Eshelby’s solution for ellipsoidal inhomogeneous inclusions with applications to compaction bands

Chunfang Meng, David D. Pollard

Department of Geological and Environmental Sciences, Stanford University, Stanford, CA 94305-2115, the U.S., email: chun-fang.meng@shell.com, dpollard@stanford.edu.

Abstract

Eshelby’s solution for an ellipsoidal inhomogeneous inclusion in an infinite elastic body is applied to compaction bands and shear enhanced compaction bands in the Aztec sandstone at Valley of Fire State Park, NV. The inclusion and matrix are linear elastic and isotropic, and a remote stress field represents tectonic loading. Uniform eigen-strain in the inclusion accounts for inelastic compaction with porosity change from 25% to 10%. Differences in elastic moduli between the matrix and inclusion are based on laboratory data. We generalize earlier results, limited to 2D and axisymmetric geometries, by considering an ellipsoid with three unequal axes with intermediate and greatest axes different by a factor of 10, accounting for field observations. Stiffness contrasts between inclusion and matrix produce a modest concentration or diminution of the remote stress components, but plastic strains of 1% to 10%, due to compaction, produce a significant triaxial compressive stress concentration, which presumably is responsible for band propagation. We use an iterative algorithm to estimate the plastic strain and find that it is triaxial, but dominated by the normal strain acting across the inclusion. The stress diminution on the flank of a band is easily overcome by minor increases in the tectonic loading, enabling bands to be closely spaced relative to intermediate axis lengths. We use an iterative algorithm to identify the plastic strain in the shear-enhanced inclusion. If the plastic shear and normal strains are approximately equal, the ratio of shear to normal stress is about 1.3 at the tip.

Keyword:

Ecompaction band shelby solution heterogeneity Aztec sandstone

1. Introduction

Compaction bands are one category of narrow, zones of deformation found in porous granular sedimentary rock: the broader classification of such structures is known as deformation bands (Aydin (1978)). Those deformation bands that formed primarily by shearing (usually with measurable shear offset of crossing markers in outcrop) are called shear bands; those formed primarily by volume reduction (with no measurable shear offset) are called compaction bands (Aydin et al. (2006)). Another family of deformation band is known as dilation band introduced in Aydin et al. (2006). The occurrence of deformation bands is widespread across the U.S. Southwest (Davis (1999)), and they have been called “the most common strain localization feature found in deformed porous sandstones and sediments” (Fos-sen et al. (2007)). Their role in hydrocarbon reservoir permeability has been related to tectonic regime (Solum et al. (2012)), and microstructural characteristics (Ballas et al. (2012)). Here we focus on compaction bands (Mollema and Antonellini (1996)), but include some analysis relevant to bands that may have been enhanced by shearing (Baud et al. (2004) and Eichhubl et al. (2010)).

The sub-parallel compaction bands (Eichhubl et al. (2004), Sternlof et al. (2005) and Aydin and Ahmadov (2009)) in Valley of Fire State Park, NV are up to tens of meters in length (often terminated at dune boundaries), taper from up to 15 mm in thickness to zero at tips, and are spaced up to several meters apart. The Aztec Sandstone has a porosity of about 25% outside bands and about 10% inside bands. The bands have lower permeability than the surrounding matrix (Taylor and Pollard (2000) and Main et al. (2001)), and as a result, compaction bands can significantly affect the performance of sandstone reservoirs and aquifers (Sternlof et al. (2004), Sternlof et al. (2006) and Sun et al. (2011)).

In previous work (Katsman and Aharonov (2005), Katsman et al. (2006b) and Katsman et al. (2006a)) compaction bands are modeled using a tabular inclusion and spring network methods. Alternatively, the initiation of compaction bands has been modeled using bifurcation theory (Rudnicki and Rice (1975), Olsson (1999), Issen and Rudnicki (2000), Wong et al. (2001), Besuelle (2001), Borja and Aydin (2004), Rudnicki (2004) and Issen and Challa (2008)). In another conceptualization, compaction bands have been modeled using a combination of crack and dislocation to exploit elastic fracture mechanics results (Rudnicki and Sternlof (2005), Rudnicki (2007), Schultz (2009), and Tembe et al. (2008)). These theoretical studies have been motivated by and compared to laboratory experiments on compaction phenomena in porous rock (Olsson (1999), Baud et al. (2000), Mair et al. (2000), Olsson and Holcomb (2000), Baud et al. (2004) and Vajdova and Wong (2003)). To investigate their development and propagation,
we focus here on the ellipsoidal inhomogeneous inclusion of Eshelby, which allows one to assess the stress perturbation caused by the presence of a compaction band. We present Eshelby’s solution for (inhomogeneous) inclusion problems and its application in modeling compaction bands in porous sedimentary rock to form compaction bands and shear-enhanced compaction bands. Comparison between our results and Katsman et al. (2006a) is provided. We use empirical relationships between porosity and elastic moduli along with measured porosities of compaction bands to specify the contrast in the elastic properties of the ellipsoidal inhomogeneity. Remote stress fields are based on the geologic and tectonic history of the Aztec Sandstone. The volumetric plastic strain for the inhomogeneous inclusion is scaled to the porosity loss. The local stress perturbation is examined to understand the propensity for propagation along the elliptical tip-line of the inclusion and variations in the direction of pure compaction band propagation.

The true tip-line shape of compaction bands is seldom recorded because of lack of exposure. Nevertheless, field observations suggest that compaction bands are likely to have some eccentric shapes rather than penny shape. 3D Eshelby’s solutions allow one to investigate the effect of aspect ratio (i.e. axial ratio of an ellipsoidal inhomogeneity) to a compaction band’s formation.

For shear-enhanced compaction bands, we scale the shear component of the plastic strain to the shear stress at the tip. The resulting plastic shear strain matches the field measurements (Eichhubl et al. (2010)).

1.1. Eshelby’s solutions

Eshelby’s solutions enable one to evaluate the quasi-static elastic fields (stress, strain and displacement) inside and outside an arbitrary ellipsoidal inclusion (same elastic moduli as surrounding) that has experienced a prescribed uniform strain, or an ellipsoidal inhomogeneous inclusion (different moduli than surroundings) subjected to a prescribed remote state of stress (Eshelby (1957), Eshelby (1959) and Eshelby (1961)). These solutions have been applied in studies of rock deformation associated with fractures in sedimentary strata (Reches (1998)), pebbles in a conglomerate (Eidelman and Reches (1992)), faulting (Rudnicki (1977)), localized volume reduction structures such as compaction bands (Sternlof et al. (2005)), and alterations of regional stress by a reservoir with different elastic constants than the surrounding rock (Rudnicki (1999)). Eshelby’s solutions have also been used in modeling brittle fracture formation(Healy et al. (2006b), Healy et al. (2006a) and Meng et al. (2013)).

Because Eshelby’s solutions are rather complex and lack analytical form for an arbitrary ellipsoid, researchers have degenerated the geometry to special shapes such that analytical solutions could be formulated. Typically in previous research one of the ellipsoid’s semi-axes was set much shorter than the other two that were set equal, i.e. a penny-shaped spheroid. Reches (1998) and Healy (2009) developed computer codes to evaluate the solution for spheroids. Recently, we developed a Matlab™ code for arbitrary 3D ellipsoidal geometries (Meng et al. (2011)).

The inclusion problem described by Eshelby (1957) considers an ellipsoidal region in an infinite homogeneous, isotropic and elastic material known as the matrix (Fig. 1 A). When an inclusion undergoes a change in shape and size, its misfit with the surrounding matrix causes both regions to attain new strain/stress states. The definition of eigenstrain, $\epsilon^*$, summarized by Mura (1987)(p. 1), can be regarded as a uniform strain state that the inclusion will enter if removed from the constraint of the matrix. Eshelby (1957) referred to this as stress-free strain. Eshelby solved for the elastic fields inside and outside the ellipsoidal inclusion for a given eigenstrain and zero remote stress.

![Figure 1](image-url)
We denote the sub-domain occupied by the inclusion Ω in the infinite domain D, and the uniform eigenstrain $\epsilon^*_i$ is prescribed to be non-zero throughout Ω and zero in $D - \Omega$. The elastic moduli in the inclusion Ω and the matrix $D - \Omega$ are the same. The displacement, $u_i$, strain, $\epsilon_{ij}$, and stress, $\sigma_{ij}$ for both the inclusion and matrix are given by Mura (1987)(p. 11).

Eshelby (1957), Eshelby (1959) and Eshelby (1961) derived solutions for the interior and exterior elastic strain fields for the inclusion problem.

$$\epsilon_{ij} = S_{ijkl} \epsilon^*_{kl} \text{, for } x \in \Omega,$$

$$\epsilon_{ij}(x) = D_{ijkl}(x) \epsilon^*_{kl} \text{, for } x \in D - \Omega,$$

where $S_{ijkl}$ is known as the Eshelby tensor, Mura (1987) (p. 77); $D_{ijkl}(x)$ in general involves some incomplete elliptical integrals. Meng et al. (2011) evaluated the elliptical integrals numerically with great precision using Igor (2005). Because of this the method is referred to as quasi-analytical.

1.2. Equivalent inclusion method

The other problem solved by Eshelby (Mura (1987)) considers the ellipsoidal region to have different elastic moduli, i.e. $C_{ijkl}$ in Ω and $C_{ijkl}$ in $D - \Omega$, and the body is subjected to remote uniform stress $\sigma^o_{ij}$ (Fig. 1B). This is referred to as the inhomogeneous inclusion problem. Eshelby (1957) concluded that the stress perturbation of a uniformly applied stress due to the presence of an ellipsoidal inhomogeneous inclusion can be determined by an inclusion problem when a fictitious eigenstrain $\epsilon^o_{ij}$ is chosen properly. This is known as the equivalent inclusion method.

To solve for $\epsilon^o_{ij}$, Meng et al. (2011) rewrites Mura (1987) (22.13):

$$\left( \Delta C_{ijkl} S_{klmn} - C_{ijkl} \right) \epsilon^o_{mn} = -\Delta C_{ijkl} \epsilon^*_{kl} - C_{ijkl} \epsilon^o_{kl},$$

where $\Delta C_{ijkl} = C_{ijkl} - C_{ijkl}$; and from the relation $\epsilon^o_{ij} = C_{ijkl} C^o_{kl}$, we have the remote strain $\epsilon^o_{ij}$. $\epsilon^o_{ij}$ is an arbitrary initial eigenstrain that may be prescribed for the ellipsoidal inhomogeneous inclusion.

The interior and exterior stress and strain fields are as follows, Mura (1987) (22.8-22.13):

$$\epsilon_{ij} = \epsilon^o_{ij} + S_{ijmn} \epsilon^o_{mn},$$

$$\sigma_{ij}(x) = \epsilon^o_{ij} + D_{ijkl}(x) \epsilon^o_{kl}, \text{ for } x \in \Omega;$$

$$\epsilon_{ij}(x) = \epsilon^o_{ij} + D_{ijkl}(x) \epsilon^o_{kl}, \text{ for } x \in D - \Omega.$$

We model the compaction band problems with an ellipsoidal inhomogeneous inclusion that has axial-ratio $a_z \gg a_x$ and $a_y \gg a_x$, which gives suitable geometries for compaction bands (Mollema and Antonellini (1996), Sterlnof et al. 2005, Eichhubl et al. (2010)). The elastic moduli of the inhomogeneous inclusion are related to the physical changes, e.g. porosity loss and grain size reduction, and in general they are different than the moduli of the matrix. The infinite elastic body is under a tri-axial stress state with greatest principal compressive stress perpendicular to the (x, y)-plane of the highly eccentric ellipsoidal structure. For shear-enhanced compaction bands the greatest compressive stress is oblique to the band plane. The inelastic volume reduction is prescribed as the initial eigenstrain, $\epsilon^o_{ij}$. Analogous to Katsman et al. (2006a), we can set $\epsilon^o_{ij}$ to be uni-axial. In general however, e.g. when shearing across the band is clearly indicated, the initial eigenstrain is not uni-axial. Determining a realistic initial eigenstrain is addressed in Sections 4 and 4.2.

1.3. Compaction bands as tabular structures

A compaction band as a tabular structure illustrated in Fig. 2 has been modeled in 2D by Katsman et al. (2006a). The prescribed uni-axial strain is uniform throughout a rectangular region of initial size $2H \times 2c$ such that $\epsilon^o_x = (H - h)/H$. The remote stresses are zero, and the tabular structure has the same elastic moduli as the matrix. This is analogous to Eshelby’s inclusion solution illustrated in Fig. 1, but the shape of the inclusion is rectangular not ellipsoidal. Katsman et al. (2006a) cites the laboratory experiments by Haimson (2003) and Baud et al. (2004) and concludes: these test results “demonstrate that compaction bands would have rectangular geometries.” It has been found however, that an elliptical approximation of the compaction band cross-section produces a better fit to field data (Sterlnof et al. 2005) and Section 2.1.

In comparing the 3D ellipsoidal inclusion and inhomogeneous inclusion models to the 2D tabular inclusion model, we set the parameters such that the 3D model emulates the 2D model: the uni-axial eigenstrain $\epsilon^o_z = -(H - h)/H = -0.136$; the aspect ratio of the rectangular inclusion $H/c = 8 \times 10^{-4}$; and the axial-ratios of the ellipsoid $a_x/a_z = 20$, $a_y/a_z = 8 \times 10^{-4}$. Strictly speaking, $a_z$ should be infinite to match the 2D case, but we find as $a_z/a_x \geq 10$, the result does not vary significantly.

The rectangular and ellipsoidal inclusions have the same elastic moduli as the matrix, i.e. $C_{ijkl} = C_{ijkl}^*$ while
the ellipsoidal inhomogeneous inclusion has different moduli. The elastic moduli are sampled from porosity-modulus relations for given porosities (see Section 2.2). In the inclusion cases, we let both inclusion and matrix have a porosity of 25%. For the inhomogeneous inclusion case (see field data by Sternlof et al. (2005)), we set the porosities to \( \phi_i = 10\% \) for the ellipsoid and \( \phi_m = 25\% \) for the matrix. Similar to Sternlof et al. (2005) we set the remote stress \( \sigma_{x,x}^\infty = [-2 \times 10^5, -2 \times 10^5, -4 \times 10^5] \text{Pa} \), where \( i = x, y, z \).

Fig. 3 plots the normal stress \( \sigma_{zz} \) along the \( a_x \) axis for the 2D tabular model, elliptical inclusion model and elliptical inhomogeneous inclusion model. In both the 2D tabular and ellipsoidal inclusion cases the inside and outside stresses have opposite signs, tensile inside and compressive outside. Even though the prescribed plastic strain is stress free, the elastic matrix resists this uni-axial contraction and pulls back on the inclusion, inducing the tensile stress. The compression of the inclusion causes a local contraction of the matrix distal from the tips, thereby inducing a compressive stress.

Unlike the 2D tabular inclusion, the elliptical inclusion has uniform inside stress given by Eq. (3). Also, the stress immediately distal from the tips is not singular. The 2D tabular region with a uniform uni-axial strain is equivalent to two edge dislocations (Pollard and Fletcher (2006) p. 305), where the near-tip normal stresses on either side of the dislocation line are concentrated as \( \sigma_{zz} \propto r^{-1} \) and have opposite signs. In contrast, the stress at the tip of the ellipsoidal inclusion is finite due to the finite radius of curvature of the tip profile (see Section 6.1).

For the ellipsoidal inhomogeneous inclusion, the stress curve is increased by a constant value relative to that of the ellipsoidal inclusion. The normal stress inside is less than the remote stress because the plastic deformation induces some stress relief, but it remains compressive. The near-tip stress is finite and more compressive than the remote stress. Unlike the ellipsoidal inclusion, the ellipsoidal inhomogeneous inclusion is more consistent with the concept that the stress inside the region of compaction band should be compressive.

2. Compaction bands as ellipsoidal structures

Based on the field data for single bands that show significant loss of porosity (Sternlof et al. (2005)), we suggest that the ellipsoidal inhomogeneous inclusion is more suitable for modeling compaction bands than the tabular or ellipsoidal inclusions. In this section we review field data that document the cross-sectional shape of compaction bands and justify using the ellipsoidal shape. Also, we relate the changes in elastic moduli to the porosity losses (Avseth et al. (2005)) and justify using Eshelby’s inhomogeneous inclusion model.

2.1. Cross-sectional shape of compaction bands: field data

Sternlof et al. (2005) report data on the measured thickness distributions of sixteen compaction bands exposed tip to tip in outcrops of Astec Sandstone in the Valley of Fire State Park, NV. More than 1700 measurements on bands ranging in length from 1 m to 62 m show that a typical compaction band is thicker in the middle and tapers toward the tip. In Fig. 4, we apply elliptical and piece-wise linear fits to the length-width relations for eight measured bands.

For some of the compaction bands (left column in Fig. 4), the elliptical fit performs better, with an average correlation coefficient 0.89, while for some bands (right column in Fig. 4) the linear fit performs better, with an average correlation coefficient 0.86. Both elliptical and piece-wise linear fits perform significantly better statistically than the tabular shapes (Katsman et al. (2006a)), with uniform thickness approximated as the mean measured values, yielding correlation coefficients of only about 0.15.

Brgmann et al. (1994) suggest various mechanisms that may result the tapered shape of compaction profiles such as remote stress gradients, interaction with neighboring compaction bands and stiffer host rock. More tapered band profiles usually associate with lesser stress concentrations.

2.2. Elastic moduli of compaction bands and surroundings

The compaction bands from the Astec Sandstone that were examined petrographically (Sternlof et al. (2005)) have mean porosity of 11.9 ± 3%, whereas the sandstone outside the bands yields 24.5 ± 2.9%. This suggests that the material inside the bands could have different elastic moduli than outside (Gassmann (1951)). The elastic moduli (bulk modulus \( K \), Young’s modulus \( E \)) and Poisson’s ratio \( \nu \) as

\[
\frac{E}{\rho} = \frac{1}{K} + \frac{2}{\mu},
\]

\[
\nu = \frac{K - \mu}{2\mu},
\]

\[
\mu = \frac{K - \frac{4}{3}E}{2},
\]

\[
\rho = \text{mass density}.
\]
Figure 4: Compaction band thickness $t$ as a function of normalized length $x/a$; elliptical and piece-wise linear fits for the width-length relation and associated correlation coefficients, $c_e$ and $c_l$, as the fitness measurements (Yule (1919)).
functions of porosity $\phi$ for dry (subscript $d$) and wet (subscript $w$) sandstones are (Avseth et al. (2005)):

$$K_d = \frac{2G(1 + \nu_d)}{3(1 - \nu_d)}$$

where $G = G_0(1 - \phi/\phi_0)$,

$$K_w = K_d + \frac{1 - (K_d/K_0)^2}{\phi/K_1 + (1 - \phi)/K_0 - K_d/K_0},$$

where $K_0 = K_{d\phi=\phi_0}$,

$$E_d = 3K_d(1 - 2\nu_d), \quad E_w = 3K_d(1 - 2\nu_d),$$

$\nu_d = 0.07, \quad \nu_w = \frac{z - 1}{1 + 2z}$, where $z = \frac{3K_w}{2G}$,

where $\phi_0 = 0.4$ is the maximum porosity of the sandstone, $G_0 = 44$[GPa] is the shear modulus of solid quartz, and $K_1 = 2.2$[GPa] is the effective bulk modulus of water.

Fig. 11 plots the moduli and Poisson’s ratio of wet sandstone as functions of porosity $\phi$. Given these variations in elastic moduli and the measured reductions in porosity, a compaction band is more appropriately modeled as an inhomogeneous inclusion problem, using the equivalent inclusion method (see Section 1.2) than as an inclusion problem.

For demonstration purposes, we choose the curves of wet sandstone to sample the elastic moduli and Poisson’s ratios for given porosities. As a result, the porosity change has an impact on both elastic moduli and Poisson’s ratio. Laboratory data (Eichhubl et al. (2010)) show that the inelastic properties is likely to vary along the band by small amount.

In the following sections, we set the axial ratio, $a_z = 10^{-3} \sqrt{a_x a_y}$ (given the thickness-to-length ratios measured by Sternlof et al. (2005)). The elastic moduli are sampled from Fig. 11 for porosities inside and outside the compaction band $\phi_0 = 10\%$ and $\phi_m = 25\%$ respectively. The resulting elastic moduli are $E_m = 37.31$[GPa], $\nu_m = 0.13$ for the matrix and $E_h = 71.52$[GPa], $\nu_h = 0.084$ for the inhomogeneous inclusion. The Poisson’s ratio is below what is usually measured from Aztec Sandstone (Sternlof et al. (2005)), because by Eq. (4) (Fig. 11) the Poisson’s ratio of a wet sandstone is sensitive to its porosity.

3. Near-tip stress state for the ellipsoidal inhomogeneity

In order to evaluate the near-tip stress state we consider the flat ellipsoidal inhomogeneity with semi-axes $a_x \geq a_y \Rightarrow a_z$ and calculate the near-tip stress state at the end of the longest and intermediate semi-axes, $a_x$ and $a_y$, respectively. Following Sternlof et al. (2005) and Rudnicki (2007) continuity of the traction vector, $\mathbf{t}$, across the interface at the end of the longest semi-axis requires $t_x(a_x^+, 0, 0) = -t_x(a_x^-, 0, 0)$. The superscripts $+$ and $-$ indicate points an infinitesimal distance on either side of the interface at $x = a_x$. If follows that the normal stress components acting perpendicular to the interface at those points are equal: $\sigma_{xx}(a_x^+, 0, 0) = \sigma_{xx}(a_x^-, 0, 0)$. To simplify the notation, we name the normal stress on the left side of this equation $\sigma_{xx}^-$, using the superscript $b$ to refer to the stress in the model compaction band at the designated point. We name the normal stress on the right side of this equation $\sigma_{xx}^+$, using the superscript $t$ to refer to the stress adjacent to the tip of the band in the matrix. By continuity of the traction and these definitions:

$$\sigma_{xx}^b = \sigma_{xx}^t \quad \text{at} \quad x = a_x, \quad y = z = 0$$

The stress state in the ellipsoid is uniform, so $\sigma_{xx}^b$ is the normal stress in the $x$-direction throughout. The stress state in the matrix is highly inhomogeneous, so $\sigma_{xx}^t$ only refers to the stress at the designated point.

Displacement vectors, $\mathbf{u}$, of adjacent points across the bonded interface must be equal, so the tangential components of these displacements are equal. Using the same superscript notation as in Eq. (5) we have $u_x^u = u_x^t$ and $u_y^u = u_y^t$. It follows that spatial derivatives of these displacement components taken tangential to the interface must be continuous across the interface. These spatial derivatives are equivalent to the longitudinal strains parallel to the interface, so:

$$\epsilon_{yy}^b = \epsilon_{yy}^t \quad \text{and} \quad \epsilon_{zz}^b = \epsilon_{zz}^t \quad \text{at} \quad x = a_x, \quad y = z = 0$$

Hooke’s Law for the isotropic linear elastic matrix provides relations among the strain and stress components at the tip in terms of the shear modulus, $G$, and Poisson’s ratio, $\nu$ for the matrix (Pollard and Fletcher (2006), eqs. 8.18, 8.26):

$$\epsilon_{yy}^t = \frac{1}{2(1+\nu)}[\sigma_{yy}^t - \nu(\sigma_{xx}^t + \sigma_{yy}^t + \sigma_{zz}^t)]/(1 + \nu),$$

$$\epsilon_{zz}^t = \frac{1}{2(1+\nu)}[\sigma_{zz}^t - \nu(\sigma_{xx}^t + \sigma_{yy}^t + \sigma_{zz}^t)]/(1 + \nu)]$$

Rearranging these equations to put known quantities on the left sides, and simplifying terms with Poisson’s ratio:

$$2\mu(1 + \nu)(\epsilon_{yy}^b - \nu\epsilon_{zz}^b) = \sigma_{yy}^t - \nu\sigma_{zz}^t$$

$$2\mu(1 + \nu)(\epsilon_{zz}^b - \nu\epsilon_{yy}^b) = -\nu\sigma_{yy}^t + \sigma_{zz}^t$$

Employing the continuity conditions, Eq. (5) and (6), to eliminate the tip strains and stress from the left sides, and naming these known quantities $A$ and $B$, we have two equations in two unknowns, $\sigma_{xx}^t$ and $\sigma_{yy}^t$:

$$2\mu(1 + \nu)(\epsilon_{yy}^b + \nu\epsilon_{zz}^b) = A = \sigma_{yy}^t - \nu\sigma_{zz}^t$$

$$2\mu(1 + \nu)(\epsilon_{zz}^b + \nu\epsilon_{yy}^b) = B = -\nu\sigma_{yy}^t + \sigma_{zz}^t$$

Solving for the unknown stress components and using Eq (5), the three normal stress components are:

$$\sigma_{xx}^t = \sigma_{xx}^b, \quad \sigma_{yy}^t = \frac{A + \nu B}{1 - \nu^2},$$

$$\sigma_{zz}^t = \frac{A - \nu B}{1 - \nu^2}, \quad \text{at} \quad x = a_x^t, \quad y = z = 0$$

By symmetry, these are principal stresses. A similar derivation leads to the principal stresses at the end of the intermediate semi-axis, where $x = z = 0, \quad y = a_y^t$.

To investigate the near-tip stress state we use model parameters from Rudnicki (2007) as modified from Sternlof.
true triaxial loading, how can we interpret laboratory data the propagation of compaction bands. Since laboratory

\[
\begin{align*}
\sigma_{xx}^b, \sigma_{yy}^b, \sigma_{zz}^b &= [-29.1935, -29.1935, -40.0097] \text{ MPa} \\
\sigma_{xx}^t, \sigma_{yy}^t, \sigma_{zz}^t &= [-29.1935, -19.1905, -24.6033] \text{ MPa}
\end{align*}
\]

The x- and y-components of normal stress in the band are somewhat more compressive than the remotely applied stress, because the band is stiffer than the matrix. The z-component is nearly unchanged, because the band is so thin in that direction. At the tip the x-component matches that in the band, as anticipated from the first equation of Eq. (10), and is somewhat more compressive than the applied stress. Both the y- and z-components at the tip are somewhat less compressive than the applied stress, and less compressive than the band stress, because of the shielding effect of the stiffer band. Although the band is nearly twice as stiff as the matrix, the stress concentrations in the band due to this stiffness contrast are modest. Although the aspect ratio of the band is very large, \(a_x/a_z = 2 \times 10^3\), the concentration and diminution of remotely applied stress at the tip are modest. This result is dramatically different than the significant near-tip stress concentration of a crack with similar aspect ratio, (Meng et al. (2013)).

In contrast to the modest effects of stiffness contrast and aspect ratio, the effect of the plastic strain can be very significant. In Fig. 12 the stress components normalized by their counterparts in the remote field are plotted versus the plastic strain for the same model parameters defined above. For reference, the normalized stress components at zero eigenstrain are:

\[
\begin{align*}
\sigma_{xx}^b/\sigma_{xx}^t, \sigma_{yy}^b/\sigma_{yy}^t, \sigma_{zz}^b/\sigma_{zz}^t &= [1.4597, 0.9595, 0.6151].
\end{align*}
\]

The x-component is somewhat enhanced, whereas the y-component is diminished slightly and the z-component is somewhat diminished.

As the magnitude of the plastic strain goes to \(\epsilon_x^p = -0.1\), the normalized x-component decreases slightly, the y-component increases to more than 25 times the remote stress, and the z-component increases to more than 95 times the remote stress:

\[
\begin{align*}
\sigma_{xx}^b/\sigma_{xx}^t, \sigma_{yy}^b/\sigma_{yy}^t, \sigma_{zz}^b/\sigma_{zz}^t &= [1.3878, 25.5812, 95.4695].
\end{align*}
\]

A plastic strain of 10% within the model band overweights the stress changes due to the stiffness contrast and aspect ratio. The near-tip stress components are principal stresses and the state of stress is truly triaxial: \([\sigma_1^t, \sigma_2^t, \sigma_3^t] = [-27.7552, -511.6247, -3818.8] \text{ MPa}\). This brings up a number of questions about the propagation of compaction bands. Since laboratory testing is generally done with axisymmetric loading, not true triaxial loading, how can we interpret laboratory data to provide insight about compaction bands? The plastic (eigen) strain used here, and based upon conclusions of Sternlof et al. (2005), is uniaxial. Is uniaxial strain a likely outcome of true triaxial loading with principal stress differences as in this example? This result also emphasizes the need to estimate the plastic strain, which we address in the next section. Note, neither in the fields nor in experiments tips of compaction bands can be well observed.

4. Plastic strain in compaction bands: a heuristic and iterative estimation

The initial eigenstrain, \(\epsilon_{ij}^0\), is defined as the plastic strain caused by the compaction, as revealed for example, by porosity losses from the field samples (Sternlof et al. (2005)). Resolving this plastic strain from the porosity change is not a straightforward matter. The three principal components \(\epsilon_{ij}^p\) can be related to the porosity change by:

\[
(1 + \epsilon_{xx}^p)(1 + \epsilon_{yy}^p)(1 + \epsilon_{zz}^p) = 1 - (\phi_m - \phi_h) \tag{12}
\]

To determine all six components of \(\epsilon_{ij}^p\), we need five equations in addition to Eq. (12).

4.1. Plastic strain in pure compaction bands

Here, we focus on pure compaction bands, i.e. the ones formed without plastic shear deformation, Eichhubl et al. (2010). The strain components are \(\epsilon_{ij}^p = 0\), for \(i \neq j\), which reduces the unknowns to three. Still, two additional equations are needed to resolve the tri-axial strain.

An approach in previous research, e.g. Katsman et al. (2006a) and Sternlof et al. (2005), is to assume a uni-axial plastic strain that is non-zero in the compaction direction, e.g. \(\epsilon_{zz}^p \neq 0\) and the other components are zero. Eq. (12) is then reduced to:

\[
1 + \epsilon_{zz}^p = 1 - (\phi_m - \phi_h). \tag{13}
\]

For the parameters given in Section 2.2, the resulting uniaxial strain is \(\epsilon_{zz}^p = -0.15\).

If we assume an isotropic tri-axial strain such that \(\epsilon_{xx}^p = \epsilon_{yy}^p = \epsilon_{zz}^p\), Eq. (12) becomes:

\[
(1 + \epsilon_{zz}^p)^3 = 1 - (\phi_m - \phi_h). \tag{14}
\]

The resulting tri-axial strain is then \(\epsilon_{xx}^p = \epsilon_{yy}^p = \epsilon_{zz}^p = -0.0527\).

The thin section photos of compaction bands (from Sternlof et al. (2005) and Eichhubl et al. (2010)) suggest that the plastic deformation is indeed tri-axial, but more noticeable in the compaction (z) direction, e.g. \(\epsilon_{zz}^p < \epsilon_{xx}^p \approx \epsilon_{yy}^p < 0\). Neither the uni-axial strain nor the isotropic tri-axial strain can capture this plastic deformation thoroughly.

We propose an iterative algorithm that takes the tip-incipient stress and the plastic strain as input and output in a loop and seeks a convergent tri-axial strain. The rationale (illustrated in Fig. 13) is explained as follow: As

Stanford Rock Fracture Project Vol. 24, 2013 D-7
a compaction band forms and propagates, the tip-incipient matrix will join the band by undergoing the plastic strain. The tip-incipient stress is responsible for this plastic strain.

Here, we formulate a heuristic relationships between the tip stress and the plastic strain. One simple relationship is truncated linear, by which we scale the magnitudes of three principal strains $\epsilon^{0}_{ii}$ as:

$$
\epsilon^{*}_{xx} = \epsilon^{0}_{xx} - \sigma^{c} \frac{H(\sigma^{xx})}{\sigma^{xx} - \sigma^{c}} - \epsilon^{p},
$$
$$
\epsilon^{*}_{yy} = \epsilon^{0}_{yy} - \sigma^{c} \frac{H(\sigma^{xx})}{\sigma^{xx} - \sigma^{c}} - \epsilon^{p},
$$
$$
\epsilon^{*}_{zz} = \epsilon^{0}_{zz} - \sigma^{c} \frac{H(\sigma^{xx})}{\sigma^{xx} - \sigma^{c}} - \epsilon^{p},
$$

(15)

where $\sigma^{c}$ are the average principal stresses at the band tip. $\sigma^{c}$ is the critical principal stress above which plastic compaction will happen, $H(\cdot)$ is a Heaviside function that makes the strain state zero when $\sigma^{c}$. We use the uni-axial strain and tri-axial strain given by Eq. (13) and Eq. (14) as the initial guesses. For the model parameters given in Section 2.2 and damping factor $c$.

We use average values of $N$ discrete tip-incipient points to approximate $\sigma^{c}_{ij}$ by:

$$
\sigma^{c}_{ij} = \frac{\sum_{m}^{N} \epsilon^{(m)}_{ij} \Delta s_{m}}{\sum_{m}^{N} \epsilon^{(m)}_{ij} \Delta s_{m}} = \frac{\sum_{m}^{N} \epsilon^{(m)}_{ij} \Delta s_{m}}{\sum_{m}^{N} \epsilon^{(m)}_{ij} \Delta s_{m}},
$$

(16)

where $\epsilon^{(m)}_{ij}$ and $\Delta s_{m}$ are the m-th discrete point (Fig. 13).

With Eq. (12), Eq. (15) and Eq. (16), the principal plastic strains $\epsilon^{p}_{ii}$ can be resolved. When the new plastic strain is fed back to the model, a new tip stress is produced. By solving for $\epsilon^{p}_{ii}$ and $\sigma^{c}_{ij}$ iteratively we seek the convergent tip stress and plastic strain.

If we feed the new plastic strain directly to the iteration, the results will oscillate and never converge. Therefore, we combine the old plastic strain and the new one with a damping factor $\alpha \in [0, 1]$. The flow diagram for such an algorithm is given in Fig. 14.

We use the uni-axial strain and tri-axial strain given by Eq. (13) and Eq. (14) as the initial guesses. For the model parameters given in Section 2.2 and damping factor $\alpha = 0.3$, the convergent curves of the three principal strains are given in Fig. 15. The results suggest that no matter which initial condition we take, the strains always converge to the same three values:

$$
[\epsilon^{p}_{xx} \epsilon^{p}_{yy} \epsilon^{p}_{zz}] = [-0.0119 - 0.0149 - 0.1268]
$$

(17)

The resulting plastic strain is tri-axial with the normal component $\epsilon^{p}_{zz}$ greater than the other two principal components by a factor 10, suggesting this uni-axial component dominates the in-band plastic strain. We also find (not shown here) the near-tip strain component $\epsilon^{p}_{z}$ is not sensitive to the axial ratio $a_{i}/a_{j}$ of the ellipsoid.

4.2. Plastic strain in shear-enhanced compaction bands

We could scale the shear components of $\epsilon^{p}_{ij}$ linearly with the average tip shear stresses $\sigma^{c}_{ij}$ that exceeds a critical shear stress $\sigma^{c}_{th}$, similarly to Eq. (15). Unlike the principal plastic strains however, the shear strains do not affect the volume, and there is no analogous relation to Eq. (12) that would provide enough equations to resolve the shear strains. Instead, we simulate the shear deformation as an incremental process with a varying shear modulus. This allows one to relate the in-band plastic shear strain and the near-tip stress explicitly.

During the plastic shear deformation the effective shear modulus $G_{p}$ increases from the matrix shear modulus $G_{m}$ to the in-band shear modulus $G_{th}$ linearly with the plastic shear strain $\epsilon^{p}_{ij}$ (Fig. 16 for the remote stress rotated around the y-axis). Field measurements (Eichhubl et al. (2010)) suggest that the in-band plastic shear strain $\epsilon^{p}_{zz}$ has about the same magnitude as the plastic normal strain due to compaction $\epsilon^{p}_{zz}$. We let the effective shear modulus equal the in-band shear modulus $G_{p}$ when $\epsilon^{p}_{zz}$ reaches $\epsilon^{p}_{zz}$.

We relate the in-band plastic shear strain to the near-tip stress as follow:

$$
\epsilon^{p}_{ij} = \epsilon^{(n-1)}_{ij} + \frac{(\bar{\sigma}^{c}_{ij} - \sigma^{(n)}_{sh})}{G_{p}^{(n)}} H(\bar{\sigma}_{ij} - \sigma^{(n)}_{sh}),
$$
$$
\epsilon^{(n)}_{sh} = \epsilon^{(n-1)}_{sh} + \frac{G_{p}^{(n)}}{G_{p}^{(n-1)}},
$$

(18)

where $n$ is the iteration number (see Section 4); $H(\cdot)$ is the Heaviside function. The initial values of the plastic shear strain $\epsilon^{p}_{ij}$ (i.e. shear components of the eigenstrain) and the critical shear stress $\sigma^{(th)}_{sh}$ (for plastic shear deformation to happen) must be specified. We incorporate Eq. (18) into the iteration (Fig. 13 in Section 4) resolving all six components of the plastic strain $\epsilon^{p}_{ij}$.

The relative orientations between the pure and shear-enhanced compaction bands (Eichhubl et al. (2010)) suggest that the angle between the most compressive remote stress $\sigma^{(c)}_{sh}$ and the normal vector of the shear-enhanced compaction band is about 45° (Eichhubl et al. (2010) Figure 8, Fig. 5A). We rotate the tri-axial remote stress ($\sigma^{(c)}_{xx} = \sigma^{(c)}_{yy} = -20$ MPa, $\sigma^{(c)}_{zz} = -40$ MPa) around the y-axis, which creates shear stress $\sigma^{c}_{xz}$ at the tip. The initial critical shear stress is $\sigma^{(th)}_{sh} = 20(\sigma^{(c)}_{sh} - \sigma^{(c)}_{xx})/2$, i.e. twenty times the maximum remote shear stress, and we gradually increase the assumed initial shear strain $\epsilon^{p}_{zz}$ in $[0.01, 0.25]$. The resulting in-band plastic strain components $\epsilon^{p}_{ij}$ as functions of the iteration number $n$ are plotted Fig. 17.

The three normal components of $\epsilon^{p}_{ij}$, similarly to Fig. 15, quickly converge to a constants, and the shear components, except $\epsilon^{p}_{zz}$, are zero. When the initial value of the shear strain $\epsilon^{p}_{zz}$ is too small, it will not vary with $n$. This means that the shear stress at the tip is less than the critical shear stress required for plastic shear deformation. When the initial shear strain is too large, it will increase with $n$ without converging to a constant: the shear stress at the tip always triggers even larger in-band plastic shear deformation. The in-band plastic shear strain we seek is the one that
would cause the correct amount shear stress concentration, which in turn produces the exact shear strain, i.e. the largest strain converging to a constant \( \varepsilon_{xz} \approx 0.15 \) by Fig. 17. Field measurements (Eichhubl et al. (2010)) suggest that \( \varepsilon_{xz} \) and \( \varepsilon_{x} \) are of the same magnitude, which is consistent with this result.

5. Compaction bands under symmetric remote stress

Arrays of sub-parallel to anastamosing compaction bands crop out in the upper half of the Aztec Sandstone in the Valley of Fire State Park (Fig. 6). Their outcrop traces vary from a few to more than 100m, and spacings vary from a few decimeters to more than 10m. In some cases the bands are clustered with spacings from zero (touching) to a few centimeters. Band segments may curve in proximity to adjacent segments forming eye shaped structures (Fig. 6 B). Despite these local irregularities, in many outcrops the bands form a systematic and symmetric set with approximately parallel members. The approximate parallelism of the members of a set indicates that if the remote stress field at a scale larger than the band set is homogeneous, the bands were formed approximately parallel and perpendicular to the principal stresses. This motivates consideration of a model based on the Eshelby inhomogeneous inclusion under symmetric remote stresses.

In the following examples we investigate the ellipsoidal inhomogeneous inclusion with \( a_x \ll a_z \) and \( a_x \ll a_y \) subjected to a remote stress (Sternlof et al. (2005)) \( \sigma_{ii}^{\infty} = [-2, -2, -4] \times 10^7 \) Pa, where \( i = x, y, z \). The initial plastic strain \( \varepsilon_{ij}^p \) within the inclusion is determined by the iterative algorithm in Fig. 14 with all shear (off-diagonal) components zero. The elastic constants are determined as in Section 2.2 for the inclusion and matrix.

We evaluate the elastic fields in three different observation grids illustrated in Fig. 18. On the cross-section grid in the \( (x-o-z) \) plane we illustrate how the stress is concentrated near the tip and diminished near the flank; in the \( (x-o-y) \) plane we illustrate how the stress varies along the tip-line, and on the near-tip grid \( (r-o-x) \) we focus on stress perturbation around the tip.

5.1. Three-dimensionality of a pure compaction band

In previous research, Healy (2009) and Reches (1998) evaluated the special case of the spheroidal inhomogeneities, where two of the three semi-axes are equal, \( a_x = a_y = a_z \). Under symmetric remote stress states (principal stresses parallel to the semi-axes of the ellipsoid), the resulting stress is uniformly distributed along the circular tip line. The three-dimensionality of the ellipsoidal inhomogeneous inclusion allows one to investigate the stress variation along the tip when \( a_y/a_z \neq 1 \) (Meng et al. (2011)).

We plot the normalized major (compressive) principal stress \( \sigma_3/\sigma_3^\infty \) for different axial-ratios \( a_y/a_z = [0.5 \ 1 \ 2 \ 20] \) about the tip (at longitude \( \theta = 0 \), i.e. an end of the semi-axis \( a_x \) on the cross-section plane \( y = 0 \) (first row in Fig. 7). As the ellipsoid stretches in the \( a_x \) direction, the near-tip (\( \theta = 0 \)) stress becomes more compressive, while the near-flank stress becomes less compressive. This indicates that the regions near a band tip are more likely to host new bands. Also, a band is somewhat more likely to propagate from the tip with a shorter axis (e.g. \( a_y \) in the fourth column of

**Figure 5:** A, Wavy compaction bands (large arrows indicating the compaction directions) and shear-enhanced compaction bands (small arrow indicating the shear directions); B, Mutual joining relations for one set of shear-enhanced bands and wavy bands indicate both were active under the same stress field; C, Mutual joining relations for two sets of shear-enhanced bands suggest both were active under the same stress field.
Figure 6: Pure compaction bands at Valley of Fire State Park, NV (Sternlof et al. (2005)): A, sub-parallel set with some anastamosing and typical spacing of several decimeter (some of them are very closely spaced); B, sub-parallel compaction bands with eye-structures.

Fig. 7). Furthermore, new parallel bands are more likely to form adjacent to a band with lesser axial ratio, because the region of stress diminution is smaller (e.g. first column Fig. 8).

We plot the normalized principal stress about the tip at $\theta = 0$ on the circle ($\alpha', r = 10^{-2} \sqrt{a_x a_y}, \lambda$) (first row in Fig. 9). The bi-modal distributed stress is symmetric in $\lambda$ for a symmetric remote stress. The stress concentration is less along the extension of the band ($\lambda = 0$) and increases to maxima to either side of the band plane ($\phi = \pm 61^\circ$). The stress concentration is greater at the ends of the intermediate semi-axis (e.g. fourth column Fig. 7) than at the ends of the major semi-axis (e.g. first column Fig. 9). Note, when $a_y/a_x$ shifts from 0.5 to 2, $a_y$ becomes the intermediate semi-axis, and $a_x$ becomes the major semi-axis.

5.2. Influence region of a pure compaction band

Because of the stress relieved by the plastic deformation (Fig. 7), the rock on the flanks of a pure compaction band is under less compressive stress than in the remote field, whereas in front of the tips it is under greater compression. The concentration immediately in front of the tip provides the stress required for propagation (see Section 4). Here we focus on the diminution to evaluate band spacing.

When the compaction band heights are much less than the lengths, e.g. because they are confined by the dune boundaries Fig. 19, the regions of diminution scale with the heights rather than the lengths. On some nearly horizontal outcrops bands are closely spaced relative to their lengths, but not too closely spaced relative to their heights, e.g. in Fig. 19, the bands are less than two meters in height while tens of meters in length. In these instances the region of stress diminution may determine the spacing. In other examples bands are spaced more closely than even their shortest in-plane dimension. In these instances we must seek another explanation for spacing.

Although, the isolated (existing) band would have a greater stress near its tip, but the increasing remote stress would not cause it to propagate if the tip has stopped because of interaction with other parallel bands or with a dune boundary. Thus it is reasonable to consider the stress increase away from the flank and ask, as we have, what increase in remote stress in necessary to promote the initiation and propagation of a closely spaced new band?

For a new band to form parallel to the flank of an existing one, the compressive stress there must exceed the critical value $\sigma'$. The close spacing of the sub-parallel compaction bands observed in the field (6) suggests that the stress diminution on a band flank must be overcome easily by an increasing tectonic stress.

We use a remote compressive stress acting normal to the band that increases in time to simulate an increase in tectonic compression, perhaps due to slip on the major thrust faults in the region. The principal stress $\sigma_{zz}$ along the z-axis under increasing $\sigma_{zz}'$ is plotted in Fig. 20. When
Figure 7: Normalized principal stress $\sigma_3/\sigma_{\infty}^3$ evaluated the cross-section plane $y = 0$ for axial-ratio $a_y/a_x = [0.5 1 2 20]$ (column), under a pure tensile remote stress ($\eta_y = 0$) and under the stress rotated around $y$-axis by $\eta_y = [15 30 45]^\circ$ (row).
Figure 8: Normalized major principal stress $\sigma_3^\infty/\sigma_3^\infty$ and circumferential stress $\sigma_\lambda^\infty/\sigma_3^\infty$ evaluated on the near-tip grid about $[a, 0, 0]$ (i.e. an end of the long semi axis) under remote stress $\sigma_\infty = [20 \ 20 \ 40]$ MPa and under this stress rotated round $y$-axis by angle $\eta_y = [15^\circ \ 45^\circ]$. 
Figure 9: Normalized principal stress $\sigma_3/\sigma_\infty$ evaluated on a circle ($r = 10^{-2} \sqrt{a_x a_y}, \lambda$) around the tip at longitude $\theta = 0$ for axial-ratio $a_y/a_x = [0.5 1 2 20]$ (column), under a pure tensile remote stress ($\eta_y = 0$) and under the stress rotated around y-axis by $\eta_y = [15 30 45]^\circ$ (row)
the normal remote stress becomes slightly more compressive, e.g. by 10%, the compaction condition $\sigma_{zz} = \sigma'$ comes much closer to the band flank. This indicates that while the strain accommodation capability of a single compaction band is quickly consumed in a tectonic process, a closely spaced parallel band (or band set) may form to further accommodate the remote stress build up.

5.3. Propagation of a pure compaction band

We investigate propagation of the compaction band under symmetric loading based on the near-tip principal stresses (first rows of Fig. 7 and Fig. 9). Such a near-tip stress distribution is similar to that of a mode I crack (Williams (1957) and Lawn and Wilshaw (1975), P.55), but of opposite sign (compressive instead of tensile).

The bimodal distribution raises a question: why do compaction bands choose not to branch following the two maximum compression planes, but rather propagate in between these planes along a more-or-less straight path ($\lambda = 0$)? This can be explained by asserting that a compaction band can only propagate from its tip. This assertion has been employed (e.g. Erdogan and Sih (1963)) in studies of mixed-mode I II crack propagation. By Lawn and Wilshaw (1975) (Fig. 3.5), a mode I crack tip, analogous to pure compaction bands, has the circumferential stress $\sigma_{\lambda\lambda}$ peaking at $\lambda = 0$. However, the above question remains since $\sigma_{\lambda\lambda} < \sigma_3$. The crack here is referred to as main crack rather than the micro-cracks developed in from of process zone away from the tip (P.T. Delaney (1986)).

For each instance of the principal (most compressive) stress $\sigma_3$ evaluated on a circle about the tip (Fig. 18B) using the polar coordinates ($\alpha', r, \lambda$), there is a plane $L$ passing through the evaluation point and perpendicular to $\sigma_3$. Among these planes there is a special one, in between the two planes associated with the two peak compressive stresses, that contains the origin $\alpha'$. In other words $|\alpha'\alpha'| = 0$, where $\alpha''$ is the perpendicular projection of $\alpha'$ on the plane $L$. By the foregoing assertion, this special plane contains the next increment of the propagating band originated from $\alpha'$. For symmetric loadings the special plane aligns with the plane of the band, i.e. the $x-o-y$ plane. Deviation from this path could be caused by local heterogeneity of the rock properties (e.g. cross bedding, dune boundaries and other bands) or changes in the local stress orientations due to mechanical interaction with nearby bands.

6. Compaction bands under asymmetric remote stresses

Under symmetric remote stress, a pure compaction band, once formed, will tend to propagate in plane, as described in Section 5.3. Field observations suggest however, that some bands offset the sedimentary layering or older bands, indicating they accommodated some shearing (Eichhubl et al. (2010)). This brings up the possibility that in the process of compaction band formation and propagation, the plane on which the major principal (compressive) stress acts would not align well with the band plane. We model this misalignment by rotating the remote stress $\sigma_0^{\infty}$ around the $y$-axis by an angle $\eta$. Localized stress perturbations on scale of 1 to 10 meters would provide more realistic conditions for this misalignment, however this is beyond capability of Eshelby’s solution.

We assume that a compaction band already is formed within the major principal stress plane, and its plastic strain $\epsilon_0^{\infty}$ does not change with the remote stress rotations. Thus, we let the band have a fixed plastic strain generated under the symmetric remote stresses (produced by the iteration scheme shown in Fig. 14). For this plastic strain and different rotation angles $\eta$, we evaluate the near-tip stress fields and try to relate them to observed band geometries. This addresses the possible change in propagation direction of a pure compaction band that experiences a change in the remote stress orientation.

In addition to the model parameters given in Section 2.2, we investigate the case of an axial ratio $a_x/a_z = 0.5$ in more detail. The remote stress rotation around the $y$-axis would create the local stress at $[a_x, 0, 0]$ analogous to a crack tip under mixed mode I II loading (Lawn and Wilshaw (1975), Chapter 3).

6.1. Three-dimensionality of a shear-enhanced compaction band

Similarly to the symmetric case, we plot the near-tip normalized principal stresses under different asymmetric loadings, i.e. for the rotation angle $\eta = [15, 30, 45]^\circ$, on the Cartesian and circular coordinates (Fig. 18). The stress distributions on the cross section plane $y = 0$ (Fig. 7 second through fourth rows) tilt following the remote stress rotation, and the magnitudes slightly decrease. This indicates that under asymmetric stresses a compaction band would tend to propagate off-plane, or stop due to the lesser stress.

The tip stress variations are also shown on the polar coordinates (Fig. 9 second through fourth rows). The most compressive stress distribution shifts toward in positive $\lambda$ with the remote stress rotation.

For $a_x/a_z = 0.5$, we zoom in to the tip at $\theta = 0$ and plot the normalized principal stress $\sigma_3/\sigma_0^{\infty}$ (first column in Fig. 8). To have a quantitative indication of the off-plane growth tendency we plot the normalized circumferential stress $\sigma_{\lambda\lambda}/\sigma_0^{\infty}$ with respect to the tip tangent vector $n_\eta = [0, 0, 1]$ (second column in Fig. 8). The stress contour tilts following the rotation. This is consistent with an off-plane growth tendency which is better indicated by the maximum traces of the contours of circumferential stress $\sigma_{\lambda\lambda}$ such that the off-plane angle is roughly equal to $\eta$.

6.2. Propagation of shear-enhanced compaction band

If we analyze the near-tip stress analogous to a mode I II crack (see Erdogan and Sih (1963)), the band, with certain conditions met, will propagate perpendicular to the
maximum circumferential stress. For example the tip (second and third rows in Fig. 8) would kink along the maximum trace of the \( \sigma_{x,z} \) contours.

The fact that the tip-incipient material undergoes a plastic strain and joins the compaction band, and that the stress along the tip-line is finite, motivate a propagation criterion different than the classic fracture mechanics criterion for a crack.

Eichhubl et al. (2010) suggest that shear stress may **enhance** the compaction band formation. In Fig. 10, we plot the near-tip maximum shear stress \((\sigma_3 - \sigma_1)/2\) and circumferential shear stress \((\sigma_{r,z})\) normalized by the remote maximum shear stress on the near-tip grid (at longitude \( \theta = 0 \)). The maximum shear stress has distributions similar to the major principal stress in Fig. 8 (comparing the first columns).

When a compaction band forms, the band plane may not be aligned precisely with the plane normal to the greatest compression because the Aztec Sandstone is anisotropic due to cross bed layering, and it is heterogeneous due to sand dune boundaries. As a result, the stress \( \sigma_{x,z} \) (\( \sigma_{r,z} \)) will not be symmetric (anti-symmetric), e.g. about \( \alpha_i \) in Fig. 8 (Fig. 10). For example, one of the two anti-symmetric wings of \( \sigma_{x,z} \) becomes weaker. This, combined with a kinked \( \sigma_{x,z} \), could enhance the compaction about the other stronger wing.

If the shear stress indeed enhances the compaction deformation, the direction in which the band tip would advance off-plane is likely to be in between the maximum trace of \( \sigma_{x,z} \) and the stronger wing of \( \sigma_{x,z} \). This leads to a kink angle that overshoots the stress rotation angle \( \eta_i \). When the new tip grows, the remote stress may **correct** this overshoot by turning the tip back toward its former orientation.

The stress fields calculated using Eshelby’s inhomogeneous inclusion combined with some propagation criteria can be used to predict the incipient direction of out-of-plane propagation. Unfortunately because of its geometrical limitation, if the compaction band develops into some non-ellipsoidal geometry, Eshelby’s solution will cease to apply.

7. Stress condition for shear-enhanced compaction bands

Plastic shear deformation is inferred by Eichhubl et al. (2010) along some planar compaction bands in Valley of Fire, based on offset bedding and force chains. Relationships among shear-enhanced compaction bands suggest that two orientations of these bands, presumably symmetric and oblique to the principle stress plane, developed at the same time (Fig. 5). In particular individual bands from one set can be traced continuously through a kinked path to become parallel to the second set, and vice versa. The in-band plastic shear strain and normal strain are both about 10\% as inferred by Eichhubl et al. (2010). In Section 4.2, an iterative method is used to find the in-band plastic shear, and the result is consistent with Eichhubl et al. (2010).

Here, we focus on the tip-incipient stress that would cause this shear-compaction as observed. We impose the relation \( e_{c,z}^p/\sigma_{z}^p = c \) for \( c \in [0.1, 1.5] \) and set the remote stress the same as in Section 4.2, i.e. \( \sigma_{z}^\infty \) in 45° to the band. A penny-shaped band shape is assumed in addition to other parameters as in Section 2.2.)

The resulted shear stress \( \sigma_{x,z} \) and stress ratio \( \sigma_{x,z}/\sigma_{z} \) behave linearly with the strain ratio \( e_{x,z}^p/\sigma_{z}^p \). If (as mentioned by Eichhubl et al. (2010)) \( e_{x,z}^p/\sigma_{z}^p \approx 1 \), the corresponding tip stress ratio \( \sigma_{x,z}/\sigma_{z} \) is approximately 1.3.

8. Conclusion and discussion

Eshelby’s solution for the ellipsoidal inhomogeneous inclusion offers unique insight in modeling localized compaction in porous rock, e.g. compaction and shear-enhanced compaction bands. Varying the aspect ratio \((a_y/a_z)\) in the plane of the band enables us to investigate the formation of compaction bands in 3D.

The compaction bands observed and mapped at Valley of Fire, NV are depicted as highly eccentric ellipsoids, which show more resemblance to the field observations than the tabular inclusion model of Katsman et al. (2006a). Also, the inhomogeneous inclusion model keeps in-band stresses compressive unlike the tabular inclusion, where the in-band normal stress is tensile. Further studies are needed to explain why some compaction bands have more tapered profiles than others.

We formulate truncated linear relationships between the plastic strains and the tip-incipient stresses, Eq. (15) and Eq. (18). These heuristic relationships are subject to improvements, as one further explores the plastic stress-strain relationships, but they admit an estimation of the plastic strain state within the band, which is required as input for the Eshelby inhomogeneous inclusion model.

This model is consistent with a shear-compression stress ratio \((c = 1.3)\) under which shear-enhanced compaction bands are favored.

**Acknowledgments:** We thank Atilla Aydin for introducing us to compaction bands in the field; Peter Eichhubl for the concept of shear-enhanced compaction bands; Kurt Sternlof and John Rudnicki for field and theoretical guidance with respect to compaction bands. This research was supported by grant DE-FG02-04ER15588 from the Department of Energy, Basic Energy Sciences Program and the Stanford University Rock Fracture Project.

**References**


Figure 10: Near-tip maximum shear stresses (left) and circumferential shear stresses (right) normalized by remote maximum shear stress under remote stress $\sigma^p_{ii} = -[20 20 40]$ MPa and under this stress rotated about y-axis for angle $\eta_y = [15^o 45^o]$.


Figure 11: A. Bulk modulus $K_{d(w)}$ and Young’s modulus $E_{d(w)}$ plotted versus porosity $\phi$; B. Poisson’s ratio $\nu_{d(w)}$ versus $\phi$ for dry (d) and wet (w) sandstones.
Figure 12: Near-tip stress components plotted versus the plastic (eigen) strain for the penny-shaped inhomogeneity with model parameters defined in the text.

Figure 13: Cross-section of the model compaction band: A, (x, z)-plane; B, (x, z)-plane. When the band tip-line advances, marked by the dashed lines, the tip-incipient areas under stress $\sigma^{m}_{ij}$ will join the compaction band by undergoing a plastic strain $\epsilon^{p}_{ij}$.

Figure 14: Iterative scheme resolving the tip-incipient stress $\sigma_{ij}$ and plastic strain $\epsilon^{p}_{ij}$.

Figure 15: Convergent behaviors of the three principal plastic strains $\epsilon^{p}_{ii}$ for different guesses, i.e. uni-axial and tri-axial.
**Figure 16:** Effective shear modulus $G_p$ during the plastic shear deformation increase linearly with the in-band shear strain $\varepsilon_{xz}^p$.

**Figure 17:** Upper: for initial in-band shear strain $\varepsilon_{xz}^{p(0)} \in [0.01, 0.25]$, evolution of the shear strain $\varepsilon_{xz}^p$ as a function of the iteration number $n$. Lower: convergence of other components of $\varepsilon_{ij}^p$ that are independent to the initial shear strain.

**Figure 18:** A. Cartesian coordinates on cross-section planes of the ellipsoidal inhomogeneous inclusion. B. polar coordinate $(\lambda, r)$ about the ellipsoid tip-line at longitude angle $\theta$. 
Figure 19: When the band heights are limited by the dune boundary, the regions of influence are mainly determined the heights rather than the lengths.

\[ \sigma_{zz} = \frac{a_y}{a_x} = 0.5, \quad \sigma_{yy} = \sigma_{xx} = -20 \times 10^6 [\text{Pa}], \quad x = y = 0, \quad z \in [0 \ 1] \ (a_x \cdot a_y)^{1/2} \]

Figure 20: Normal stress \( \sigma_{zz} \) evaluated along \( z \)-axis when the remote normal stress \( \sigma_{zz}^{\infty} \) increases by up to 10%