one on \( \mathfrak{U}_1 \) is contractible, \( h_{\mathfrak{U}_1} \) and \( h_{\mathfrak{U}_2}|_{\mathfrak{U}_1} \) are homotopic. By the fact that the two compositions depend continuously on the metric on \( h_E|_{\mathfrak{U}_1} \), the lemma follows.

Lemma 4.5.2. For any \( \alpha \) and any open substack \( \mathfrak{U} \subset \mathfrak{X}_\alpha \), \( \psi_{\mathfrak{U}} \) is a \( K(n) \)-equivalence.

Proof. Since \( \mathfrak{X}_\alpha \simeq [M_\alpha/G_\alpha], \mathfrak{U} \simeq [U/G_\alpha] \) for some \( G \)-invariant open subset \( U \) of \( M_\alpha \). For simplicity, we write \( G = G_\alpha, \mathfrak{X}' = \mathfrak{X}'_\alpha, M' = M'_\alpha \).
Consider the following diagram. We will give the definitions of maps in it and show its commutativity below.

\[
\begin{align*}
K(n)_*(\text{Ho}(\mathcal{X})/(\text{Ho}(\mathcal{X}) - \text{Ho}(\mathcal{U}))) & \approx \psi_1 K(n)_*(\text{Ho}(\mathcal{X}')/(\text{Ho}(\mathcal{X}') - \text{Ho}(\mathcal{U}))) \rightarrow K(n)^-*(\text{Ho}(\mathcal{U})^{-T\mathcal{X}}) \\
K(n)_*(\text{EG} \times_G M'/(\text{EG} \times_G (M' - U))) & \approx \psi_2 K(n)_*(\text{EG} \times_G U) \rightarrow K(n)^-*((\text{EG} \times_G U)^{-TM}) \\
K(n)_*(\text{EG}_+ \wedge_G U^c) & \approx \psi_3 \end{align*}
\]

In the top triangle, the vertical map is induced by the obvious homeomorphism. \(\psi_1\) is obtained by applying Pontryagin-Thom construction to the diagonal maps \(\mathcal{U} \rightarrow \mathcal{X}' \times \mathcal{U}\) using the same data used in the construction of \(\psi_{\mathcal{U}}\), except that we replace \(\mathcal{W}|_{\mathcal{X} \times \mathcal{U}} \rightarrow \mathcal{X} \times \mathcal{U}\) by \(\mathcal{W}|_{\mathcal{X}' \times \mathcal{U}} \rightarrow \mathcal{X}' \times \mathcal{U}\) and perturb \(\iota|_{\mathcal{U}}\) outside the unit disc bundle \(D_{\mathcal{U}}(\mathcal{E}|_{\mathcal{U}})\) to make sure that the image of the new \(\iota|_{\mathcal{U}}\) lies in \(\mathcal{W}|_{\mathcal{X} \times \mathcal{U}}\). Nevertheless, this perturbation of \(\iota|_{\mathcal{U}}\) outside \(D_{\mathcal{U}}(\mathcal{E}|_{\mathcal{U}})\) has no effect on the Pontryagin-Thom map of the diagonal. Hence, the top triangle commutes.

In the bottom triangle, the vertical isomorphism is induced by

\[
(\text{EG} \times_G M)/(\text{EG} \times_G (M' - U)) = \text{EG}_+ \wedge_G (M'/((M' - U)) \approx \text{EG}_+ \wedge_G U^c.
\]

The last map is a homeomorphism because \(U\) is relatively compact in \(M'\). Both \(\psi_2\) and \(\psi_3\) are obtained from the Spanier-Whitehead maps for global quotient stacks described in the previous section, by applying Pontryagin-Thom construction to the diagonal maps \(\mathcal{U} \rightarrow \mathcal{X}' \times \mathcal{U}\) and \(\mathcal{U} \rightarrow \mathcal{U} \times \mathcal{U}\) respectively. The bottom triangle commutes for reason similar to that for the top triangle.

Then, we look at the middle square. Both \(\text{Ho}(\mathcal{X}') \rightarrow \mathcal{X}'\) and \(\text{EG} \times_G M' \rightarrow \mathcal{X}'\) represent the homotopy type of \(\mathcal{X}\). The left vertical map in the middle square is the
canonical isomorphisms induced by the weak equivalences

$$\text{Ho}(\mathcal{X}') \cong \text{Ho}(\mathcal{X}') \times_{\mathcal{X}'} (EG \times_G M') \cong EG \times_G M'.$$

Similarly, the right vertical map is the canonical $K(n)$-cohomology isomorphism induced by two homotopy type representatives of $\mathcal{U}$. Since both $\psi_1$ and $\psi_2$ are obtained from the Pontryagin-Thom construction of the map $\mathcal{U} \to \mathcal{X}' \times \mathcal{U}$, they commute with the canonical isomorphisms. This completes the proof of the commutativity of the diagram.

By the partial result lemma 4.3.3, $\psi_3$ is an isomorphism. The commutativity of the diagram implies that $(\psi_\mathcal{U})_*$ is an isomorphism.

We now complete the proof the main theorem.

**Proof of theorem 1.0.1.** First of all, we want to show the map $\psi$ constructed in section 4.2 reduces to (1.1) if $\mathcal{X}$ is a differentiable manifold and (1.4) if $\mathcal{X}$ is the quotient stack $[*/G]$ for a finite group $G$.

If $\mathcal{X}$ is a closed differentiable manifold $M$, we can choose the open cover $\{\mathcal{X}_\alpha\}_{\alpha \in I}$ to be the singleton $\{[M/1]\}$ and $\mathfrak{V}$ be the zero vector bundle in the construction of $\psi$. Then it is clear from section 2.3 that $\psi$ reduces to the Spanier-Whitehead duality for manifolds (1.1) in this case.

If $\mathcal{X} = [*/G]$ for a finite group $G$, $\psi$ is equal to $\psi_{[*/G]}^c$ of (4.12). Using the notations in section 4.4, the adjoint of (1.4) is the preimage of $\delta \circ \zeta_0$ under the bijection (4.15) in the special case $M = *$. By the analysis in the proof of proposition 4.3.2, the preimage is $\delta \circ \zeta_2$ and so (1.4) is the adjoint of $\delta \circ \zeta_2$, which is equal to $\psi_{[*/G]}^c$. This proves part (a).

For part (b), suppose $\mathcal{X}$ is a closed Deligne Mumford stack. Hence, there exists a finite open cover $\{\mathcal{X}_\alpha\}_{\alpha = 1}^n$ of $\mathcal{X}$ consisting of global quotients $\mathcal{X}_\alpha \simeq [M_\alpha/G_\alpha] \subset \mathcal{X}'_\alpha \simeq [M'_\alpha/G_\alpha] \subset \mathcal{X}$ such that $M_\alpha \subset M'_\alpha$ is relatively compact and $G_\alpha$ is finite.

By lemma 4.5.2, for any $1 \leq \alpha, \beta \leq n$, $\psi_{\mathcal{U}_\alpha}, \psi_{\mathcal{U}_\beta}$ and $\psi_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$ are all $K(n)$-equivalence. By looking at the Mayer-Vietoris sequence for the pair $\mathcal{U}_\alpha, \mathcal{U}_\beta$ and
applying five lemma, we prove that $\psi_{\mathcal{U}_\alpha \cup \mathcal{U}_\beta}$ is also a $K(n)$-equivalence. By induction, we can show that $\psi_{\mathcal{U}}$ is a $K(n)$-equivalence for $\mathcal{U}$ equal to a finite union of $\mathcal{U}_\alpha$. Since $\mathcal{X} = \cup_{\alpha=1}^n \mathcal{U}_\alpha$, part (b) is proved.

Part (c) follows immediately from part (b) by the Thom isomorphism of $K(n)$. See [17].
Chapter 5

Some Examples

In this chapter we will discuss a few examples of $K(n)$-duality of stacks. In section 5.1 our focus will be on stacks defined by finite groups. The intersection product on their $K(n)$-homology will be studied and theorem 1.0.7 will be proved. In section 5.2 we will look at the moduli stack of Riemann surfaces and define a fundamental class in the relative homology by bordification of the moduli stack.

5.1 Finite Groups

First let us fix our notations. Let $i : H \to G$ be an inclusion of finite groups. The same symbol $i$ is also used to denote the induced morphisms on stacks $[*/H] \to [*/G]$, on homotopy types $BH \to BG$ and on spectra $\Sigma^\infty BH_+ \to \Sigma^\infty BG_+$. The transfer map $\Sigma^\infty BG_+ \to \Sigma^\infty BH_+$ and its induced map $K(n)_*(BG) \to K(n)_*(BH)$ will both be denoted by $i_!$. Also, $[BH]$ represents both the $K(n)$-fundamental class of $BH$ in $K(n)_0(BH)$ and its image in $K(n)_0(BG)$ under $i_*$. We write $\psi_H : \Sigma^\infty BH_+ \to F(\Sigma^\infty BH_+, S)$ for the map (1.8) in the equivariant homotopy theory version of our main theorem for the case of trivial $H$-action on a point.

Next, we look at intersection of subgroups. Recall from definition 1.0.5 that two subgroups $H$ and $K$ of a finite group $G$ are said to intersect transversely if $HK := \{hk | h \in H, k \in K\} = G$. 

Lemma 5.1.1. Let $G$ be a finite group and $H, K$ be subgroups of $G$. The following are equivalent:

(i) $H, K$ intersect transversely;

(ii) $|H||K|/|H \cap K| = |G|$;

(iii) The quotient map $\pi : G/(H \cap K) \to G/H \times G/K$ is an isomorphism of left $G$-sets.

Proof. We will show each of (i) and (ii) is equivalent to (ii) by counting argument. To show (i)$\Leftrightarrow$(iii), let $H \times K$ acts on $HK$ by $(h, k)x = hxk^{-1}$. It is clear that it is a transitive action with the stabilizer of $e \in HK$ equals to $\{(a, a^{-1})|a \in H \cap K\}$, which has cardinality $|H \cap K|$. Hence, $|H||K|/|H \cap K| = |HK|$. As a result, $HK = G$ if and only if $|H||K|/|H \cap K| = |G|$.

For (ii)$\Leftrightarrow$(iii), it is clear that $\pi$ is injective. Hence, $\pi$ is a bijection, and hence an isomorphism of $G$-sets if and only if $|G/(H \cap K)| = |G/H \times G/K|$. The last condition is equivalent to (ii).

An important tool for us is the Mackey property. It is useful for showing the commutativity of diagrams involving transfer maps. Let $\pi : P \to Y$ be a finite covering. Consider $f : X \to Y$ and the pullback bundle diagram

$$
\begin{array}{ccc}
    f^* P & \xrightarrow{g} & P \\
    \downarrow f^* \pi & & \downarrow \pi \\
    X & \xrightarrow{f} & Y
\end{array}
$$

The Mackey property states that the diagram

$$
\begin{array}{ccc}
    \Sigma^\infty (f^* P) & \xrightarrow{\Sigma^\infty g} & \Sigma^\infty P \\
    \downarrow (f^* \pi)^! & & \downarrow (\pi)^! \\
    \Sigma^\infty X & \xrightarrow{\Sigma^\infty f} & \Sigma^\infty Y
\end{array}
$$

commutes.
We will be mainly interested in applying the Mackey property to covering map arising from subgroups of finite groups.

**Proposition 5.1.2.** Suppose $H$ and $K$ are transverse subgroups of a finite group $G$. Then there is a pullback diagram

$$
\begin{array}{ccc}
\ast/(H \cap K) & \xrightarrow{p} & \ast/K \\
q \downarrow & & \downarrow j \\
\ast/H & \xrightarrow{i} & \ast/G
\end{array}
$$

where all the maps in it are induced by inclusions of subgroups. There is also a commutative diagram of spectra

$$
\begin{array}{ccc}
\Sigma^\infty B(H \cap K)_+ & \xrightarrow{p} & \Sigma^\infty BK_+ \\
q^! \uparrow & & \uparrow j^! \\
\Sigma^\infty BH_+ & \xrightarrow{i} & \Sigma^\infty BG_+
\end{array}
$$

**(5.1)**

**Proof.** By lemma 3.1.14 and lemma 5.1.1, it is easy to see that $[\ast/H] \times_{[\ast/G]} [\ast/K]$ is equivalent to $[\ast/(H \cap K)]$ and all the maps in the pullback diagram are induced by inclusions. The commutativity of the second diagram follows from the Mackey property.

The Mackey property can be used to show the relation between $\psi_G$ and $\psi_H$ for $H \subset G$.

**Proposition 5.1.3.** Suppose $H$ is a subgroup of a finite group $G$ and $i : H \to G$ is the inclusion. Then there is a commutative diagram

$$
\begin{array}{ccc}
K(n)_*(BH) & \xrightarrow{(\psi_H)_*} & K(n)^{-*}(BH) \\
i^! \downarrow & & \downarrow i^* \\
K(n)_*(BG) & \xrightarrow{(\psi_G)_*} & K(n)^{-*}(BG)
\end{array}
$$

In particular, $i^!(\lbrack BG\rbrack) = [BH]$. 
Proof. Consider the following commutative diagram of group monomorphisms and its induced diagram on classifying spaces

\[
\begin{array}{ccc}
H & \xrightarrow{i} & G \\
\downarrow^{(i,1)} & & \downarrow^{\Delta_G} \\
G \times H & \xrightarrow{1 \times i} & G \times G
\end{array}
\quad \begin{array}{ccc}
BH & \xrightarrow{i} & BG \\
\downarrow^{(i,1)} & & \downarrow^{\Delta_G} \\
BG \times BH & \xrightarrow{1 \times i} & BG \times BG
\end{array}
\]

Since \(G \times H, \Delta_G(G)\) are transverse subgroups of \(G \times G\), proposition 5.1.2 implies that the second diagram is a pullback diagram. Also, the left vertical map of the first diagram is equal to the composite \(H \xrightarrow{\Delta_H} H \times H \xrightarrow{i \times 1} G \times H\) of monomorphisms. Hence, \((i, 1)^t = \Delta'_H(i \times 1)^t = \Delta'_H(i^t \times 1)\). By Mackey property, there is a commutative diagram

\[
\begin{array}{ccc}
\Sigma^\infty BH_+ & \xrightarrow{i} & \Sigma^\infty BG_+ \\
\Delta'_H \downarrow & & \Delta'_G \downarrow \\
\Sigma^\infty BH_+ \wedge \Sigma^\infty BH_+ & & \Sigma^\infty BG_+ \wedge \Sigma^\infty BG_+
\end{array}
\]

(5.2)

Return to the diagram in the proposition and consider the two composites in it. The map \((\psi_H)_* i^!\) is obtained from the composite

\[
\begin{array}{ccc}
\Sigma^\infty BG_+ \wedge \Sigma^\infty BH_+ & \xrightarrow{\Delta'_H(i^t \times 1)} & \Sigma^\infty BH_+ \\
\varepsilon_H \downarrow & & \varepsilon_H \downarrow \\
\Sigma^\infty BG_+ \wedge \Sigma^\infty BH_+ & \xrightarrow{i^t 1} & \Sigma^\infty BG_+ \wedge \Sigma^\infty BG_+
\end{array}
\]

by taking adjoint and \(K(n)\)-homology. Similarly, the other map \(i^*(\psi_G)_*\) is obtained from the composite

\[
\begin{array}{ccc}
\Sigma^\infty BG_+ \wedge \Sigma^\infty BH_+ & \xrightarrow{\Delta'_G (1 \times i)} & \Sigma^\infty BG_+ \\
\varepsilon_G \downarrow & & \varepsilon_G \downarrow \\
\Sigma^\infty BG_+ \wedge \Sigma^\infty BH_+ & \xrightarrow{1 \times i} & \Sigma^\infty BG_+ \wedge \Sigma^\infty BG_+
\end{array}
\]

by the same procedure. By the commutative diagram (5.2) and the fact that \(\varepsilon_G i = \varepsilon_H\), the two maps \(\Sigma^\infty BG_+ \wedge \Sigma^\infty BH_+ \to \mathbb{S}\) above are equal, so are \((\psi_H)_* i^!\) and \(i^*(\psi_G)_*\).

The last part of the proposition follows easily from the commutative diagram:

\[
i^1([BG]) = (\psi_H)_*^{-1} i^*(\psi_G)_* ([BG]) = (\psi_H)_*^{-1} i^*(1) = (\psi_H)_*^{-1}(1) = [BH].
\]
Let $\alpha \in K(n)_i(BG)$ and $\beta \in K(n)_j(BG)$. Recall that the intersection product $\alpha \cap \beta \in K(n)_{i+j}(BG)$ is defined by

$$\alpha \cap \beta = (\psi_G)_*^{-1}((\psi_G)_*(\alpha) \cup (\psi_G)_*(\beta)).$$

By Künneth theorem, $\alpha$ and $\beta$ defines an element $\alpha \otimes \beta \in K(n)_{i+j}(BG \times BG)$. It is related to $\alpha \cap \beta$ by the formula below.

**Proposition 5.1.4.** $\alpha \cap \beta = \Delta^!(\alpha \otimes \beta)$.

**Proof.** By proposition 5.1.3 and definitions of $\psi_G$ and $\psi_{G \times G}$, we have the commutative diagram

$$
\begin{array}{ccc}
K(n)_*(BG) & \xrightarrow{(\psi_G)_*} & K(n)^{-*}(BG) \\
\Delta^! \downarrow & & \Delta^! \downarrow \\
K(n)_*(BG \times BG) & \xrightarrow{(\psi_{G \times G})_*} & K(n)^{-*}(BG \times BG) \\
\cong \downarrow & & \cong \downarrow \\
K(n)_*(BG) \otimes K(n)_*(BG) & \xrightarrow{(\psi_G)_* \otimes (\psi_G)_*} & K(n)^{-*}(BG) \otimes K(n)^{-*}(BG)
\end{array}
$$

By the fact that the right vertical composite in the diagram defines cup product, the formula we want to prove follows from the commutativity of the diagram.

**Lemma 5.1.5.** Let $G$ be a finite group and $i : H \subset G$ be a subgroup. Then the composite

$$K(n)_*(BG) \xrightarrow{i_*} K(n)_*(BH) \xrightarrow{i_*} K(n)_*(BG)$$

maps $\alpha \in K(n)_*(BG)$ to $i_*i^!(\alpha) = \alpha \cap i_*([BH])$.

**Proof.** By taking $K(n)$-homology of (5.2) and applying proposition 5.1.4, there is a
CHAPTER 5. SOME EXAMPLES

commutative diagram

\[
\begin{array}{c}
K(n)_*(BH) \xrightarrow{i_*} K(n)_*(BG) \\
\cap_H \\
\downarrow \\
K(n)_*(BH) \times K(n)_*(BH) \\
\cap_G \\
\downarrow \\
K(n)_*(BG) \times K(n)_*(BH) \xrightarrow{1 \times i_*} K(n)_*(BG) \times K(n)_*(BG)
\end{array}
\]

Here we add subscripts to \(\cap\) to distinguish between the two intersection products in \(K(n)_*(BG)\) and \(K(n)_*(BH)\). The commutative diagram implies that

\[
\alpha \cap_G i_*([BH]) = i_*(i^j(\alpha \cap_H [BH])) = i_*i^j(\alpha).
\]

After all the preparation work, it is now easy to prove the intersection product formula \([BH] \cap [BK] = [B(H \cap K)]\) for transverse subgroups \(H, K\) of \(G\).

**Proof of theorem 1.0.7.** By proposition 5.1.2 and applying \(K(n)\)-homology to the diagram (5.1), we have the commutative diagram

\[
\begin{array}{c}
K(n)_*(B(H \cap K)) \xrightarrow{p_*} K(n)_*(BK) \\
\downarrow q^j \\
K(n)_*(BH) \xrightarrow{i_*} K(n)_*(BG)
\end{array}
\]

Together with proposition 5.1.3 and lemma 5.1.5, the commutativity of the diagram implies that

\[
j_*p_*([B(H \cap K)]) = j_*p_*q^j([BH]) = j_*j^1i_*([BH]) = i_*([BH]) \cap j_*([BK]).
\]

\[\square\]
5.2 Applications to Mapping Class Groups

Let $\mathcal{M} = \mathcal{M}_{g,r}$ denote the moduli stack of Riemann surfaces of genus $g$ and $r$ marked points. It is the quotient stack represented by the action of the mapping class group $\Gamma = \Gamma_{g,r}$ on the Teichmüller space $\mathcal{T} = \mathcal{T}_{g,r}$. $\mathcal{T}$ has a complex manifold structure of real dimension $6g - 6 + 2r$ and is contractible. Harvey [9] and Harer [8] constructed a Borel-Serre bordification of $\mathcal{T}$ for $n = 0$ and $n > 0$ respectively. It is a non-compact $(6g - 6 + 2r)$-dimension PL $\Gamma$-manifold $W$ with corners such that its interior is $\Gamma$-homeomorphic to $\mathcal{T}$. Also, the $\Gamma$-action on $W$ is properly discontinuous with compact quotient $W/\Gamma$.

It is known that $\Gamma$ is virtually torsion free, i.e. it contains a torsion free subgroup $N$ with finite quotient $G$

$$1 \rightarrow N \rightarrow \Gamma \rightarrow G \rightarrow 1.$$ 

Since the $\Gamma$-action on $W$ has finite stabilizer subgroups, $N$ acts freely on $W$. As a result, $W/N$ is a manifold with corners, and as in [3] we can write

$$B\Gamma \simeq (EG \times W)/\Gamma = EG \times_G (W/N).$$

Applying the version of theorem 1 for manifolds with boundary, we have

$$K^*(n)((EG \times W)/\Gamma) \cong K^*(n)(EG \times_G (W/N))$$

$$\cong K(n)_{6g-6+2r-*}(EG \times_G (W/N), EG \times_G \partial(W/N))$$

$$\cong K(n)_{6g-6+2r-*}((EG \times W)/\Gamma, (EG \times \partial W)/\Gamma).$$

The first and last term in the isomorphism above can be interpreted as $K(n)^*(\mathcal{M})$ and $K(n)_{6g-6+2r-*}(\mathcal{M}, \partial\mathcal{M})$ respectively. Since $W/N$ is compact and $G$ is finite, the terms have finite rank over $K(n)_*$. The image of $1 \in K^0(n)(\mathcal{M})$ defines a $K(n)$-fundamental class of $\mathcal{M}$ in $K(n)_{6g-6+2r}(\mathcal{M}, \partial\mathcal{M})$. To summarize,

**Proposition 5.2.1.** $K(n)^*(\mathcal{M}_{g,r}) \cong K(n)^*(B\Gamma_{g,r})$ has finite rank over $K(n)_*$. A $K(n)$-fundamental class $[\mathcal{M}_{g,r}]$ of $\mathcal{M}_{g,r}$ can be defined in $K(n)_{6g-6+2r}(\mathcal{M}, \partial\mathcal{M})$. 

Bibliography


